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DETERMINING THE NUCLEOLUS OF COMPROMISE STABLE GAMES

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Abstract

The main goal is to illustrate that the so-called indirect function of a cooperative game in characteristic function form is applicable to determine the nucleolus for a subclass of coalitional games called compromise stable transferable utility (TU) games. In accordance with the Fenchel–Moreau theory on conjugate functions, the indirect function is known as the dual representation of the characteristic function of the coalitional game. The key feature of a compromise stable TU game is the coincidence of its core with a box prescribed by certain upper and lower core bounds. For the purpose of the determination of the nucleolus, we benefit from the interrelationship between the indirect function and the prekernel of coalitional TU games. The class of compromise stable TU games contains the subclasses of clan games, big boss games and 1- and 2-convex n-person TU games. As an adjunct, this paper reports the indirect function of clan games for the purpose of determining their nucleolus.

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1. Compromise stable TU games

Fix the finite player set N and its power set $\mathcal{P}(N) = \{S \mid S \subseteq N\}$ consisting of all the subsets of N (including the empty set $\emptyset$). A *cooperative transferable utility game*, or TU game for short, is given by the so-called *characteristic function* $v : \mathcal{P}(N) \to \mathbb{R}$ satisfying $v(\emptyset) = 0$. That is, the TU game v assigns to each coalition $S \subseteq N$ its *worth* $v(S)$ amounting to the monetary benefits achieved by cooperation among the members of S.

In the framework of set-valued solution concepts for TU games, we aim to determine the prekernel for a special subclass of TU games called compromise stable TU games [10] using a new mathematical tool called the indirect function [1, 6]. The economic interpretation of this function is the following. An employer has to select among the players those who will produce the maximum profit to him. In case

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the nonempty coalition $S \subseteq N$ is selected, then its members will produce, using the resources that are available to the employer, a total amount of output whose monetary utility is represented by the worth $v(S)$. The expression $e^v(S, \vec{y}) = v(S) - \sum_{k \in S} y_k$, called the *excess* of coalition S at the payoff vector $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ in the TU game v, is thus the net profit the employer would obtain from the coalition S if the (possibly negative) salary required by the player i amounts to y_i for $i \in N$. Write $e^v(\emptyset, \vec{y}) = 0$. In accordance with the Fenchel–Moreau theory on conjugate functions, the indirect function provides a dual representation to TU games in the sense that indirect functions provide the same information as characteristic functions because a simple formula permits us to recover any characteristic function from its associated indirect function.

Definition 1.1 [6, page 292]. With every TU game $v : \mathcal{P}(N) \to \mathbb{R}$, there is associated the *indirect function* $\pi^v : \mathbb{R}^N \to \mathbb{R}$, given by

$$\pi^v(\vec{y}) = \max_{S \subseteq N} e^v(S, \vec{y}) = \max_{S \subseteq N} \Big[v(S) - \sum_{k \in S} y_k \Big] \quad \text{for all } \vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N.$$

Definition 1.2. The Core(v) of the TU game $v : \mathcal{P}(N) \to \mathbb{R}$ consists of *efficient* salary vectors of which all the excesses are nonpositive, that is,

$$\text{Core}(v) = \{\vec{y} \in \mathbb{R}^N \mid e^v(N, \vec{y}) = 0 \text{ and } e^v(S, \vec{y}) \leq 0 \text{ for all } S \subsetneq N, S \neq \emptyset\}. \tag{1.1}$$

Equivalently, $\vec{y} \in \text{Core}(v)$ if and only if $e^v(N, \vec{y}) = 0$ and $\pi^v(\vec{y}) = 0$.

Concerning the definition of compromise stable TU games, we follow the notation used in [10].

Definition 1.3. Let $v : \mathcal{P}(N) \to \mathbb{R}$ be a TU game.

(i) The *utopia demand* vector $\vec{M}^v = (M_k^v)_{k \in N} \in \mathbb{R}^N$ is given by $M_i^v = v(N) - v(N \setminus \{i\})$ for all $i \in N$.

