Canad. Math. Bull. Vol. 17 (3), 1974.

THE STABILITY OF SOLUTIONS OF GENERALIZED EMDEN-FOWLER EQUATIONS

BY
HUGO TEUFEL, JR.

ABSTRACT. This paper gives several monotonicity properties of all oscillatory solutions of equations with separable and non-separable nonlinearities which are more general than the Emden-Fowler equations

$$(*) \quad x''+a(t)\,|x|^{\gamma}\,\mathrm{sgn}\,x = 0, \quad a(t) > 0, \quad 0 < \gamma < \infty.$$

Principally, if $x(t)$ is an oscillatory solution, conditions are given such that; if $a(t)\uparrow\infty$ as $t\to\infty$, then $x(t)\to 0$; and, if $a(t)\downarrow 0$ as $t\to\infty$, then $\limsup |x(t)| = \infty$.

1. **Introduction.** The equations considered in this paper are of the form

$$(1) \qquad x''+F(t, x) = 0, \qquad xF > 0 \quad \text{for } t, x \neq 0;$$

where, if $x\neq 0$, either ($\uparrow$) $xF\uparrow\infty$ as $t\to\infty$, or ($\downarrow$) $xF\downarrow 0$ as $t\to\infty$. The objective is to state general dual sufficient conditions additional to ($\uparrow$), ($\downarrow$), such that, given any solution $x(t)$ of (1); if ($\uparrow$), then $x(t)\to 0$ as $t\to\infty$, and, if ($\downarrow$), then $\limsup|x(t)|=\infty$ as $t\to\infty$. Also, corollaries are given about the asymptotic distribution of the zeros.

These questions have been studied intensively for the linear version of (1); for instance, Hartman [4] gave refined sufficient conditions for ($\uparrow$) and ($\downarrow$); and, recently, De Kleine [3] gave a counter-example for ($\uparrow$).

The study of the indicated limiting behaviors for nonlinear equations (1) where ($\uparrow$) or ($\downarrow$) holds is intrinsically more difficult. However, Hinton [6] has discussed both these cases, as well as the distribution of zeros, for instances of (1) where $F\equiv a(t)x^{\gamma}$, $\gamma>1$, a ratio of odd, positive integers. Another recent work on ($\uparrow$) is by Burton and Grimmer [1]. The present work improves these by not requiring F to be separable, and by easing the smoothness required of the t-dependance of F.

Of course, if, as in Wong [8], the t-dependance of F is roughly intermediary between ($\uparrow$) and ($\downarrow$); then, Liapunov methods apply and, typically, all solutions are asymptotically stable, e.g., both $x(t)\to 0$ and $x'(t)\to 0$ as $t\to\infty$.

2. **Preliminaries.** Since the conditions and the arguments are dual, the case ($\uparrow$) is considered in the main text, and the case ($\downarrow$) is considered contiguously, in brackets.

(Received February 18, 1972 and, in revised form, July 26, 1972.)

AMS 1970 subject classification. Primary 34C10, 34D99.

Key words and phrases. Emden-Fowler equation, oscillation, monotonicity, stability, convexity.

All functions are required to be continuous at each (t, x) in $[0, \infty)\times(-\infty, \infty)$ unless otherwise designated. Moreover,

i). $xF>0$ for $t, x\neq 0$, $F(t, -x)=-F(t, x)$, for each x-value F is a non-decreasing [non-increasing] function of t which is absolutely continuous on each finite t-interval, while for each t-value F is a non-decreasing function of x.

Furthermore, it is supposed for each $T>0$ that there are functions h, H, and G, $h(t)\equiv h(T, t)$, $H(t, x)\equiv H(T, t, x)$, and $G(x)\equiv G(T, x)$, such that $h(t)H(t, x)\equiv F(t, x)$ for $t\geq T$ where $h(t)$ is absolutely continuous on each finite t-interval. Also,

ii). $h(t)>0$, $h'(t)\geq 0$ [$\leq$] almost everywhere, $h(t)\to\infty$ [0] as $t\to\infty$, $h'(t)$ is non-decreasing [non-increasing] almost everywhere;

iii). $xH_t(t, x)\leq 0$ [$\geq$] almost everywhere, and $\int_0^x H(t, s)\, ds\to G(x)$ as $t\to\infty$, where $G(x)>0$ if $x\neq 0$.

