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SINGULAR INTEGRALS SUPPORTED BY SUBVARIETIES FOR VECTOR-VALUED FUNCTIONS

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Abstract. In this paper, we show that singular integrals supported by subvarieties are bounded on $L^p(\mathbb{R}^n;\mathbf{X})$ for $1<p<\infty$ and some UMD space $\mathbf{X}$. In the terminology from operator space theory, we prove that singular integrals supported by subvarieties are completely L^p-bounded.

§1. Introduction

Let $\mathbf{X}$ be a Banach space, $\Gamma:\mathbb{R}^k\to\mathbb{R}^n$ be a C^∞ mapping. Let K be a Calderón–Zygmund kernel in $\mathbb{R}^k$, that is, K is homogeneous of degree $-k$, smooth away from the origin, and

$$\int_{\mathbf{S}^{k-1}} K(t)\,d\sigma(t)=0.$$

To Γ we associate vector-valued singular integrals T_Γ defined for $f\in\mathscr{S}(\mathbb{R}^n;\mathbf{X})$ as follows:

$$T_\Gamma f(x)=\text{p.v.}\int_{\mathbb{R}^k} f(x-\Gamma(t))K(t)\,dt.$$

In addition to the classical case $n=k$ and $\Gamma(t)=(t_1,t_2,\ldots,t_n)$, much effort has been devoted to the question of whether the mapping properties of singular integral operators could be extended to the Lebesgue–Bôhner spaces $L^p(\mathbb{R}^n;\mathbf{X})$ $(1<p<\infty)$ of vector-valued functions. In [1], Benedek *et al.* proved that the boundedness on $L^{p_0}(\mathbb{R}^n;\mathbf{X})$ for one $1<p_0<\infty$ of a singular integral operator, together with Hörmander's condition, implies its boundedness on $L^p(\mathbb{R}^n;\mathbf{X})$ for all $1<p<\infty$. But, it is significantly difficult to get the $L^{p_0}(\mathbb{R}^n;\mathbf{X})$-boundedness except in the case $\mathbf{X}=L^{p_0}(\Omega)$ for some measure space Ω. In this direction, Burkholder [4] showed that if the underlying Banach space $\mathbf{X}$ satisfies the so-called UMD-property, that is,

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$\mathbf{X}$ is an UMD space, then the Hilbert transform is bounded on $L^p(\mathbb{R};\mathbf{X})$ for any $1<p<\infty$. Moreover, the UMD-property was shown by Bourgain [3] to be necessary for the boundedness of the Hilbert transform. The analogue statement also holds for Riesz transforms, which can be found in [12]. It is well known that Riesz transforms are prototypes of singular integral operators and Fourier multipliers, their boundedness motivates McConnell's [9] and Zimmermann's [19] results on vector-valued Marcinkiewicz–Mihlin multipliers, and motivates Hytönen and Weis's [6] results on vector-valued singular convolution integrals.

On the other hand, T_Γ is well understood when $\mathbf{X}=\mathbb{R}$. T_Γ belongs to the class of singular Radon transforms which is connected with the $\overline{\partial}$-Neumann problem for pseudo-convex domains [10] and the boundary behavior of Poisson integral on symmetric spaces [16]. An exposition of the background and related references are to be found in [14]. In particular, Stein [17] proved T_Γ is also bounded on $L^p(\mathbb{R}^n)$, where $\Gamma(t)=(P_1(t),P_2(t),\ldots,P_n(t))$ with P_j being polynomials in $t\in\mathbb{R}^k$.

The purpose of my project is to extend the singular integrals theory to a more general setting. More precisely, I am interested in the boundedness of T_Γ on Lebesgue–Bôhner spaces $L^p(\mathbb{R}^n;\mathbf{X})$ for some UMD spaces $\mathbf{X}$ and smooth surfaces Γ. In fact, I was motivated by the work of Rubio de Francia *et al.* [12]; they obtained the boundedness of the Hilbert transform along well-curved curve on $L^p(\mathbb{R}^n;l^q)$ for $1<p,q<\infty$. So, I start my project by studying the higher- dimensional version and proved the following result:

THEOREM A. (See [7]) *Let $\mathcal{P}(t)=(P_1(t),P_2(t),\ldots,P_n(t))$ with P_j being polynomials in $t\in\mathbb{R}^k$. $T_{\mathcal{P}}$ is bounded on $L^p(\mathbb{R}^n;l^q)$, $1<p,q<\infty$; the bounds for the operator do not depend on the coefficients of the polynomial $\mathcal{P}$, but only on the total degree of $\mathcal{P}$.*

Note that l^q is a prototype of UMD spaces. Then, I continued the project and considered the boundedness of the Hilbert transform along homogeneous curves and convex curves on $L^p(\mathbb{R}^n;\mathbf{X})$ for some UMD spaces $\mathbf{X}$; see [8]. Naturally, it is an interesting problem to extend Theorem A to some more general UMD spaces. Therefore, I study the boundedness of singular integrals $T_{\mathcal{P}}$ on $L^p(\mathbb{R}^n;\mathbf{X})$ for $1<p<\infty$ and some UMD spaces $\mathbf{X}$ in this paper.

To present our main result, we need to give some notations. The first definition was introduced by Burkholder [4].