(ii) The *minimum right* vector $\vec{m}^v = (m_k^v)_{k \in N} \in \mathbb{R}^N$ is given by

$$m_i^v = \max \Big[v(S) - \sum_{k \in S \setminus \{i\}} M_k^v \mid S \subseteq N, \ i \in S \Big] \quad \text{for all } i \in N. \tag{1.2}$$

(iii) The *core cover* $CC(v) \subseteq R^N$ consists of efficient payoff vectors representing compromises between utopia demands as well as minimum rights, that is,

$$CC(v) = \{\vec{y} \in \mathbb{R}^N \mid e^v(N, \vec{y}) = 0 \text{ and } m_i^v \leq y_i \leq M_i^v \text{ for all } i \in N\}.$$

(iv) The TU game v is called *compromise stable* if $CC(v) = \text{Core}(v)$.

We remark that the inclusion Core$(v) \subseteq CC(v)$ holds in general because the utopia demand vector $\vec{M}^v$ and the minimum right vector $\vec{m}^v$ are well known to be an upper and a lower bound for the core, respectively. As a first main contribution, we provide an alternative proof of the following characterisation of compromise stable TU games. For any nonempty coalition $T \subseteq N$ and any payoff vector $\vec{z} = (z_k)_{k \in N} \in \mathbb{R}^N$, write $\vec{z}(T) = \sum_{k \in T} z_k$, where $\vec{z}(\emptyset) = 0$.

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THEOREM 1.4. *A TU game $v : \mathcal{P}(N) \to \mathbb{R}$ is compromise stable if and only if*

$$v(S) \leq \max\Big[\sum_{k\in S} m_k^v,\ v(N) - \sum_{k\in N\setminus S} M_k^v\Big] \quad \text{for all } S \subseteq N,\ S \neq \emptyset. \tag{1.3}$$

PROOF. (i) Suppose that (1.3) holds. We prove the coincidence $CC(v) = \mathrm{Core}(v)$. It suffices to prove the inclusion $CC(v) \subseteq \mathrm{Core}(v)$. Suppose $\vec{y} = (y_k)_{k\in N} \in CC(v)$. Then $m_i^v \leq y_i \leq M_i^v$ for all $i \in N$. Let $S \subseteq N$, $S \neq \emptyset$. Clearly, $\vec{y}(S) \geq \vec{m}^v(S)$, whereas $\vec{y}(S) = v(N) - \vec{y}(N\setminus S) \geq v(N) - \vec{M}^v(N\setminus S)$. Hence, $\vec{y}(S) \geq \max[\vec{m}^v(S), v(N) - \vec{M}^v(N\setminus S)]$. From (1.3), $\vec{y}(S) \geq v(S)$ for all $S \subseteq N$, $S \neq \emptyset$, and so $\vec{y} \in \mathrm{Core}(v)$, provided $\vec{y} \in CC(v)$.

(ii) For the converse statement, suppose that the coincidence $CC(v) = \mathrm{Core}(v)$ holds. We aim to prove (1.3). Let $S \subseteq N$, $S \neq \emptyset$. We distinguish two cases.

Case 1. Assume $v(N) - \vec{M}^v(N\setminus S) < \vec{m}^v(S)$. We prove that $v(S) \leq \vec{m}^v(S)$. For that purpose, construct the efficient payoff vector $\vec{y} = (y_k)_{k\in N} \in \mathbb{R}^N$ such that $y_i = m_i^v$ for all $i \in S$ and

$$y_i = m_i^v + \frac{v(N) - \vec{m}^v(N)}{(\vec{M}^v - \vec{m}^v)(N\setminus S)} \cdot (M_i^v - m_i^v)$$

for all $i \in N\setminus S$. Then $m_i^v \leq y_i \leq M_i^v$ for all $i \in N\setminus S$, by our assumption. So, $\vec{y} \in CC(v)$ and so $\vec{y} \in \mathrm{Core}(v)$; thus, $\vec{y}(S) \geq v(S)$ or, equivalently, $\vec{m}^v(S) \geq v(S)$.

Case 2. Assume $v(N) - \vec{M}^v(N\setminus S) \geq \vec{m}^v(S)$. We prove that $v(S) \leq v(N) - \vec{M}^v(N\setminus S)$. We distinguish two subcases. Put $g^v(N) = \vec{M}^v(N) - v(N)$.