REMARK. The conditions on F in case ($\uparrow$) include such non-separable functions as, for example, $F(t, x)=tx+t^2x^3$. Here, for instance, $h(t)=t^2$, $H(t, x)=t^{-1}x+x^3$, and $G(x)=\frac{1}{4}x^4$.

Several properties of solutions of (1) prerequisite to the main results are stated next. Proofs are not given, for these results are routine extensions of what is known.

Observe that the continuity condition on F ensures that for any initial triple (T, x_0, x_1), $T\geq 0$, $-\infty<x_0, x_1<\infty$, there is a solution of (1) which exists on some maximal, non-trivial interval $[T, T_1)$. It will, without loss of generality, be assumed that $T>0$ in what follows.

Given a specific solution $x(t)$ of (1) defined on $[T, T_1)$, define the functions $E(t)$ and $W(t)$ on $[T, T_1)$ by

$$E(t) = \tfrac{1}{2}h^{-1}(t)(x'(t))^2+\int_0^{x(t)} H(t, s)\, ds, \tag{2}$$

$$W(t) = \tfrac{1}{2}(x'(t))^2+\int_0^{x(t)} F(t, s)\, ds. \tag{3}$$

LEMMA. *$E(t)$ is a non-increasing [non-decreasing] function, absolutely continuous on finite subintervals of $[T, T_1)$. $W(t)$ is a non-decreasing [non-increasing] function, absolutely continuous on finite subintervals of $[T, T_1)$.*

Proof. Let $(\int_0^{x(t)} H(t, s)\, ds)_t$ denote the t-derivative with respect to the first argument (it is not necessary to differentiate under the integral sign). By writing difference quotients it is readily shown that

$$E' = x'h^{-1}(x''+F)+\tfrac{1}{2}(h^{-1})'(x')^2+\left(\int_0^x H\right)_t \quad \text{a.e.} \tag{4}$$

Since $x(t)$ is a solution of (1), the first term of (4) is zero. Assumptions ii) and iii), respectively, imply the second and third terms are non-positive [non-negative] a.e. The assumptions on h and H ensure E is absolutely continuous on finite subintervals of $[T, T_1)$; whence, $E'(t)\leq 0$ [$\geq$] a.e., and the first assertion of the lemma is proved.

Similar arguments show that

$$W' = x'(x''+F)+\left(\int_0^{x(t)} F(t, s)\, ds\right)_t \geq 0 \quad [\leq], \quad \text{a.e.}$$

Thus, the second assertion of the lemma is proved.

a. It follows in a straightforward way from the monotonicity of E in case (↑) and of W in case (↓) that any solution of (1) exists throughout the positive half-line to the right of its initial t-value.

b. In case (↑) works such as Wong's [9] show that every solution of (1) is oscillatory (has a zero in each positive half-line). However, the situation in case (↓) is more complex.

Here, it is possible that all solutions are oscillatory (necessary and sufficient conditions are given in [9]); or, all solutions are non-oscillatory (sufficient conditions are given by Chiou [7]), or, (1) may have both oscillatory and non-oscillatory solutions (sufficient conditions for the existence of an oscillatory solution are discussed by Heidel and Hinton [5]). Thus, the results for case (↓) hold only for all oscillatory solutions of (1), if they exist.

c. The monotonicity of E implies the trivial solution of (1) is right [left] unique; and, the monotonicity of W implies the trivial solution of (1) is left [right] unique. Hence, a continuity argument shows the zeros of any oscillatory solution of (1) are discrete and may be enumerated as an unbounded, strictly increasing sequence $\{t_{2n}\}$, $n=0, 1, 2, \ldots$. The monotonicity of W implies $|x'(t_{2n})| \leq |x'(t_{2n+2})|$ $[\geq]$.

d. In each interval (t_{2n}, t_{2n+2}) there is a single zero of $x'(t)$, call it t_{2n+1}. Arguments like those given by Das [2] show that the monotonicity of F in its t-variable implies $t_{2n+1}-t_{2n} \geq t_{2n+2}-t_{2n+1}$ $[\leq]$ and $|x(t_{2n+1})| \geq |x(t_{2n+3})|$ $[\leq]$.