DEFINITION 1.1. A Banach space $\mathbf{X}$ is an UMD space if for some (equivalently, all) $p \in (1, \infty)$ there is a positive constant C such that

$$\left\| \sum_{k=1}^{N} \epsilon_k d_k \right\|_{L^p(\mu,\mathbf{X})} \leqslant C \left\| \sum_{k=1}^{N} d_k \right\|_{L^p(\mu,\mathbf{X})},$$

whenever $(d_k)_{k=1}^N \in L^p(\mu, \mathbf{X})^N$ is a martingale difference sequence and $(\epsilon_k)_{k=1}^N \in \{-1, 1\}^N$.

The second notation is due to Berkson and Gillespie [2].

DEFINITION 1.2. The class $\mathcal{I}$ consists of those UMD spaces $\mathbf{X}$ which are isomorphic to a closed subspace of a complex interpolation space $[\mathbf{H}, \mathbf{Y}]_\theta$, $0 < \theta < 1$, between a Hilbert space $\mathbf{H}$ and another UMD space $\mathbf{Y}$.

REMARK 1.3.

(1) $\mathcal{I}$ contains almost all standard examples of UMD spaces. In 1986, Rubio de Francia [13] proved that for any UMD lattice $\mathbf{X}$ of functions on a σ-finite measure space there exists $\theta \in (0, 1)$, Hilbert space $\mathbf{H}$ and another UMD lattice $\mathbf{Y}$ such that $\mathbf{X} = [\mathbf{H}, \mathbf{Y}]_\theta$. So, every UMD lattice belongs to $\mathcal{I}$. But $\mathcal{I}$ also contains the Schatten–von Neumann ideals $\mathscr{C}^p = [\mathscr{C}^2, \mathscr{C}^q]_\theta$, $1/p = (1 - \theta)/2 + \theta/q$. $\mathscr{C}^p$, $p \neq 2$, do not have local unconditional structure, then, they are not Banach lattices.
(2) In [13], Rubio de Francia posed an open problem, "Is every $B \in UMD$ intermediate between a 'worse' B_0 and a Hilbert space?". Its significance is results in UMD spaces can be proved by interpolating with the estimates available in arbitrary UMD spaces and the stronger ones that one can get in a Hilbert space. Thus it is interesting to know if $\mathcal{I}$ actually contains all UMD spaces.

The main result of this paper is the following theorem:

THEOREM 1.4. *Let $\mathbf{X} \in \mathcal{I}$, $\mathcal{P}(t) = (P_1(t), P_2(t), \ldots, P_n(t))$ with P_j being polynomials in $t \in \mathbb{R}^k$. Then for $1 < p < \infty$ there exists a constant $C_p > 0$ such that*

$$\|T_{\mathcal{P}} f\|_{L^p(\mathbb{R}^n;\mathbf{X})} \leqslant C_p \|f\|_{L^p(\mathbb{R}^n;\mathbf{X})}.$$

The constant C_p may depend on $\mathbf{X}$, k, n and the total degree of $\mathcal{P}$, but it is independent of the coefficients of $\mathcal{P}$.

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Remark 1.5. Obviously, Theorem 1.4 is an extension of Stein's result [17] and Theorem A; the extension is not trivial. In fact, in [17] and [7], it is significant to get the L^p boundedness for the analytic family of operators T_z, where $\mathrm{Re}(z)$ is negative. The boundedness on L^2 obtained with the help of the Fourier transform and Plancherel theorem, together with Hörmander condition implies the boundedness on L^p for all $1<p<\infty$. For general Banach-valued functions, we cannot use Fourier transform and Plancherel theorem to get the boundedness on $L^2(\mathbb{R}^n;\mathbf{X})$ as *a priori* estimate. It turned out to be a significantly more difficult task to get the boundedness without *a priori* estimate even for the single p. So, we have to appeal to new techniques.

The paper is organized as follows. In Section 2, by using method of descent for vector-valued functions, we imply Theorem 1.4 under the assumption that singular integrals associated to monomials are bounded on $L^p(\mathbb{R}^n;\mathbf{X})$. Finally, in Section 3, we show the boundedness of singular integrals associated to monomials on $L^p(\mathbb{R}^n;\mathbf{X})$.

§2. Proof of Theorem 1.4

Let $\mathcal{P}(t)=(P_1(t),P_2(t),\ldots,P_n(t))$, d denotes the maximum degree of the polynomials $P_j(t)$. We consider the collection of all monomials t^α with $1\leqslant|\alpha|\leqslant d$, let N denote the number of these monomials. We work in $\mathbb{R}^N$, whose coordinates are labeled by the multi-indices α with $1\leqslant|\alpha|\leqslant d$, that is, $\mathbb{R}^N=\{(x_\alpha)_{1\leqslant|\alpha|\leqslant d}\}$. Let $\mathfrak{p}$ be a polynomial map from $\mathbb{R}^k$ to $\mathbb{R}^N$, precisely, $\mathfrak{p}(t)=(t^\alpha)_{1\leqslant|\alpha|\leqslant d}$. For $f\in\mathscr{S}(\mathbb{R}^N;\mathbf{X})$, we define singular integrals by

$$T_{\mathfrak{p}}f(x)=\mathrm{p.v.}\int_{\mathbb{R}^k}f((x_\alpha-t^\alpha)_{1\leqslant|\alpha|\leqslant d})K(t)\,dt.$$

Clearly, it can be looked as a model case of $T_{\mathcal{P}}$. To prove Theorem 1.4, we accept the following inequality for a moment,

$$\|T_{\mathfrak{p}}f\|_{L^p(\mathbb{R}^N;\mathbf{X})}\leqslant C_p\|f\|_{L^p(\mathbb{R}^N;\mathbf{X})},\tag{2.1}$$

which will be proved in the next section.

