ON OUTWARDLY SIMPLE LINE FAMILIES

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In **(5)** Hammer and Sobczyk defined an *outwardly simple line family* in the plane as a family of lines in the plane having the property that each point outside some given circle lies on exactly one line of the family, and they characterized planar outwardly simple line families as follows **(5)**: the extended diameters[1] of a convex body, whose boundary has no pair of parallel line segments in it, form an outwardly simple line family; moreover, each outwardly simple line family is the family of extended diameters of a convex body having constant width. (In the following we shall accordingly consider only convex bodies without parallel line segments in their boundaries and for expediency such bodies will be referred to as convex bodies.)

Given an outwardly simple line family $\mathfrak{L}$ let us define a function N, called the incidence function of $\mathfrak{L}$, by putting, for $x \in E_2$, $N(x)$ equal to the number of lines in $\mathfrak{L}$ which pass through x. (If this is infinite, then $N(x) = \infty$.) We denote $N^{-1}(k)$ by N_k where $k = 0, 1, \ldots, \infty$. The purpose of this paper is to study the nature of the sets N_k.

1. Some general theorems. As notation to be used throughout the following, if A is any set, then the symbols $H(A)$, A^0, $\bar{A}$, and Br A will denote the convex hull, interior, closure, and boundary respectively of A. In particular, if $z, w \in E_2$, $[z, w] = H(\{z, w\})$ and $(z, w) = [z, w] - \{z, w\}$. We begin by citing the following analytical characterization of an outwardly simple line family.

THEOREM 1 (Hammer and Sobczyk, **4**). *$\mathfrak{L}$ is an outwardly simple line family if and only if $\mathfrak{L}$ is the family of lines whose equations are given by*

$$x \sin \alpha - y \cos \alpha = p(\alpha), \qquad \alpha \in [0, \pi],$$

where p is a function defined on $[0, \pi]$ such that $p(0) = -p(\pi)$ and p satisfies the following two Lipschitz conditions: there exists a $k > 0$ such that

$$|p(x) - p(y)| \leqslant k\,|x - y|$$

and

$$|p(x) + p(y)| \leqslant k\,|\pi - (x - y)|$$

whenever

$$0 \leqslant y \leqslant x \leqslant \pi.$$

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[1]A chord of a convex body is a diameter if there exists a (distinct) pair of parallel support lines at its ends.

We shall call the above function p the *distance function* of $\mathfrak{L}$. Then using this analytical representation of $\mathfrak{L}$, it is easy to prove the following two facts (cf. Hammer and Sobczyk, **5**; **6**):

THEOREM 2. $N_0 = \emptyset$.

THEOREM 3. *For some* $k \geqslant 3$, $N_k \neq \emptyset$.

In **(2)** Grünbaum states without proof that both the centroid and the critical point of a convex body with respect to the Minkowski measure of symmetry have (diameter) incidence $\geqslant 3$. In fact, Neumann **(7)** has shown that there is only one such critical point and there are at least three chords which are divided by the critical point into the critical ratio. It is easily shown that each such chord is a diameter.

From Theorem 1 we also obtain the following lemma.

LEMMA 1. *Let N and p be the incidence and distance functions of an outwardly simple line family. Then $N_k \neq \emptyset$ if and only if there exists a $\beta \in (0, \pi]$ and c such that $c \sin(\alpha - \beta) = p(\alpha)$ for exactly k values of α in $(0, \pi]$.*

Using this lemma and the fact that a non-zero analytic function (i.e. representable in a Taylor series) on a finite interval can have only a finite number of zeros, we can easily derive the following facts, which are useful in constructing examples:

1. If p is a *piecewise analytic* (i.e. it consists of a finite number of analytic arcs) distance function whose pieces are not of the form $c \sin(\alpha - \beta)$ for any c and β, then $N_\infty = \emptyset$.

2. If p is a piecewise analytic distance function whose pieces are linear or of the form $c \sin(\alpha - \beta)$, then N has a finite maximum (i.e.,

$$\max\{k : N_k \neq \emptyset \quad \text{and} \quad k \neq \infty\}$$

exists).

3. If p is a piecewise analytic distance function, then N_∞ is finite.

In particular, if p consists of finitely many non-horizontal line segments, then N has a finite maximum and $N_\infty = \emptyset$. Moreover, for each odd number $n \geqslant 3$ we can easily construct such a distance function whose incidence function has a finite maximum equal to n.