Subcase 2.1. Suppose that there exists at least one player $i \in S$ with $M_i^v - m_i^v \geq g^v(N)$. Construct the efficient payoff vector $\vec{y} = (y_k)_{k\in N} \in \mathbb{R}^N$ such that $y_i = M_i^v - g^v(N)$ and $y_j = M_j^v$ for all $j \in N\setminus\{i\}$. Then $m_j^v \leq y_j \leq M_j^v$ for all $j \in N$. So, $\vec{y} \in CC(v)$ and so $\vec{y} \in \mathrm{Core}(v)$. Thus, $\vec{y}(S) \geq v(S)$ or, equivalently, $v(S) \leq \vec{M}^v(S) - g^v(N) = v(N) - \vec{M}^v(N\setminus S)$.

Subcase 2.2. Suppose $M_i^v - m_i^v < g^v(N)$ for all $i \in S$. Without loss of generality, write $S = \{i_1, i_2, \ldots, i_s\}$ such that $M_{i_1}^v - m_{i_1}^v \leq M_{i_2}^v - m_{i_2}^v \leq \cdots \leq M_{i_s}^v - m_{i_s}^v$. Then there exists $2 \leq t \leq s$ such that

$$\sum_{k=1}^{t-1}[M_{i_k}^v - m_{i_k}^v] < g^v(N) \quad \text{and} \quad \sum_{k=1}^{t}[M_{i_k}^v - m_{i_k}^v] \geq g^v(N).$$

Construct the efficient payoff vector $\vec{y} = (y_k)_{k\in N} \in \mathbb{R}^N$ such that $y_i = M_i^v$ for all $i \in N\setminus S$, and

$$y_{i_k} = \begin{cases} m_{i_k}^v & \text{for all } i_k \in S, k < t, \\ M_{i_k}^v & \text{for all } i_k \in S, k > t, \\ M_{i_t}^v + \displaystyle\sum_{k=1}^{t-1}[M_{i_k}^v - m_{i_k}^v] - g^v(N) & \text{for } k = t. \end{cases}$$

Then $m_j^v \leq y_j \leq M_j^v$ for all $j \in N$. So, $\vec{y} \in CC(v)$ and so $\vec{y} \in \mathrm{Core}(v)$. Thus, $\vec{y}(S) \geq v(S)$ or, equivalently, $v(S) \leq \vec{M}^v(S) - g^v(N) = v(N) - \vec{M}^v(N\setminus S)$. This completes the proof. □

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Remark 1.5. With every TU game $v : \mathcal{P}(N) \to \mathbb{R}$, there is associated its *gap function* $g^v : \mathcal{P}(N) \to \mathbb{R}$ defined by $g^v(S) = \vec{M}^v(S) - v(S)$ for all $S \subseteq N$, where $g^v(\emptyset) = 0$. An adapted version of (1.3) is well known as the so-called 1-convexity constraint as follows:

$$v(S) \leq v(N) - \vec{M}^v(N\backslash S) \quad \text{or, equivalently,} \quad g^v(N) \leq g^v(S) \quad \text{for all } S \subseteq N,\ S \neq \emptyset.$$

In words, the TU game v is said to be 1-convex if its corresponding (nonnegative) gap function g^v attains its minimum at the grand coalition. Clearly, the class of compromise stable TU games contains the subclass of 1-convex n-person games [3, 4], as well as the 2-convex n-person games [3, 5] and the big boss and clan games [2, 8, 9].

Further, from (1.2), we deduce that $M_i^v - m_i^v = \min[g^v(S) \mid S \subseteq N,\ i \in S]$ for all $i \in N$. Thus, $m_i^v \leq M_i^v$ if and only if $g^v(S) \geq 0$ for all $S \subseteq N$ with $i \in S$. In particular, $\vec{m}^v \leq \vec{M}^v$ if and only if $g^v(S) \geq 0$ for all $S \subseteq N$, $S \neq \emptyset$. Throughout the next section we tacitly assume a nonnegative gap function.