We also need the method of descent for vector-valued functions that allows us to pass from results about operators in one $\mathbb{R}^N$ to corresponding operators on another $\mathbb{R}^n$. The following lemma is a vector-valued version of Ricci and Stein's "method of transference" (see [11, Proposition 5.1] and [18, p. 483]); it can be proved in a similar way.

Let T be given by convolution with a distribution $d\mu$, $Tf = f * d\mu$, where f is an appropriate vector-valued function on $\mathbb{R}^N$. We fix a linear mapping L from $\mathbb{R}^N$ to $\mathbb{R}^n$; the operator T^L is defined by

$$T^L f(x) = \int_{\mathbb{R}^N} f(x - L(z))\, d\mu(z),$$

where f is an appropriate vector-valued function on $\mathbb{R}^n$. In fact, T^L is a convolution operator on $\mathbb{R}^n$ with the $d\mu^L$, where $d\mu^L$ is defined by

$$\int_{\mathbb{R}^n} f(x)\, d\mu^L(x) = \int_{\mathbb{R}^N} f(Lz)\, d\mu(z).$$

LEMMA 2.1. *Suppose $L : \mathbb{R}^N \to \mathbb{R}^n$ is a fixed linear mapping as above. Then the norm of the operator T^L acting on $L^p(\mathbb{R}^n; \mathbf{X})$ does not exceed the norm of T acting on $L^p(\mathbb{R}^N; \mathbf{X})$ for $1 < p < \infty$.*

To prove Theorem 1.4, we define a distribution $d\mu$ on $\mathbb{R}^N$ by

$$\int_{\mathbb{R}^N} f(x)\, d\mu(x) = \text{p.v.} \int_{\mathbb{R}^k} f(\mathfrak{p}(t)) K(t)\, dt.$$

Then, $T_{\mathfrak{p}} f = f * d\mu$. We set

$$P_j(t) = \sum_{1 \leqslant |\alpha| \leqslant d} a_{j\alpha} t^\alpha.$$

Using the coefficients $a_{j\alpha}$, we define a linear mapping $L : \mathbb{R}^N \to \mathbb{R}^n$, where the coordinates of $L(x)$ are given by

$$L(x)_j = \sum_{\alpha} a_{j\alpha} x_\alpha$$

with $x = (x_\alpha) \in \mathbb{R}^N$. $d\mu^L$ is given by

$$\int_{\mathbb{R}^n} f(x)\, d\mu^L(x) = \text{p.v.} \int_{\mathbb{R}^k} f(L(\mathfrak{p}(t))) K(t)\, dt = \text{p.v.} \int_{\mathbb{R}^k} f(\mathcal{P}(t)) K(t)\, dt.$$

By Lemma 2.1, we have

$$\|T_{\mathcal{P}}\|_{L^p(\mathbb{R}^n;\mathbf{X}) \to L^p(\mathbb{R}^n;\mathbf{X})} \leqslant \|T_{\mathfrak{p}}\|_{L^p(\mathbb{R}^N;\mathbf{X}) \to L^p(\mathbb{R}^N;\mathbf{X})}.$$

Finally, the boundedness of $T_{\mathfrak{p}}$ on $L^p(\mathbb{R}^N; \mathbf{X})$ implies our main result.

§3. Proof of (2.1)

First, we define a one-parameter family of dilations relevant $\mathfrak{p}$, which is given by

$$x \mapsto \delta \circ x = (\delta^{|\alpha|} x_\alpha)_{1 \leqslant |\alpha| \leqslant d}$$

for $x \in \mathbb{R}^N$. We also define related norm

$$\rho(x) = [|x_{\alpha^1}|^{2a_1} + |x_{\alpha^2}|^{2a_2} + \cdots + |x_{\alpha^N}|^{2a_N}]^{\frac{1}{2|\alpha^1| \cdots |\alpha^N|}},$$

where $a_i = \prod_{j \neq i} |\alpha^j|$, the quasi-distance is $\rho(x, y) = \rho(x - y)$. It is trivial that $\rho(\delta \circ x) = \delta \rho(x)$, we also have the following properties:

LEMMA 3.1. *If $\rho(x) \geqslant 1$, then $\rho(x) \leqslant |x|$; If $\rho(x) \leqslant 1$, then $|x| \leqslant \rho(x)$.*

For $z \in \mathbb{C}$, we define an analytic family of operators $T_{\mathfrak{p}}^z$ by

$$\widehat{T_{\mathfrak{p}}^z f}(\xi) = m^z(\xi) \widehat{f}(\xi),$$

where m^z are given by

$$m^z(\xi) = \{\rho(\xi)\}^z \text{p.v.} \int_{\mathbb{R}^k} e^{-2\pi i \xi \cdot \mathfrak{p}(t)} |t|^z K(t)\, dt.$$

Obviously, $T_{\mathfrak{p}}^0$ is the original operator $T_{\mathfrak{p}}$.