It is readily seen that the distance function of an outwardly simple line family generated by the extended diameters of a convex polygon is piecewise analytic whose pieces are all of the form $c \sin(\alpha - \beta)$ (see § 2), and hence, N_∞ is non-empty and finite and N has a finite maximum. In the general case, however, N_∞ may be empty even if all other N_k are non-empty; and N_∞ may be infinite and non-closed even when the finite maximum is 3. The following examples illustrate this.

Example 1. N_∞ may be empty even with $N_k \neq \emptyset$ for infinitely many k.

For each $n \geqslant 3$ and $0 \leqslant m < n$ we define I_{nm} to be the line segment $[a, b]$ where

(i) if n is even and m even,

$$a = \left(\left(1 + \frac{m}{n}\right)2^{-n-1}, \left(\left(1 + \frac{m}{n}\right)2^{-n-1}\right)^3\right),$$

$$b = \left(\left(1 + \frac{m+1}{n}\right)2^{-n-1}, \left(\left(1 + \frac{m+1}{n}\right)2^{-n-1}\right)^3\right);$$

(ii) if n is even and m odd,

$$a = \left(\left(1 + \frac{m}{n}\right)2^{-n-1}, \left(\left(1 + \frac{m}{n}\right)2^{-n-1}\right)^3\right),$$

$$b = \left(\left(1 + \frac{m+1}{n}\right)2^{-n-1}, \left(\left(1 + \frac{m+1}{n}\right)2^{-n-1}\right)^3\right);$$

(iii) if n is odd,

$$a = (2^{-n}, (2^{-n})^2) \quad \text{and} \quad b = (2^{-n-1}, (2^{-n-1})^2).$$

In addition let I_0 be the line segment joining $(0, \pi)$ with $(1/8, 1/64)$. Now let p be the "saw-toothed" function defined by $p = \bigcup \{I_{nm} : n \geqslant 3$ and $0 \leqslant m < n\} \cup I_0$ To show that p is a valid distance function we need only verify the two Lipschitz conditions in Theorem 1. The first condition follows from the easily proved fact that the absolute values of the slopes of the segments I_{nm} are bounded by 1. For the second condition let $0 \leqslant y \leqslant x \leqslant \pi$. Whenever $x - y \leqslant \pi - 1/8$ we have $p(y) + p(x) \leqslant \frac{1}{4} \leqslant 2(\pi - (x - y))$. And whenever $x - y \geqslant \pi - 1/8$ we have

$$p(y) + p(x) \leqslant y^2 + \frac{1}{64\pi - 8}(\pi - x) \leqslant \pi - (x - y).$$

Hence $p(x) + p(y) \leqslant 2(\pi - (x - y))$ for all $0 \leqslant y \leqslant x \leqslant \pi$.

To show $N_\infty = \emptyset$ we must show by Lemma 1 that there are no c and β for which $c \sin(\alpha - \beta) = p(\alpha)$ for infinitely many $\alpha \in (0, \pi]$. Let g be the function given by $g(\alpha) = c \sin(\alpha - \beta)$, $\alpha \in [0, \pi]$. If $g(0) \neq 0$, then g is bounded away from the origin and clearly can only intersect finitely many of the line segments in p. In the case $g(0) = 0$, we must have $c = 0$ or $\beta = 0$ or $\beta = \pi$. If $c = 0$ or $\beta = \pi$, or if $\beta = 0$ and $c < 0$, then g intersects p only once in $(0, \pi]$. If $\beta = 0$ and $c > 0$, then clearly there exists a $\delta > 0$ such that $\alpha^3 < \alpha^2 < g(\alpha)$ for $0 < \alpha \leqslant \delta$. Consequently, g does not intersect p over the interval $(0, \delta]$ and, hence, can only intersect p finitely many times in $(0, \pi]$. Therefore, $N_\infty = \emptyset$.

Now we show that $N_k \neq \emptyset$ for infinitely many k. (Actually it can be shown that each $N_k \neq \emptyset$ for $k \neq \infty$.) Let m be any even number $\geqslant 4$; then by taking $c = -2^{-m-2}$ and $\beta = \pi/2 + 3 \cdot 2^{-m-2}$ we see that $\alpha^3 \leqslant c \sin(\alpha - \beta) \leqslant \alpha^2$ for $2^{-m-1} \leqslant \alpha \leqslant 2^{-m}$. But this implies $N((c, \beta)) \geqslant m$, which completes the proof.

Example 2. N_∞ may be infinite and non-closed even when the finite maximum is 3.