3. The Main Results

THEOREM. *If $x(t)$ is any oscillatory solution of (1); then, the conditions for case (↑) imply $|x(t_{2n+1})| \downarrow 0$, and $h^{-1}(t_{2n})(x'(t_{2n}))^2 \downarrow 0$ as $n \to \infty$; and the conditions for case (↓) imply $|x(t_{2n+1})| \uparrow \infty$, and $h^{-1}(t_{2n})(x'(t_{2n}))^2 \uparrow \infty$ as $n \to \infty$.*

Proof. Some information about the convexity of $|x'(t)|$ is required in the ensuing. Thus, differentiate (1) to obtain

$$x''' \equiv -F_t(t, x)-F_x(t, x)x', \quad \text{a.e.} \tag{5}$$

According to condition i) $x''' \leq 0$ $[\geq]$, a.e., if $x>0$, $x'>0$ $[x>0, x'<0]$ and $x''' \geq 0$ $[\leq]$, a.e., if $x<0$, $x'<0$ $[x<0, x'>0]$. Hence, $|x'|$ is convex upward on $[t_{2n}, t_{2n+1}]$ in case (↑) [and on $[t_{2n+1}, t_{2n+2}]$ in case (↓)].

If the theorem is false; then, there is a constant $C_0>0$ $[<\infty]$ such that, by 2.c., $|x(t_{2n+1})| \downarrow C_0$ [↑] as $n \to \infty$. In view of the definition of $G(x)$, it follows from the Lemma, and the monotonicity of $E(t)$, that there is a constant $C>0$ $[<\infty]$ such that $E(t_{2n+1}) \downarrow C$ [↑] as $n \to \infty$.

Consider the derivative of $E(t)$ in (4), and by neglecting $(\int_0^{x(t)} H(t, s)\, ds)_t$ obtain

$$E'(t) \le \tfrac{1}{2}(h^{-1}(t))'(x'(t))^2 \quad [\ge], \quad \text{a.e.} \tag{6}$$

Let $\varepsilon>0$ be given. Then, there is an $m=m(\varepsilon)$ so large that $t\ge t_{2m}$ implies $C<E(t)<C+\varepsilon[C-\varepsilon<E(t)<C]$.

Since E and h are absolutely continuous, (6) may be integrated over $[t_{2m}, t_{2n}]$ to produce

$$\varepsilon > \mp\frac{1}{2}\int_{t_{2m}}^{t_{2n}} (h^{-1})'(x')^2 = \mp\frac{1}{2}\sum_{k=m}^{n-1}\int_{t_{2k}}^{t_{2k+2}} (h^{-1})'(x')^2 \tag{7}$$

where the minus [plus] sign holds in case (↑) [↓].

According to 2.c. $|x'(t_{2k})|\ge|x'(t_{2m})|$ $[\ge|x'(t_{2n})|, m\le k\le n]$. Moreover, if (↑), the convexity of $|x'(t)|$ implies $|x'(t)|\ge\frac{1}{2}|x'(t_{2k})|$ on $[t_{2k}, t'_{2k}]$, $t_{2k}=\frac{1}{2}(t_{2k}+t_{2k+1})$ [and, if (↓), $|x'(t)|\ge|x'(t_{2k+2})|$ on $[t''_{2k}, t_{2k+2}]$, $t''_{2k}=\frac{1}{2}(t_{2k+1}+t_{2k+2})$]. These bounds substantiate the inequalities

$$\begin{gathered}\int_{t_{2k}}^{t_{2k+2}} (h^{-1})'(x')^2 \le \int_{t_{2k}}^{t'_{2k+2}} (\quad) \le \tfrac{1}{4}(x'(t_{2m}))^2\int_{t_{2k}}^{t'_{2k}} (h^{-1})' \\ \left[\int_{t_{2k}}^{t_{2k+2}} (h^{-1})'(x')^2 \ge \int_{t''_{2k}}^{t_{2k+2}} (\quad) \ge \tfrac{1}{4}(x'(t_{2n}))^2\int_{t''_{2k}}^{t_{2k+2}} (h^{-1})'\right].\end{gathered} \tag{8}$$

Since h' is non-decreasing [non-increasing] it follows from the inequalities stated for the t-intervals in 2.d. that

$$\begin{gathered}\int_{t_{2k}}^{t'_{2k}} (h^{-1})' \le \frac{1}{4}\int_{t_{2k}}^{t_{2k+2}} (h^{-1})' \\ \left[\int_{t''_{2k}}^{t_{2k+2}} (h^{-1})' \ge \frac{1}{4}\int_{t_{2k}}^{t_{2k+2}} (h^{-1})'\right].\end{gathered} \tag{9}$$