3.1 The boundedness of $T_{\mathfrak{p}}^z$ on $L^2(\mathbb{R}^N; \mathbf{H})$

In this subsection, we prove that

$$\|T_{\mathfrak{p}}^z f\|_{L^2(\mathbb{R}^n; \mathbf{H})} \leqslant C(z) \|f\|_{L^2(\mathbb{R}^n; \mathbf{H})},$$

where $\mathrm{Re}(z) < \frac{1}{d}$ and $C(z)$ grows at most polynomially in $(1 + |\mathrm{Im}(z)|)$. Clearly, the boundedness of $T_{\mathfrak{p}}^z$ on $L^2(\mathbb{R}^N; \mathbf{H})$ is equivalent to the uniform boundedness of $m^z(\xi)$. Thus, we just need to show that

$$|m^z(\xi)| \leqslant C(z). \tag{3.1}$$

Suppose that $\psi \in C_0^\infty(\mathbb{R}^k)$ is radial, vanishes near the origin and satisfies

$$\sum_{j \in \mathbb{Z}} \psi(2^j t) = 1, \quad t \in \mathbb{R}^k \backslash \{0\}.$$

For $j \in \mathbb{Z}$, we also define m_j^z by

$$m_j^z(\xi) = \{\rho(\xi)\}^z \int_{\mathbb{R}^k} e^{-2\pi i \xi \cdot \mathfrak{p}(t)} |t|^z K(t) \psi(2^j t)\, dt.$$

Then we have equations

$$m^z(\xi) = \sum_{j\in\mathbb{Z}} m_j^z(\xi) = \sum_{j\in\mathbb{Z}} m_0^z(2^{-j}\circ\xi).$$

With above preparation, we begin the proof of (3.1).

Proof. For fixed ξ with $\rho(\xi)\leqslant 1$, we have

$$\begin{aligned}|m_0^z(\xi)| &= \{\rho(\xi)\}^{\mathrm{Re}(z)}\left|\int_{\mathbb{R}^k} e^{-2\pi i\mathfrak{p}(t)\cdot\xi}|t|^z K(t)\psi(t)\,dt\right|\\ &= \{\rho(\xi)\}^{\mathrm{Re}(z)}\left|\int_{\mathbb{R}^k} [e^{-2\pi i\mathfrak{p}(t)\cdot\xi}-1]|t|^z K(t)\psi(t)\,dt\right|\\ &\leqslant C(z)|\xi| \leqslant C(z)\rho(\xi).\end{aligned}$$

Note that $\mathfrak{p}(t)$ is of finite type at each point, and is indeed of type at most d. For $\rho(\xi)>1$, by [18, Theorem 2 of Chapter 8] and Lemma 3.1, we have

$$\begin{aligned}|m_0^z(\xi)| &= \{\rho(\xi)\}^{\mathrm{Re}(z)}\left|\int_{\mathbb{R}^k} e^{-2\pi i\mathfrak{p}(t)\cdot\xi}|t|^z K(t)\psi(t)\,dt\right|\\ &\leqslant C(z)\{\rho(\xi)\}^{\mathrm{Re}(z)}|\xi|^{-1/d} \leqslant C(z)\rho(\xi)^{\mathrm{Re}(z)-1/d}.\end{aligned}$$

Finally, we use above two estimates and the fact $\mathrm{Re}(z)<\frac{1}{d}$, then

$$\begin{aligned}|m^z(\xi)| = \left|\sum_{j\in\mathbb{Z}} m_0^z(2^{-j}\circ\xi)\right| &\leqslant \sum_{\rho(2^{-j}\circ\xi)\leqslant 1} |m_0^z(2^{-j}\circ\xi)|\\ &\quad + \sum_{\rho(2^{-j}\circ\xi)>1} |m_0^z(2^{-j}\circ\xi)|\\ &\leqslant C(z)\rho(\xi)\sum_{\rho(\xi)\leqslant 2^j} 2^{-j} + C(z)\{\rho(\xi)\}^{\mathrm{Re}(z)-1/d}\\ &\quad \times \sum_{\rho(\xi)>2^j} 2^{-j[\mathrm{Re}(z)-1/d]}\\ &\leqslant C(z).\end{aligned}$$

□

3.2 The boundedness of $T_{\mathfrak{p}}^z$ on $L^p(\mathbb{R}^n;\mathbf{Y})$

The next goal is to prove that

$$\|T_{\mathfrak{p}}^z f\|_{L^p(\mathbb{R}^N;\mathbf{Y})} \leqslant C(z)\|f\|_{L^p(\mathbb{R}^N;\mathbf{Y})}, \quad 1<p<\infty, -\beta<\mathrm{Re}(z)<0, \tag{3.2}$$

where β is some positive constant and $C(z)$ grows at most polynomially in $(1 + |\mathrm{Im}(z)|)$.