Let p be defined as follows: for odd n and $2^{-n-1} \leqslant \alpha \leqslant 2^{-n}$ define

$$p(\alpha) = 2^{-2n-2} \sin(\alpha - \pi/2 - 3\cdot 2^{-n-2}).$$

Then we extend this p linearly over the interval $[0, \pi]$ so that $p(0) = p(\pi) = 0$. It is easily shown that p satisfies the two Lipschitz conditions and hence is a distance function. Using Lemma 1, it is clear that the maximum finite incidence is 3 and that the only points having incidence ∞ are of the form (c_n, β_n), where $c_n = 2^{-n-2}$ and $\beta_n = \pi/2 + 3\cdot 2^{-n-2}$. Moreover, the origin has incidence 1 and belongs to $\bar{N}_\infty$.

The set N_∞ is countable whenever N has a finite maximum (see Theorem 5), but in the general case may be uncountable as shown by the following example.

Example 3. N_∞ may be uncountable. Define $p(\alpha) = \sin\alpha \sin(\log\alpha)$ for $\alpha \in (0, \pi]$ and $p(0) = 0$. Then it is easily checked that p has a bounded derivative and, hence, will be a distance function. But for each $c \in [-1, +1]$, $c \sin\alpha = p(\alpha)$ for infinitely many values of α. Therefore, by Lemma 1, $[-1, +1] \subset N_\infty$.

However, N_∞ always has measure zero as shown by the following theorem.

THEOREM 4 (Hammer and Sobczyk, **6**). *The measure of N_k where k is even or ∞ is zero.*

Of special interest is the case when N has a finite maximum. To investigate this case we first establish the following preliminaries. Let $\mathfrak{L}$ be an outwardly simple line family and let $x \in E_2$. Suppose L_α and L_β are two distinct lines in $\mathfrak{L}$, with directions α and β respectively, which pass through x and such that $x \notin L_\gamma$ whenever $\gamma \in I_{\alpha,\beta}$, where $I_{\alpha,\beta} = (\alpha, \beta)$ if $\beta > \alpha$ and

$$I_{\alpha,\beta} = [0, \beta) \cup (\alpha, \pi] \text{ if } \alpha > \beta.$$

Such an ordered pair of lines in $\mathfrak{L}$ through x will be called a *consecutive* pair. If $L_\alpha, L_\beta, L_\gamma$ are distinct lines in $\mathfrak{L}$ we let $T_{\alpha,\beta,\gamma}$ denote the closed triangle determined by them. Now, if (L_α, L_β) is a consecutive pair of lines through x, we define $S_{\alpha,\beta}(x) = (\bigcup\{T_{\alpha,\beta,\gamma} : \gamma \in I_{\alpha,\beta}\})^0$ and call it the *strip* at x determined by the consecutive pair (L_α, L_β). It is easy to see that $S_{\alpha,\beta}(x)$ must lie entirely in one of the quadrants determined by the lines L_α and L_β. Finally, by a *wedge* at x we mean any one of the four open quadrants, determined by a consecutive pair of lines in $\mathfrak{L}$ at x, which does not intersect any L_α that contains x. Now with the aid of Theorem 1 and Lemma 2 of **(6)** we can prove the following lemma.

LEMMA 2. (i) *If $y \in S_{\alpha,\beta}(x)$, then there exist two distinct angles $\gamma, \xi \in I_{\alpha,\beta}$ such that $y \in L_\gamma \cap L_\xi$.*

(ii) *If $y \in L_\gamma \cap \operatorname{Br} S_{\alpha,\beta}(x)$ and $x \in L_\gamma$* with *$x \neq y$, then there exists a $\xi \in I_{\alpha,\beta}$ such that $y \in L_\xi$.*

(iii) *If y belongs to a wedge determined by a consecutive pair (L_α, L_β), then there exists a $\gamma \in I_{\alpha,\beta}$ such that $y \in L_\gamma$.*

Now we can prove the following theorem.

THEOREM 5. *Suppose the incidence function N has a finite maximum equal to m. Then*
(i) *m is odd;*
(ii) *N_m is open, provided $N_m \cap \bar{N}_\infty = \emptyset$;*
(iii) *N_∞ is countable.*

Proof. For (i) let $x \in N_m$. Then there are m strips and $2m$ wedges determined by the m lines passing through x. Each strip intersects exactly $m - 1$ wedges, so the total number of intersections of wedges with strips is $m(m - 1)$. Now suppose m were even. Then the maximum number of strips intersecting any given wedge W would be $\leqslant \frac{1}{2}m - 1$; for if the n strips $S_1(x), \ldots, S_n(x)$ where $n \geqslant \frac{1}{2}m$ all intersected W, then by Lemma 2 any point in

$$W \cap \left(\bigcap_{i=1}^{n} S_i(x) \right)$$

would have incidence $\geqslant 2n + 1 \geqslant m + 1$. But since N_∞ has measure zero, there would be a point of finite incidence $\geqslant m + 1$ in

$$W \cap \left(\bigcap_{i=1}^{n} S_i(x) \right),$$

which is a contradiction. Now since there are $2m$ wedges, the maximum number of intersections of strips with wedges is thus $\leqslant 2m(\frac{1}{2}m - 1) = m^2 - 2m$, which is less than $m(m - 1)$, the actual number. Hence, m must be odd.