2. The indirect function as a tool for the determination of the nucleolus of compromise stable TU games

Theorem 2.1. *Let the TU game $v : \mathcal{P}(N) \to \mathbb{R}$ be compromise stable. Then its indirect function $\pi^v : \mathbb{R}^N \to \mathbb{R}$ has the following properties:*

(i) $\pi^v(\vec{y}) = \max[0, v(N) - \sum_{k\in N} y_k]$ *for all* $\vec{y} = (y_k)_{k\in N} \in \mathbb{R}^N$ *with* $\vec{m}^v \leq \vec{y} \leq \vec{M}^v$*;*

(ii) *for all* $\vec{y} = (y_k)_{k\in N} \in \mathbb{R}^N$ *such that there exist unique* $i, j \in N$ *with* $y_i < m_i^v$, $y_j > M_j^v$ *and* $m_k^v \leq y_k \leq M_k^v$ *for all* $k \in N\backslash\{i, j\}$,

$$\pi^v(\vec{y}) = \max\Big[m_i^v - y_i,\ v(N\backslash\{j\}) - \sum_{k\in N\backslash\{j\}} y_k\Big] \tag{2.1}$$

$$= \max\Big[m_i^v - y_i,\ v(N) - \sum_{k\in N} y_k + y_j - M_j^v\Big]; \tag{2.2}$$

(iii) *with any efficient payoff vector* $\vec{x} = (x_k)_{k\in N} \in \mathbb{R}^N$ *satisfying* $\vec{m}^v \leq \vec{x} \leq \vec{M}^v$*, any pair* $i, j \in N$ *of players and any transfer* $\delta \geq 0$ *from* i *to* j*, there is associated the adapted payoff vector* $\vec{x}^{ij\delta} = (x_k^{ij\delta})_{k\in N} \in \mathbb{R}^N$ *given by* $x_i^{ij\delta} = x_i - \delta$, $x_j^{ij\delta} = x_j + \delta$ *and* $x_k^{ij\delta} = x_k$ *for all* $k \in N\backslash\{i, j\}$*. Then, for* $\delta \geq 0$ *sufficiently large,*

$$\pi^v(\vec{x}^{ij\delta}) = \delta + \max[m_i^v - x_i, x_j - M_j^v] \quad \text{for all } i, j \in N,\ i \neq j; \tag{2.3}$$

(iv) *for* $\delta \geq 0$ *sufficiently large, the pairwise equilibrium condition* $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$ *is equivalent to*

$$\min[x_i - m_i^v, M_j^v - x_j] = \min[x_j - m_j^v, M_i^v - x_i] \quad \text{for all } i, j \in N,\ i \neq j. \tag{2.4}$$

Proof. From Theorem 1.4, we see that for every vector $\vec{y} \in \mathbb{R}^N$ with $\vec{m}^v \leq \vec{y} \leq \vec{M}^v$ and every coalition $S \subseteq N$, $S \neq N$, $S \neq \emptyset$,

$$v(S) - \vec{y}(S) \leq \max[(\vec{m}^v - \vec{y})(S), v(N) - \vec{y}(N) + (\vec{y} - \vec{M}^v)(N\backslash S)] \leq \max[0, v(N) - \vec{y}(N)]$$

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and so $\pi^v(\vec{y}) = \max[0, v(N) - \vec{y}(N)]$ for all $\vec{m}^v \le \vec{y} \le \vec{M}^v$. This completes the proof of part (i).

In order to prove part (ii), let $\vec{y} = (y_k)_{k\in N} \in \mathbb{R}^N$ be such that there exist $i, j \in N$ with $y_i < m_i^v$, $y_j > M_j^v$ and $m_k^v \le y_k \le M_k^v$ for all $k \in N\backslash\{i, j\}$. In order to study the excesses $e^v(S, \vec{y})$, $S \subseteq N$, $S \neq N$, $S \neq \emptyset$, we distinguish three cases.

Case 1. Assume $\vec{m}^v(S) \le v(N) - \vec{M}^v(N\backslash S)$. Then $v(S) \le v(N) - \vec{M}^v(N\backslash S)$ and so

$$\begin{aligned} v(S) - \vec{y}(S) &\le v(N) - \vec{M}^v(N\backslash S) - \vec{y}(S) = v(N) - \vec{y}(N) + (\vec{y} - \vec{M}^v)(N\backslash S) \\ &\le v(N) - \vec{y}(N) + (y_j - M_j^v) = v(N\backslash\{j\}) - \vec{y}(N\backslash\{j\}). \end{aligned} \tag{2.5}$$

By (2.5), $e^v(S, \vec{y}) \le e^v(N\backslash\{j\}, \vec{y})$ for all $S \subseteq N$ with $\vec{m}^v(S) \le v(N) - \vec{M}^v(N\backslash S)$.