In [18, p. 514], the L^2 boundedness of analytic singular integrals, together with Hörmander condition, implies the boundedness on L^p for all $p \in (1, \infty)$. This argument is also true for a general situation of vector-valued functions. In Hilbert space, the L^2 boundedness is obtained for free with the help of the Fourier transform and Plancherel's theorem. However, for general UMD space $\mathbf{Y}$, Plancherel's theorem does not hold anymore; it is a difficult task to get the boundedness of analytic singular integrals on $L^2(\mathbb{R}^N; \mathbf{Y})$, even for any single p_0. The following theorem gives the conditions on the singular kernel to yield a singular integral operator bounded on $L^p(\mathbb{R}^N; \mathbf{Y})$, which is a partial generalization of [5, Corollary 4.2] and can be prove in a similar way.

THEOREM 3.2. *Let $\mathbf{Y}$ be an UMD space, and $\mathbf{K} \in \mathscr{S}'(\mathbb{R}^N)$. The anisotropic dilations are given by $\delta_\lambda x = (\lambda^{b_1} x_1, \lambda^{b_2} x_2, \ldots, \lambda^{b_N} x_N)$ with $b_i > 0$, the respected norm function is denoted by $\rho(x)$. Suppose $\mathbf{K}$ has the homogeneity property $\lambda^b \mathbf{K}(\delta_\lambda x) = \mathbf{K}(x)$ with $b = \sum b_i$, $\mathbf{K}$ also satisfies the following conditions*

$$|\widehat{\mathbf{K}}(\xi)| \leqslant A_0,$$

and

$$\int_{\rho(x) \geqslant C_0 \rho(y)} (\mathbf{K}(x-y) - \mathbf{K}(x)) \log^N(e + \rho(x))\, dx \leqslant A_1 \log^N(2 + \rho(y)) \tag{3.3}$$

for some $C_0 \geqslant 6$. Then $f \in \mathscr{S}(\mathbb{R}^N; \mathbf{Y}) \mapsto \mathbf{K} * f$ extends to a bounded linear operator

$$f \in L^p(\mathbb{R}^N; \mathbf{Y}) \mapsto \mathbf{K} * f \in L^p(\mathbb{R}^N; \mathbf{Y})$$

with norm at most $C(A_0 + A_1)$, where C is a geometric constant.

We now use Theorem 3.2 to prove (3.2). We set $\Delta = \sum_{1 \leqslant |\alpha| \leqslant d} |\alpha|$. For $-\Delta < \mathrm{Re}(z) < 0$, $h_z(x)$ and $K_z(x)$ are given by

$$\hat{h}_z(\xi) = \{\rho(\xi)\}^z \qquad \text{and} \qquad \widehat{K_z}(\xi) = m^z(\xi),$$

respectively. Obviously,

$$K_z(x) = \text{p.v.} \int_{\mathbb{R}^k} h_z(x - \mathfrak{p}(t)) |t|^z K(t)\, dt.$$

For $f \in \mathscr{S}(\mathbb{R}^N; \mathbf{Y})$, we see that

$$T_{\mathfrak{p}}^z f(x) = K_z * f(x).$$

It suffices to verify K_z satisfying conditions in Theorem 3.2. So, we need the following lemma which can be found in [14, p. 1272].

LEMMA 3.3. *Let h_z and K_z be given as above. Then $h_z(x)$ is a locally integrable function, C^∞ away from the origin satisfying*

$$h_z(\lambda \circ x) = \lambda^{-\Delta - z} h_z(x), \qquad \lambda > 0, x \neq 0.$$

Moreover, each derivative of $h_z(x)$ is bounded by a polynomial in $|z|$, if $\rho(x) = 1$. In particular, K_z has the homogeneity property $\lambda^\Delta K_z(\lambda \circ x) = K_z(x)$.

According to the Fourier estimate in Section 3.1, we obtain that

$$|\widehat{K_z}(\xi)| = |m_z(\xi)| \leqslant C(z) \quad \text{when } \mathrm{Re}(z) < \frac{1}{d}.$$

To verify K_z satisfying (3.3), we may assume that $\rho(y) = 1$ and just need to prove that

$$\int_{\rho(x) \geqslant C_0} |K_z(x-y) - K_z(x)| \log^N(e + \rho(x))\, dx \leqslant C(z). \tag{3.4}$$

In fact, we set $\lambda = \rho(y)$ and $y' = y/\lambda$. Obviously, $\rho(y') = 1$. By a linear transformation $x = \lambda \circ x'$, notice the homogeneity of K_z, we have

$$\begin{aligned} &\int_{\rho(x) \geqslant C_0 \rho(y)} |K_z(x-y) - K_z(x)| \log^N(e + \rho(x))\, dx \\ &\quad = \int_{\rho(x') \geqslant C_0} |K_z(x'-y') - K_z(x')| \log^N(e + \lambda\rho(x'))\, dx'. \end{aligned}$$

If $\lambda = \rho(y) \geqslant 6$, it is trivial that

$$\log(e + \lambda\rho(x')) \leqslant \log(e+\lambda) + \log(e + \rho(x')) \leqslant \log(e+\lambda)\log(e+\rho(x')),$$

where we use the fact $C_0 \geqslant 6$. By (3.4), we obtain

$$\begin{aligned} &\int_{\rho(x) \geqslant C_0 \rho(y)} |K_z(x-y) - K_z(x)| \log^N(e + \rho(x))\, dx \\ &\quad \leqslant \int_{\rho(x') \geqslant C_0} |K_z(x'-y') - K_z(x')| \log^N(e + \rho(x'))\, dx' \log^N(e+\rho(y)) \\ &\quad \leqslant C(z) \log^N(e + \rho(y)). \end{aligned}$$