For (ii), let x be given in N_m. Let $L_{\alpha_1}, \ldots, L_{\alpha_m}$ denote the m lines passing through x with $\alpha_1 < \alpha_2 < \ldots < \alpha_m < \pi$. Now let C be a circle centred at x and choose $a_i \in L_{\alpha_i} \cap C$ so that $a_1, \ldots, a_m$ are consecutive and in counterclockwise order around C. We shall say that $S_i = S_{\alpha_i, \alpha_{i+1}}(x)$ is a *right* strip at x if $(a_{i+1}, x) \subset \bar{S}_i$. Otherwise it is a left strip.

Now suppose there are two consecutive right strips among the strips at x, say S_1 and S_2. Consider the family $\mathfrak{S}$ of the $2k - 3$ strips $\{S_4, S_5, \ldots, S_{2k}\}$ where $m = 2k + 1$. Then it is clear that if $\mathfrak{T}$ is a subfamily of $\mathfrak{S}$ consisting of t right (or left) strips, then the open set $(\bigcap \mathfrak{T}) \cap S_1 \cap S_2 \neq \emptyset$. From this it follows that there can be at most $k - 2$ right strips (and at most $k - 2$ left strips) in $\mathfrak{S}$, since if there were t right (or left) strips, where $t \geqslant k - 1$, then $t + 2$ strips would intersect in $E_2 - N_\infty$ giving a point of finite incidence $\geqslant 2(t + 2) + 1 \geqslant 2k + 3$, by Lemma 2. Hence the maximum number of strips, both left and right, in $\mathfrak{S}$ would be $\leqslant 2(k - 2)$, a contradiction.

Hence, the strips must alternate from right to left and it then follows that each of the $2m$ wedges at x intersects exactly k strips. Now, assuming x is the origin, pick r so that

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$$0 < r < \min_i \left| p\left(\frac{\alpha_i + \alpha_{i+1}}{2}\right) \right| .$$

Then letting D be the disk of radius r about x we have $N(y) \geqslant 2k + 1 = m$ for each $y \in D$ by Lemma 2 again. Now if $x \notin \bar{N}_\infty$, $D - \bar{N}_\infty$ will be an open set in N_m containing x. Thus N_m is open.

For the proof of (iii), let $x \in N_\infty$ and without loss of generality we may assume that x is the origin. Then $F = p^{-1}(0)$ is infinite and closed. If G, the complement of F, had infinitely many components, there would be infinitely many distinct pairs, $\{(\alpha_a, \beta_a) : a \in A\}$, for which $\alpha_a, \beta_a \in F$ and $(\alpha_a, \beta_a) \subseteq G$. Let A_1 consist of all $a \in A$ such that $p(\alpha) > 0$ for $\alpha \in (\alpha_a, \beta_a)$. Then either A_1 or $A - A_1$ is infinite. Without loss of generality we can assume that A_1 is infinite. For each $a \in A_1$ put $f(a) = \sup\{p(\alpha) : \alpha \in [\alpha_a, \alpha_b]\}$. Now pick m distinct elements from $A_1, a_1, a_2, \ldots, a_m$ such that no $\alpha_{ai} = 0$.

Put $\delta = \min\{f(a_i) : 1 \leqslant i \leqslant m\}$ and $\gamma = \min\{\alpha_{a_i} : 1 \leqslant i \leqslant m\}$. Then clearly if $(c, \beta) \in B = \{(c, \beta) : 0 \leqslant c < \delta$ and $0 \leqslant \beta < \delta\}$, we have that $c \sin(\alpha - \beta) = p(\alpha)$ for more than m values of α. Hence, $B \subseteq N_\infty$, which contradicts the zero-measure of N_∞. Therefore, G has only finitely many components and thus F contains an interval. So if $x \in N_\infty$, there exist $\alpha_x, \beta_x \in [0, \pi)$ such that $x \in L_\gamma$ for each $\gamma \in [a_x, \beta_x]$. But if $[\alpha_x, \beta_x]$ were to intersect $[\alpha_y, \beta_y]$ for $x \neq y$, we would obtain two parallel lines in the given outwardly simple line family. Since there can be at most only a countable number of disjoint intervals on the line, N_∞ must be countable.