Now, inequalities (8) and (9) can be combined in (7) to give

$$\begin{gathered}\varepsilon > 2^{-4}(x'(t_{2m}))^2(h^{-1}(t_{2m})-h^{-1}(t_{2n})) \\ [\varepsilon > 2^{-4}(x'(t_{2n}))^2(h^{-1}(t_{2n})-h^{-1}(t_{2m}))].\end{gathered} \tag{10}$$

In the case (↑) it follows from the unboundedness of h and the definitions of E and the constant C that (10) implies $\varepsilon>2^{-4}(x'(t_{2m}))^2h^{-1}(t_{2m})>2^{-3}C$. In the case (↓) note in equation (2) that the assumed boundedness of E and the zero limit of h imply $(x'(t_{2n}))^2 \downarrow 0$ as $n\to\infty$. Hence, in the limit as $n\to\infty$, the second inequality of (10) implies $\varepsilon>2^{-3}C$. In both cases the selection $\varepsilon=2^{-3}C$ provides a contradiction, and the theorem is proved.

COROLLARY. *If* (↑), *then* $t_{2n+2}-t_{2n}\to 0$ as $n\to\infty$. If (↓), *then* $t_{2n+2}-t_{2n}\to\infty$ *as* $n\to\infty$.

Proof. (↑). According to the remarks preceding inequality (8) and 2.c., $|x'(t)| \geq 2^{-1}|x'(t_{2n})| \geq 2^{-1}|x'(t_0)|$ on $[t_{2n}, t'_{2n}]$. Then, an integration over this interval leads to $|x(t_{2n+1})| \geq |x(t'_{2n})| \geq 2^{-1}\,|x'(t_0)|\,(t_{2n}-t'_{2n})$. Since $t'_{2n}-t_{2n} \geq \frac{1}{4}(t_{2n+2}-t_{2n})$, the theorem implies the desired result.

(↓). Since $|x'(t)| \leq |x'(t_{2n+2})| \leq |x'(t_0)|$, $t_{2n+1} \leq t \leq t_{2n+2}$, an integration gives $|x(t_{2n+1})| \leq |x'(t_0)|\,(t_{2n+2}-t_{2n+1})$. The theorem implies the desired result; and, the corollary is proved.

REMARK. It is noteworthy that Hinton [6] also showed, under his conditions, that, if (↑), then, $\lim\sup|x'(t)| = \infty$ as $t \to \infty$, and, if (↓), then $\lim|x'(t)| = 0$ as $t \to \infty$.

ACKNOWLEDGEMENT. The author thanks J. Macki for a remark leading to reference [4] and the editor and referee for permitting the later inclusion of the (↓) case.

REFERENCES

1. T. Burton and R. Grimmer, *On the asymptotic behavior of solutions of $x''+a(t)f(x)=0$*, Proc. Camb. Phil. Soc. **70** (1971), 77–88.

2. K. Das, *Comparison and monotonity theorems for second order non-linear differential equations*, Acta. Math. Sci. Hungar. **15** (1964), 449–456.

3. H. A. DeKleine, *A counterexample to a conjecture in second-order linear equations*, Mich. Math. J. **17** (1970), 29–32.

4. P. Hartman, *The existence of large or small solutions of linear differential equations*, Duke Math. J. **28** (1961), 421–430.

5. J. W. Heidel and D. B. Hinton, *The existence of oscillatory solutions for a nonlinear differential equation* (to appear).

6. D. B. Hinton, *Some stability conditions for a nonlinear differential equation*, Trans. Amer. Math. Soc. **139** (1969), 349–358.

7. Kuo-liang Chiou, *A second order nonlinear oscillation theorem*, SIAM J. Appl. Math. **21** (1971).

8. J. S. W. Wong, *On the global asymptotic stability of $(p(t)x')'+q(t)x^{2n-1}=0$*, J. Inst. Maths. Applics. **3** (1967), 403–405.

9. J. S. W. Wong, *On second order nonlinear oscillation*, Funkc. Ekvac. **11** (1969), 207–234.

See Combined Membership List for present address.