When $\lambda = \rho(y) < 6$, using (3.4), we have

$$\begin{aligned}
&\int_{\rho(x)\geqslant C_0\rho(y)} |K_z(x-y) - K_z(x)| \log^N(e+\rho(x))\,dx \\
&\leqslant 2^n \int_{\rho(x')\geqslant C_0} |K_z(x'-y') - K_z(x')| \log^N(e+\rho(x'))\,dx' \\
&\leqslant C(z) \leqslant C(z)\log^N(e+\rho(y)).
\end{aligned}$$

To prove (3.4), we define K_z^1 and K_z^2 by

$$K_z^1(x) = \int_{|t|\leqslant 1} h_z(x-\mathfrak{p}(t))|t|^z K(t)\,dt \qquad \text{and} \qquad K_z^2(x) = K_z(x) - K_z^1(x),$$

respectively. We split the integral as

$$\begin{aligned}
&\int_{\rho(x)\geqslant C_0} |K_z(x-y) - K_z(x)| \log^N(e+\rho(x))\,dx \\
&\leqslant \int_{\rho(x)\geqslant C_0} |K_z^1(x)| \log^N(e+\rho(x))\,dx \\
&\quad + \int_{\rho(x)\geqslant C_0} |K_z^1(x-y)| \log^N(e+\rho(x))\,dx \\
&\quad + \int_{\rho(x)\geqslant C_0} |K_z^2(x-y) - K_z^2(x)| \log^N(e+\rho(x))\,dx.
\end{aligned}$$

To estimate first two summands, we need a estimate related to h_z, which can be found in [14, p. 1273]. The homogeneity of h_z and the smoothness of h_z away from 0 imply that

$$|h_z(x-y) - h_z(x)| \leqslant C(z)\frac{|y|}{\{\rho(x)\}^{\Delta+\mathrm{Re}(z)+\mu}} \tag{3.5}$$

for some $\mu > 0$, provided $|y|/|x|$ is sufficiently small.

We set $\beta = \min\{\mu, 1\}$. For the first integral, by using Fubini theorem and (3.5), we have

$$\begin{aligned}
&\int_{\rho(x)\geqslant C_0} |K_z^1(x)| \log^N(e+\rho(x))\,dx \\
&\leqslant \int_{\rho(x)\geqslant C_0} \int_{|t|\leqslant 1} |h_z(x-\mathfrak{p}(t)) - h_z(x)||t|^{\mathrm{Re}(z)-k}\,dt \log^N(e+\rho(x))\,dx \\
&\leqslant \int_{|t|\leqslant 1} |t|^{\mathrm{Re}(z)-k} \int_{\rho(x)\geqslant C_0} |h_z(x-\mathfrak{p}(t)) - h_z(x)| \log^N(e+\rho(x))\,dx\,dt
\end{aligned}$$

$$\begin{aligned}
&\leqslant \int_{|t|\leqslant 1} |t|^{\mathrm{Re}(z)-k}|\mathfrak{p}(t)| \int_{\rho(x)\geqslant C_0} \rho(x)^{-[\Delta+\mathrm{Re}(z)+\mu]} \log^N(e+\rho(x))\,dx\,dt\\
&\leqslant \int_{|t|\leqslant 1} |t|^{\mathrm{Re}(z)-k}|\mathfrak{p}(t)| \int_{\mathcal{S}^{N-1}} \int_{C_0}^{\infty} r^{-[1+\mathrm{Re}(z)+\mu]} \log^N(e+r)\,dr\,d\omega\,dt\\
&\leqslant C(z),
\end{aligned}$$

where we use the fact that $-\beta<\mathrm{Re}(z)<0$ and coordinatize $\mathbb{R}^N$ by r and ω with $r=\rho(x)$ and $\omega=\frac{1}{r}\circ x$.

The norm function $\rho(x)$ has the property of $\rho(x+y)\leqslant c(\rho(x)+\rho(y))$ for some $c>0$ (see [14, Propositions 1–9]). Specially, we set $C_0\geqslant\max\{6,3c\}$. Note that $\rho(x-y)\geqslant\frac{1}{c}\rho(x)-\rho(y)\geqslant\frac{C_0}{c}-1\geqslant 2$ and $\rho(x)\leqslant c[\rho(x-y)+\rho(y)]\leqslant c\rho(x-y)+c$. Using a linear transformation, the second summand can be treated as the first one, then

$$\begin{aligned}
&\int_{\rho(x)\geqslant C_0} |K_z^1(x-y)| \log^N(e+\rho(x))\,dx\\
&\qquad\leqslant \int_{\rho(x)\geqslant 2} |K_z^1(x)| \log^N(e+c+c\rho(x))\,dx\\
&\qquad\leqslant C(z).
\end{aligned}$$

Finally, using Fubini theorem, we have

$$\begin{aligned}
&\int_{\rho(x)\geqslant C_0} |K_z^2(x-y)-K_z^2(x)| \log^N(e+\rho(x))\,dx\\
&\qquad\leqslant \int_{|t|>1}\int_{\rho(x)\geqslant C_0} |h_z(x-y-\mathfrak{p}(t))-h_z(x-\mathfrak{p}(t))|\\
&\qquad\quad\times \log^N(e+\rho(x))\,dx|t|^{\mathrm{Re}(z)-k}\,dt.
\end{aligned}$$