It is unknown whether the hypothesis in Theorem 5 (ii) is necessary.

A very attractive conjecture is that "the intermediate value property" holds for the incidence function; that is, if $N_{k+1} \neq \emptyset$, then $N_k \neq \emptyset$. Although unsolved in the general case, it is true whenever the distance function is analytic as shown by the following theorem (it is also true in the polygonal case; see Theorem 11).

THEOREM 6 *If p is piecewise analytic, then $N_{2k} \neq \emptyset$ implies that $N_{2k+1} \neq \emptyset$ and $N_{2k-1} \neq \emptyset$. Moreover, if p is analytic, then $N_{2k+1} \neq \emptyset$ implies that $N_{2k} \neq \emptyset$.*

Proof. First we establish the following proposition: if is p piecewise analytic and W is a wedge at x (the origin) which hits exactly m strips and $(c, \beta) \in W$, then there exists a $k \in (0, 1)$ such that $N((\delta c, \beta)) = 2m + 1$ for each $\delta \in (0, k)$. For the proof, suppose W hits the m distinct strips $S_1, \ldots, S_m$, where $S_i = S_{\alpha_i, \gamma_i}(x)$. Then the piecewise analyticity implies that for each i there exists a $k_i \in (0, 1)$ such that for $\delta \in (0, k_i)$, $c\delta \sin(\alpha - \beta) = p(\alpha)$ has exactly two roots in $[\alpha_i, \gamma_i]$. Also we find a $k_0 \in (0, 1)$ such that for each $\delta \in (0, k_0)$, $c\delta \sin(\alpha - \beta) = p(\alpha)$ has exactly one root in the interval corresponding to the wedge W. Now take

$$k = \min_i k_i$$

to complete the proof of the proposition. Also we remark that if W_1 and

W_2 are adjacent wedges at some x and if W_1 hits m strips, then W_2 must hit m, $m - 1$, or $m + 1$ strips. From this it easily follows that if one wedge at x hits m strips and another wedge hits n strips, there exist wedges at x hitting all intermediate values.

Now suppose p is piecewise analytic and $x \in N_{2k}$. Without loss of generality we can suppose that x is the origin since a translation preserves the piecewise analyticity of the distance function. Since there are $4k$ wedges at x and $2k$ strips at x with each strip hitting $2k - 1$ wedges, the total number of intersections of strips with wedges is $2k(2k - 1) = 4k^2 - 2k$. Now, if each wedge hits $\geqslant k$ strips the number of intersections would be $\geqslant 4k \cdot k = 4k^2$, a contradiction. Also, if each wedge hits $\leqslant k - 1$ strips the number of intersections would be $\leqslant 4k(k - 1) = 4k^2 - 4k$, a contradiction. Hence, there exist wedges hitting k and $k - 1$ strips respectively. Now applying the above proposition we get that $N_{2k+1} \neq \emptyset$ and $N_{2k-1} \neq \emptyset$.

For the second part, suppose p is analytic and $x \in N_{2k+1}$, where again we may assume x to be the origin. Let $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_m = \pi$, where $m = 2k + 1$, be the zeros of p. Then it is clear that one of two cases arises: either (Case I) each wedge at x hits exactly k strips or (Case II) there are adjacent wedges at x hitting k and $k - 1$ strips respectively.

For Case I, for each $0 \leqslant i < m$ let C_i consist of all points $(x(\alpha), y(\alpha))$ such that

$$x(\alpha) = p'(\alpha) \cos \alpha + p(\alpha) \sin \alpha,$$
$$y(\alpha) = p'(\alpha) \sin \alpha - p(\alpha) \cos \alpha,$$

where α ranges throughout the interval $[\alpha_i, \alpha_{i+1}]$. It is easily checked (by just using the existence of p'') that the tangent line to C_i at α exists and has slope equal to $\tan \alpha$, and hence the equation of the tangent line is