Case 2. Assume $\vec{m}^v(S) > v(N) - \vec{M}^v(N\backslash S)$. Then $v(S) \le \vec{m}^v(S)$. We distinguish two subcases.

Subcase 2.1. Assume $i \notin S$. Then we derive $v(S) - \vec{y}(S) \le (\vec{m}^v - \vec{y})(S) \le 0$.

Subcase 2.2. Assume $i \in S$. Then we derive $v(S) - \vec{y}(S) \le (\vec{m}^v - \vec{y})(S) \le m_i^v - y_i$.

In summary, $e^v(S, \vec{y}) \le m_i^v - y_i$ for all $S \subseteq N$ with $\vec{m}^v(S) > v(N) - \vec{M}^v(N\backslash S)$. In particular, $e^v(\{i\}, \vec{y}) = v(\{i\}) - y_i \le m_i^v - y_i$. Notice that $m_i^v \ge v(N) - \vec{M}^v(N\backslash\{i\})$ because $M_i^v - m_i^v \le g^v(N)$. Due to the forthcoming Remark 2.2, we claim, without loss of generality, that $m_i^v - y_i$ equals the excess $e^v(\{i\}, \vec{y})$. Hence, (2.1) holds or, equivalently, (2.2). As a direct consequence, (2.3)–(2.4) hold. □

Remark 2.2. Whenever $m_i^v \neq v(\{i\})$, the latter part of the proof of Theorem 2.1 has to be adapted by means of a slight change of the worth of player i without changing the core and nucleolus concept. Formally, with a TU game $v : \mathcal{P}(N) \to \mathbb{R}$ and a fixed player $i \in N$, there is associated the TU game $w : \mathcal{P}(N) \to \mathbb{R}$ given by $w(\{i\}) = m_i^v$ and $w(S) = v(S)$ for all $S \subseteq N$, $S \neq \{i\}$. Clearly, $\vec{M}^w = \vec{M}^v$ as well as $\vec{m}^w = \vec{m}^v$. Moreover, by (1.1), both games possess the same core because $m_i^v \ge v(\{i\})$ and as well $\vec{m}^v$ represents a lower bound for Core(v). Consequently, the intersection of the core with the prekernel is the same for both games [7] and, for the classes under consideration, it follows from the uniqueness part that both games have the same nucleolus. Finally, by Theorem 1.4, if the game v is compromise stable, then the game w is compromise stable, too: that is, $w(\{i\}) = m_i^v = m_i^w \le \max[m_i^w, w(N) - \vec{M}^w(N\backslash\{i\})]$ and, for all $S \subseteq N$, $S \neq \emptyset$, $S \neq \{i\}$,

$$w(S) = v(S) \le \max[\vec{m}^v(N), v(N) - \vec{M}^v(N\backslash S)] = \max[\vec{m}^w(N), w(N) - \vec{M}^w(N\backslash S)].$$

Without going into details, we state that the pairwise equilibrium conditions, $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$ for all pairs $i, j \in N$ of players and for $\delta \ge 0$ sufficiently large, fully determine the so-called *prekernel* of the TU game v [7]. As a matter of fact, the set of efficient solutions of the nonlinear system of equations (2.4) is unique and it is a so-called constrained equal award rule of the parametric form

$$x_i = m_i^v + \max\left[M_i^v - m_i^v - \lambda,\ \frac{M_i^v - m_i^v}{2}\right] \quad \text{for all } i \in N,$$

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where the parameter $\lambda \in \mathbb{R}$ is determined by the efficiency constraint $\vec{x}(N) = v(N)$. This unique solution within the prekernel is well known as the nucleolus of the TU game v. In [10], the approach to determine the nucleolus of compromise stable TU games is totally different and strongly based on the study of (convex) bankruptcy games [10, Theorem 4.2, pages 497–498].