We divide the inner integral above according to the distance between x and $\mathfrak{p}(t)$. Note that $\rho(y)=1$, if $|y|/|x-\mathfrak{p}(t)|$ is sufficient small, that is $|x-\mathfrak{p}(t)|$ is away from the origin, we can get that $\rho(x-\mathfrak{p}(t))\geqslant C_1$ for an appropriate constant C_1. In this case, by (3.5) and a linear transformation, we obtain the following estimate

$$\begin{aligned}
&\int_{|t|\geqslant 1}\int_{\substack{\rho(x)\geqslant C_0\\ \rho(x-\mathfrak{p}(t))\geqslant C_1}} |h_z(x-y-\mathfrak{p}(t))-h_z(x-\mathfrak{p}(t))|\\
&\qquad\times \log^N(e+\rho(x))\,dx|t|^{\mathrm{Re}(z)-k}\,dt
\end{aligned}$$

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$$
\begin{aligned}
&\leqslant C(z)\int_{|t|\geqslant 1}\int_{\substack{\rho(x)\geqslant C_0\\ \rho(x-\mathfrak{p}(t))\geqslant C_1}}\frac{|y|}{\{\rho(x-\mathfrak{p}(t))\}^{\Delta+\mu+\mathrm{Re}(z)}}\\
&\quad\times\log^N(e+\rho(x))\,dx|t|^{\mathrm{Re}(z)-k}\,dt\\
&\leqslant C(z)\int_{|t|\geqslant 1}\int_{\rho(x)\geqslant C_1}\frac{1}{\{\rho(x)\}^{\Delta+\mu+\mathrm{Re}(z)}}\\
&\quad\times\log^N(e+c\rho(x)+ct)\,dx|t|^{\mathrm{Re}(z)-k}\,dt\\
&\leqslant C(z)\int_{|t|\geqslant 1}\int_{\rho(x)\geqslant C_1}\frac{1}{\{\rho(x)\}^{\Delta+\mu+\mathrm{Re}(z)}}\\
&\quad\times\{\log^N(e+\rho(x))+\log^N(e+t)\}\,dx|t|^{\mathrm{Re}(z)-k}\,dt\\
&\leqslant C(z),
\end{aligned}
$$

where we have used the fact that for fixed $|t|\geqslant 1$, $\rho(x)\leqslant c[\rho(x-\mathfrak{p}(t))+\rho(\mathfrak{p}(t))]=c[\rho(x-\mathfrak{p}(t))+t]$.

It is trivial that $\rho\big(x+y+\mathfrak{p}(t)\big)\leqslant c^2[\rho(x)+\rho(y)+\rho(\mathfrak{p}(t))]=c^2[1+\rho(x)+t]$. Then, the remainder can be controlled by

$$
\begin{aligned}
&\int_{|t|\geqslant 1}\int_{\substack{\rho(x)\geqslant C_0\\ \rho(x-\mathfrak{p}(t))\leqslant C_1}}[|h_z(x-y-\mathfrak{p}(t))|+|h_z(x-\mathfrak{p}(t))|]\\
&\quad\times\log^N(e+\rho(x))\,dx|t|^{\mathrm{Re}(z)-k}\,dt\\
&\leqslant\int_{|t|\geqslant 1}\int_{\substack{\rho(x)\geqslant C_0\\ \rho(x-\mathfrak{p}(t))\leqslant C_1}}|h_z(x-y-\mathfrak{p}(t))|\log^N(e+\rho(x))\,dx|t|^{\mathrm{Re}(z)-k}\,dt\\
&\quad+\int_{|t|\geqslant 1}\int_{\substack{\rho(x)\geqslant C_0\\ \rho(x-\mathfrak{p}(t))\leqslant C_1}}|h_z(x-\mathfrak{p}(t))|\log^N(e+\rho(x))\,dx|t|^{\mathrm{Re}(z)-k}\,dt\\
&\leqslant C\int_{|t|\geqslant 1}\int_{\rho(x)\leqslant c(C_1+1)}|h_z(x)|\,dx|t|^{\mathrm{Re}(z)-k}\log^N(e+t)\,dt\\
&\leqslant C(z),
\end{aligned}
$$

where we have used the fact that h_z is locally integrable.

3.3 Analytic interpolation

To complete the proof of (2.1), we use a vector-valued extension of Stein interpolation theorem [15].

Note that $\mathbf{X}=[\mathbf{H},\mathbf{Y}]_\theta$ for some Hilbert space $\mathbf{H}$, UMD space $\mathbf{Y}$ and $\theta\in(0,1)$. For fixed $\mathbf{H}$ and $\mathbf{Y}$, there exists a constant $C(z)$ growing at most polynomially in $(1+|\mathrm{Im}(z)|)$ such that

$$\|T_{\mathfrak{p}}^{z}f\|_{L^2(\mathbb{R}^N;\mathbf{H})} \leqslant C(z)\|f\|_{L^2(\mathbb{R}^N;\mathbf{H})} \quad \text{when } \mathrm{Re}(z) < \frac{1}{d}, \tag{3.6}$$

and

$$\|T_{\mathfrak{p}}^{z}f\|_{L^q(\mathbb{R}^N;\mathbf{Y})} \leqslant C(z)\|f\|_{L^q(\mathbb{R}^N;\mathbf{Y})} \quad \text{when } -\beta < \mathrm{Re}(z) < 0, 1 < q < \infty. \tag{3.7}$$

Obviously, (3.7) also holds in particular with $\mathbf{Y}=\mathbf{H}$.