$$x \sin \alpha - y \cos \alpha = p(\alpha),$$

which is the same as L_α. Moreover, each L_α for $\alpha \in [\alpha_i, \alpha_{i+1}]$ intersects C_i just once, unless C_i is a line, for if C_i is not a line and L_α hits C_i in two places, then by the continuity of the tangent line to C_i we could find a L_η parallel to L_α. From this it follows that each point on C_i has incidence 1 relative to $[\alpha_i, \alpha_{i+1}]$. If a point in $S_i = S_{\alpha_i, \alpha_{i+1}}(x)$ had incidence 3 or more relative to $[\alpha_i, \alpha_{i+1}]$, then C_i would be tangent to each of three intersecting lines and hence would hit one of them more than once. Therefore, each point in S_i has incidence 2 relative to $[\alpha_i, \alpha_{j+1}]$. A similar argument shows that each point of any wedge determined by α_i and α_{i+1} has incidence 1 relative to $[\alpha_i, \alpha_{i+1}]$. Now let W be any wedge at x which intersects the k strips $S_{i_0}, S_{i_1}, \ldots, S_{i_{k-1}}$. Let C_{i_m} be the above curve associated with S_{i_m}. Then it is clear that there exists an i_m and β_1 and β_2 such that $\alpha_{i_m} < \beta_1 < \beta_2 < \alpha_{i_m+1}$ and the set $C = \{(c, \beta) \in C_{i_m} : \beta_1 < \beta < \beta_2\}$ is contained in $\bigcap_{n \neq m} S_{i_n}$. Then it follows that $C \subseteq N_{2k}$.

For Case II, we can assume that W_1 is a wedge between α_0 and α_1 and W_2

is an adjacent wedge between α_1 and α_2 such that W_1 hits $k-1$ strips and W_2 hits k strips. Since each strip, except S_0 and S_1, either hits or misses both W_1 and W_2, we see that S_1 misses both W_1 and W_2 and S_0 hits only W_2. But this implies that p has a relative extreme at α_1 so that $p'(\alpha_1) = 0$. Then we can show without difficulty that for each i for which S_i hits both W_1 and W_2 (there are $k-1$ such S_i), there exists a δ_i and $\beta_i \in (\alpha_0, \alpha_1)$ such that $c\sin(\alpha - \beta) = p(\alpha)$ has exactly two roots in $[\alpha_i, \alpha_{i+1}]$ whenever $c \in [0, \delta_i]$ and $\beta \in (\beta_i, \alpha_i)$. Also we can find a c and a β such that

$$0 < c < \min_i \delta_i, \qquad \max_i \beta_i < \beta < \alpha_i,$$

and $c\sin(\alpha - \beta) = p(\alpha)$ has exactly two roots in $[\alpha_0, \alpha_1]$, at one of which the two curves $c\sin(\alpha - \beta)$ and $p(\alpha)$ are tangent. Then we clearly have $N((c, \beta)) = 2k$, which finishes the proof of the theorem. (Actually Case II works out when p is piecewise analytic too.)

Now we end this section with the following result, which says that the points of incidence $\geqslant 2$ and $\geqslant 3$ are "somewhat star-shaped" (easy examples show that they are not necessarily star-shaped from a point).

THEOREM 7. *The sets* N_1, $E_2 - N_1$, *and* $E_2 - (N_1 \cup N_2)$ *are polygonally connected.*

Proof. Let $N(x) \geqslant 2$ and $N(y) \geqslant 2$. And let $x \in L_\alpha \cap L_\beta$ and $y \in L_\gamma \cap L_\delta$, where $\alpha \neq \beta$ and $\gamma \neq \delta$. Let P be the interior of the polygon which is the union of the four triangles determined by the four possible triples of lines selected from $\{L_\alpha, L_\beta, L_\gamma, L_\delta\}$. Then $P \cap (N_1 \cup N_2) = \emptyset$ and $x, y \in \mathrm{Br}\, P$, from which it follows that $E_2 - N_1$ and $E_2 - (N_1 \cup N_2)$ are polygonally connected. The fact that N_1 is polygonally connected follows easily from a result of Hammer and Sobczyk (**6**, Theorem 6) that says that $L - N_1$ is a finite interval for any L in the outwardly simple line family.

2. Polygonal outwardly simple line families. In this section we shall describe the incidence structure for an outwardly simple line family generated by the extended diameters of a convex polygon (without pairs of parallel sides). Such a family will be called a *polygonal outwardly simple line family.* In the following, P will denote such a polygon and V and S its sets of vertices and sides respectively. As before $\mathfrak{L}$ will be the family of extended diameters of P, and N will be the associated incidence function. The following theorems, the first two of which are straightforward and are given without proof, will then describe fairly well the incidence structure of a polygonal outwardly simple line family.