3. The indirect function and nucleolus of clan TU games

Definition 3.1 ([8, 9] and [2, page 59]). An n-person TU game $v : \mathcal{P}(N) \to \mathbb{R}$ is said to be a *clan game* if $M_i^v \geq v(\{i\})$ for all $i \in N$ and there exists a coalition $T \subseteq N$, called the *clan*, such that $v(S) = 0$ whenever $T \not\subseteq S$ and

$$v(S) \leq v(N) - \vec{M}^v(N \backslash S) \quad \text{for all } S \subseteq N,\ S \neq \emptyset \text{ with } T \subseteq S. \tag{3.1}$$

A clan game v with an empty clan reduces to an 1-convex game, provided $g^v(N) \geq 0$. A clan game in which the clan is a singleton is known as a big boss game.

Throughout this section we suppose that the clan T consists of at least two players.

Theorem 3.2. *Let the n-person TU game $v : \mathcal{P}(N) \to \mathbb{R}$ be a clan game. Then its indirect function $\pi^v : \mathbb{R}^N \to \mathbb{R}$ has the following properties:*

(i) $\pi^v(\vec{y}) = \max[0, v(N) - \sum_{k \in N} y_k]$ *for all $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ with $y_i \geq 0$ for all $i \in N$ and $y_i \leq M_i^v$ for all $i \in N \backslash T$;*

(ii) $\pi^v(\vec{y}) = \max[0, v(N \backslash \{\ell\}) - \sum_{k \in N \backslash \{\ell\}} y_k] = \max[0, v(N) - \sum_{k \in N} y_k + y_\ell - M_\ell^v]$ *for all $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ such that there exists a unique $\ell \in N \backslash T$ with $y_\ell > M_\ell^v \geq 0$, $y_i \leq M_i^v$ for all $i \in N \backslash T$, $i \neq \ell$, and $y_i \geq 0$ for all $i \in N$;*

(iii) $\pi^v(\vec{y}) = \max[-y_\ell, v(N) - \sum_{k \in N} y_k]$ *for all $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ such that there exists a unique $\ell \in N$ with $y_\ell < 0$, $y_i \geq 0$ for all $i \in N \backslash \{\ell\}$ and $y_i \leq M_i^v$ for all $i \in N \backslash T$;*

(iv) $\pi^v(\vec{y}) = \max[-y_j, v(N \backslash \{\ell\}) - \sum_{k \in N \backslash \{\ell\}} y_k] = \max[-y_j, v(N) - \sum_{k \in N} y_k + y_\ell - M_\ell^v]$ *for all $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ such that there exist unique $j \in N$, $\ell \in N \backslash T$ with $y_j < 0$, $y_i \geq 0$ for all $i \in N \backslash \{j\}$ and $y_\ell > M_\ell^v \geq 0$, $y_i \leq M_i^v$ for all $i \in N \backslash T$, $i \neq \ell$.*

Proof. Let $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$.

Suppose that $y_i \geq 0$ for all $i \in N$ and $y_i \leq M_i^v$ for all $i \in N \backslash T$. We distinguish two types of coalitions $S \subseteq N$, $S \neq \emptyset$. In case $T \not\subseteq S$, then $v(S) - \vec{y}(S) = -\vec{y}(S) \leq 0$. In case $T \subseteq S$, then we derive from (3.1),

$$v(S) - \vec{y}(S) \leq v(N) - \vec{M}^v(N \backslash S) - \vec{y}(S) = v(N) - \vec{y}(N) + (\vec{y} - \vec{M}^v)(N \backslash S) \leq v(N) - \vec{y}(N). \tag{3.2}$$

This proves part (i).

To prove part (ii), suppose that there exists a unique $\ell \in N \backslash T$ with $y_\ell > M_\ell^v \geq 0$, $y_i \leq M_i^v$ for all $i \in N \backslash T$, $i \neq \ell$, and $y_i \geq 0$ for all $i \in N$. We distinguish three types of coalitions $S \subseteq N$, $S \neq \emptyset$. In case $T \not\subseteq S$, then $v(S) - \vec{y}(S) = -\vec{y}(S) \leq 0$. In case $T \subseteq S$,