For $1<p<\infty$, we chose $\theta_1 \in (0,1)$, $0<\sigma_0<\frac{1}{d}$, $-\beta<\sigma_1<0$ and $q_1 \in (1,\infty)$ such that

$$\sigma_0(1-\theta_1)+\sigma_1\theta_1 =: \sigma_2 > 0, \quad \frac{1}{p} = \frac{1-\theta_1}{2} + \frac{\theta_1}{q_1}.$$

Interpolating between (3.6) and (3.7) with $\mathbf{Y}$ replaced by $\mathbf{H}$, we have

$$\|T_{\mathfrak{p}}^{z}f\|_{L^p(\mathbb{R}^N;\mathbf{H})} \leqslant C(z)\|f\|_{L^p(\mathbb{R}^N;\mathbf{H})} \quad \text{when } \mathrm{Re}(z) = \sigma_2 > 0. \tag{3.8}$$

For fixed θ, we choose $-\beta<\sigma_3<0$ such that $0=(1-\theta)\sigma_2+\theta\sigma_3$. In the same way, interpolating between (3.8) and (3.7) with $q=p$, we obtain

$$\|T_{\mathfrak{p}}f\|_{L^p(\mathbb{R}^N;\mathbf{X})} = \|T_{\mathfrak{p}}^{0}f\|_{L^p(\mathbb{R}^N;\mathbf{X})} \leqslant C\|f\|_{L^p(\mathbb{R}^N;\mathbf{X})}.$$

This completes the proof of (2.1).

References

[1] A. Benedek, A. P. Caldern and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Natl. Acad. Sci. USA **48** (1962), 356–365.

[2] E. Berkson and T. A. Gillespie, *An $\mathfrak{M}_q(\mathbb{T})$-functional calculus for power-bounded operators on certain UMD spaces*, Studia Math. **167** (2005), 245–257.

[3] J. Bourgain, "*Vector-valued singular integrals and the H^1-BMO duality*", in *Probability Theory and Harmonic Analysis (Cleveland, 1983)*, Pure Appl. Math. **98**, Dekker, New York, 1986, 1–19.

[4] D. L. Burkholder, "*A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions*", in *Conference on Harmonic Analysis in Honor of Antoni Zygmund, Vols I and II (Chicago, III, 1981)*, Wadsworth Math. Ser., Wadsworth, Belmont, CA, 1983, 270–286.

[5] T. Hytönen, *Anisotropic Fourier multipliers and singular integrals for vector-valued functions*, Ann. Mat. **186** (2007), 455–468.

[6] T. Hytönen and L. Weis, *Singular convolution integrals with operator-valued kernel*, Math. Z. **255** (2007), 393–425.

[7] H. Liu, *The boundedness of maximal operators and singular integrals via Fourier transform estimates*, Acta Math. Sin. (Engl. Ser.) **28** (2012), 2227–2242.

[8] G. Hong and H. Liu, *Vector-valued Hilbert transforms along curves*, Banach J. Math. Anal. **10** (2016), 430–450.
[9] T. R. McConnell, *On Fourier multiplier transformations of Banach-valued functions*, Trans. Amer. Math. Soc. **285** (1984), 739–757.
[10] D. H. Phong and E. M. Stein, *Hilbert integrals, singular integrals and Radon transforms I*, Acta Math. **157** (1986), 99–157.
[11] F. Ricci and E. M. Stein, *Harmonic analysis on nilpotent groups and singular integrals. II. Singular kernels supported on submanifolds*, J. Funct. Anal. **78** (1988), 56–84.
[12] J. L. Rubio de Francia, F. J. Ruiz and J. L. Torrea, *Calderón–Zygmund theory for operator-valued kernels*, Adv. Math. **62** (1986), 7–48.
[13] J. L. Rubio de Francia, "*Martingale and integral transforms of Banach space valued functions*", in *Probability and Banach spaces (Zaragoza, 1985)*, Lecture Notes in Mathematics **1221**, Springer, Berlin, 1986, 195–222.
[14] E. M. Stein and S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. (N.S.) **84** (1978), 1239–1295.
[15] E. M. Stein, *Interpolation of linear operators*, Trans. Amer. Math. Soc. **83** (1956), 482–492.
[16] E. M. Stein, "*Some problems in harmonic analysis suggested by symmetric spaces and semi-simple groups*", in *Actes du Congrs International des Mathmaticiens, Vol. 1 (Nice, 1970)*, Gauthier-Villars, Paris, 1971, 173–189.
[17] E. M. Stein, "*Problems in harmonic analysis related to curvature and oscillatory integrals*", in *Proceedings of the International Congress of Mathematicians, Berkeley, California, USA*, 1986, 196–221.
[18] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, NJ, 1993.
[19] F. Zimmermann, *On vector-valued Fourier multiplier theorems*, Studia Math. **93** (1989), 201–222.

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