THEOREM 8. (i) *Each vertex has incidence* 1 *or* ∞ *and each element of* N_∞ *is a vertex;* (ii) *each non-vertex boundary point has incidence* 1 *or* 2*;* (iii) $x \in N_\infty$ *if and only if there exists* $L \in S$ *such that* L *is parallel to a support line at* x*;*

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(iv) $x \in \text{Br}\, P \cap N_2$ *if and only if there exists a diameter* $[a, b] \in S$ *such that* $x \in (a, b)$.

THEOREM 9. *The outwardly simple line family generated by a convex polygon, with incidence function* N, *is the same as the outwardly simple line family generated by the polygon* $H(N_\infty)$.

From Theorem 9 and the next theorem it follows that any polygonal outwardly simple line family is the same as one generated by some polygon with an odd number of vertices each of which has incidence ∞.

THEOREM 10. *The number of points in* N_∞ *is odd.*

Proof. From Theorem 9 we can assume all vertices to have incidence ∞. Now enumerate the points of V counterclockwisely as $v_1, \ldots, v_n$ and let $L_k = [v_k, v_{k+1}]$ for $k < n$ and $L_n = [v_n, v_1]$. By Theorem 8 it readily follows that for each i there exists a unique k_i such that L_{k_i} is parallel to a support line at v_i and, moreover, that L_i and L_{i-1} are parallel to support lines at v_{k_i+1} and v_{k_i} respectively. Now letting v_2 so correspond to L_{k_2}, the $k_2 - 1$ vertices $v_2, \ldots, v_{k_2}$ will correspond to the $n - k_2 + 2$ sides $L_{k_2}, \ldots, L_n, L_1$. Hence, $k_2 - 1 = n - k + 2$ and $n = 2k_2 - 3$.

THEOREM 11. *Let* n *be the cardinality of* N_∞ *and* $m = \max\{k : N_k \neq \emptyset,\ k \neq \infty\}$. *Then*

(i) m *is odd and* $\leqslant n$;

(ii) *if* $0 < k \leqslant m$, *then* $N_k \neq \emptyset$;

(iii) *if* k *is odd and* $1 \leqslant k \leqslant m$, *then* $N_k = G_k \cup F_k$, *where* G_k *is a non-empty open set and* F_k *is a finite set, which is non-empty if and only if* $1 < k < m$;

(iv) *if* k *is even and* $1 < k < m$, *then* N_k *consists of finitely many open line segments.*

Proof. By Theorem 10 we can assume that $V = N_\infty$. And by Theorem 5 it follows that m is odd and $\leqslant n$.

For the proof of (ii), it suffices, according to Theorem 6, to show that $N_{2p+1} \neq \emptyset$ implies that $N_{2p} \neq \emptyset$. First we shall show that $N_{2p+1} \neq \emptyset$ implies that $N_{2p+1}{}^0 \neq \emptyset$. Hence, suppose that we have $x \in N_{2p+1} - N_{2p+1}{}^0$. If each wedge at x hits exactly p strips, then by the argument in Theorem 5, we would have $x \in N_{2p+1}{}^0$. The total number of intersections of wedges with strips is $4p^2 + 2p$. If each wedge hit fewer than p strips, the number of intersections would be $\leqslant 2(2p + 1)(p - 1) = 4p^2 + 2p - 2$. On the other hand, if each wedge hit more than p strips, the number of intersections would be $\geqslant 2(2p + 1)(p + 1) = 4p^2 + 6p + 2$. Hence, there exists a wedge hitting fewer than p strips and a wedge hitting more than p strips. Therefore, there exists a wedge W which hits exactly p strips. Now let O be a disk centred at x which is disjoint from all $[u, v]$, where $x \notin [u, v]$ and $u, v \in V$. Then clearly $O \cap W$ is open and is contained in N_{2p+1}. Hence $N_{2p+1}{}^0 \neq \emptyset$.

Now pick $y \in N_{2p+1}{}^0$. For each $v \in V$ let T_v be the triangle $H(\{v, a_v, b_v\})$, where $a_v, b_v \in V$ and $[a_v, b_v]$ is parallel to a line of support at v. Let $\{T_{v1}, \ldots, T_{vk}\}$ be the collection of sets T_v in whose interior y lies. Then clearly

$$Q = \bigcap_{i=1}^{k} T_{vi}$$

is a polygon such that $Q^0 \subseteq N_{2p+1}$. Moreover, it is not difficult to show that Q has the property that if $(a, b) \subseteq \mathrm{Br}\, Q$, then either $(a, b) \subseteq N_{2p} \cap \mathrm{Br}\, N_{2p-1}$ or $(a, b) \subseteq N_{2p+2} \cap \mathrm{Br}\, N_{2p+3}$. Now consider the set $A = \bigcup\{N_{2q+1} : q \geqslant p\}$, which is a union of such polygons as Q above. Let $(a, b) \subseteq \mathrm{Br}\, A$. Then by the above property we must have $(a, b) \subseteq N_{2p}$. Hence $N_{2p+1} \neq \emptyset$ implies $N_{2p} \neq \emptyset$.