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together with $\ell \in S$, then $v(S) - \vec{y}(S) \le v(N) - \vec{y}(N)$, as shown in (3.2). In case $T \subseteq S$, together with $\ell \notin S$, then we derive from (3.1)

$$\begin{aligned}
v(S) - \vec{y}(S) &= v(S) - \vec{y}(N) + y_\ell + \vec{y}(N \backslash (S \cup \{\ell\})) \\
&\le v(S) - \vec{y}(N) + y_\ell + \vec{M}^v(N \backslash (S \cup \{\ell\})) \\
&= v(S) - \vec{y}(N) + y_\ell - M^v_\ell + \vec{M}^v(N \backslash S) \\
&\le v(N) - \vec{y}(N) + y_\ell - M^v_\ell = v(N \backslash \{\ell\}) - \vec{y}(N \backslash \{\ell\}).
\end{aligned}$$

In this setting, the indirect function π^v attains its maximum for $S = N$, $S = N \backslash \{\ell\}$ or $S = \emptyset$, but $S = N$ cancels. The similar proof of part (iii) is left for the reader.

For part (iv), suppose that there exist unique $j \in N$, $\ell \in N \backslash T$ with $y_j < 0$, $y_i \ge 0$ for all $i \in N \backslash \{j\}$ and $y_\ell > M^v_\ell \ge 0$, $y_i \le M^v_i$ for all $i \in N \backslash T$, $i \ne \ell$. We distinguish three types of coalitions $S \subseteq N$, $S \ne \emptyset$. In case $T \not\subseteq S$, then $v(S) - \vec{y}(S) = -\vec{y}(S) \le -y_j$. In case $T \subseteq S$, the proof is similar to the proof of part (ii) and is left for the reader, too. □

Corollary 3.3. *For every n-person clan game $v : \mathcal{P}(N) \to \mathbb{R}$, with clan T, the following three statements concerning a payoff vector $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ are equivalent:*

(i) $\vec{y} \in \mathrm{Core}(v)$, *that is, $\vec{y}(N) = v(N)$ and $\vec{y}(S) \ge v(S)$ for all $S \subseteq N$, $S \ne \emptyset$;*
(ii) $\vec{y}(N) = v(N)$ *and* $\pi^v(\vec{y}) = 0$;
(iii) $\vec{y}(N) = v(N)$, *$y_i \ge 0$ for all $i \in N$ and $y_i \le M^v_i$ for all $i \in N \backslash T$.*

Theorem 3.4. *Let the n-person TU game $v : \mathcal{P}(N) \to \mathbb{R}$ be a clan game with clan T. From the explicit formula for the indirect function of clan games, as presented in Theorem 3.2(ii)–(iv), we conclude that, for $\delta \ge 0$ sufficiently large, the pairwise equilibrium conditions $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$ for all pairs $i, j \in N$ of players reduce to the following system of equations:*

Case	Pairwise equilibrium equation $\pi^v(\vec{x}^{ij\delta}) = \pi^v(\vec{x}^{ji\delta})$
$i \in T,\ j \in T$	$\max\{-(x_i - \delta),\ 0\} = \max\{-(x_j - \delta),\ 0\}$
$i \notin T,\ j \in T$	$\max\{-(x_i - \delta),\ 0\} = \max\{-(x_j - \delta),\ (x_i + \delta) - M^v_i\}$
$i \notin T,\ j \notin T$	$\max\{-(x_i - \delta),\ (x_j + \delta) - M^v_j\} = \max\{-(x_j - \delta),\ (x_i + \delta) - M^v_i\}$
Case	**Resulting pairwise equation for $\vec{x} = (x_k)_{k \in N} \in \mathbb{R}^N$**
$i \in T,\ j \in T$	$x_i = x_j$
$i \notin T,\ j \in T$	$x_i = \min\{x_j,\ M^v_i - x_i\}$
$i \notin T,\ j \notin T$	$\min\{x_i,\ M^v_j - x_j\} = \min\{x_j,\ M^v_i - x_i\}$

In summary, the unique solution is a so-called constrained equal reward rule of the form $x_i = \lambda$ for all $i \in T$ and $x_i = \min[\lambda, \frac{1}{2} b^v_i]$ for all $i \in N \backslash T$, where the parameter $\lambda \in \mathbb{R}$ is determined by the efficiency condition $\vec{x}(N) = v(N)$.

The indirect function is also a helpful tool for the determination of the nucleolus for the subclasses of big boss games as well as 1-convex and 2-convex n-person games [5].

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