For the proof of (iii), let k be odd, and >1. (If $k = 1$, then N_1 is easily shown to be open.) Let $\mathfrak{M}$ be the finite collection of diameters of P which contain two distinct vertices. Suppose $x \in N_k$ and $\{T_{v1}, \ldots, T_{vr(x)}\}$ is the collection of the sets T_v in whose interior x lies. Let O be a disk centred at x which misses all diameters in $\mathfrak{M}$ which do not contain x. Now if $k = r(x)$, we clearly have that

$$x \in O \cap \left(\bigcap_{i=1}^{r(x)} T_{vi}^{\ 0} \right) \subseteq N_k,$$

so that $x \in N_k^0$. In the case $k - r(x) > 1$, x must lie in two or more members of $\mathfrak{M}$. Hence, there can be only finitely many x for which $k - r(x) > 1$. In the case when $k - r(x) = 1$, x lies in exactly one $[u, v] \in \mathfrak{M}$. Then we have that

$$\emptyset \neq O \cap \left(\bigcap_{i=1}^{r(x)} T_{vi} \right) \cap T_u^{\ 0} \cap T_v^{\ 0} \subseteq N_{k+1},$$

which means that N_{k+1} has positive measure. Hence $k - r(x)$ is never 1 and we have $N_k = G_k \cup F_k$, where G_k is the open set $\{x : k = r(x)\}$ and F_k the finite set $\{x : k - r(x) > 1\}$. From the proof of (ii) it follows that each G_k is non-empty and Theorem 5 implies that $F_m = \emptyset$. To show that $F_k \neq \emptyset$ for $1 < k < m$, consider any p for which $5 \leqslant 2p + 1 \leqslant m$. Let $A = \bigcup\{N_{2q+1}: q \geqslant p\}$. Then clearly we can find two non-parallel line segments (a, b) and (b, c) in $\mathrm{Br}\, A$, which necessarily are contained in N_{2p}. From this it follows that $b \in F_{2p-1}$, which completes the proof of (iii).

For the proof of (iv) let $x \in N_{2p}$. If $x \notin \bigcup \mathfrak{M}$, then there are distinct triangles $T_{v1}, \ldots, T_{v2p}$ such that

$$x \in \bigcap_{i=1}^{2p} T_{vi}.$$

If O is a disk centred at x which misses $\bigcup M$, then we have that $N_{\cdot p}$ contains the non-empty open set

$$O \cap \left(\bigcap_{i=1}^{2p} T_{vi} \right),$$

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which contradicts the zero-measure of N_{2p}. Hence $N_{2p} \subseteq \cup\mathfrak{M}$. Moreover, by employing arguments similar to those above we can show that if $x \in N_{2p}$, then there exists (a, b) such that $x \in (a, b) \subset \mathrm{Br}\, N_{2p+1} \cap \mathrm{Br}\, N_{2p-1} \subset N_{2p}$, which completes the proof of the theorem.

With regard to Theorem 11, m is not necessarily the same as n, since one can easily construct a polygon where $n = 9$ and $m = 7$, where incidentally, N_7 is not connected.

One interesting problem is to what extent the incidence properties of polygons as exemplified in Theorem 11 can be generalized to more general outwardly simple line families. In particular, Theorems 4, 5, and 6 provide some answers to this question.

Finally we have the following result which attests to the "extreme" nature of the triangle relative to the incidence of its diameters.

THEOREM 12 (Eggleston, **1**, p. 122). *Let N be the incidence function associated with a convex body C. Then, $C^0 \subseteq N_3$ if and only if C is a triangle.*

3. Further aspects. It should be noted that outwardly simple line families occur in other connections besides the extended diameters of a convex body. For example, the family of all lines which divide a given convex curve into two equal (in arc length) arcs is an outwardly simple line family. Also, all lines which divide a given bounded, open set whose closure is connected into two equal (in measure) parts form an outwardly simple line family.

For further aspects of outwardly simple line families, including some possibilities for generalizing the notion of an outwardly simple line family to E_n, the reader is referred to Hammer **(3)**, Smith **(8)**, and Sobczyk **(9)**.

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