REMARKS ON THE ASYMPTOTIC BEHAVIOUR OF PERTURBED LINEAR SYSTEMS†

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1. Introduction. We are here concerned with following result of Trench:

THEOREM. (Trench [5]). *Let v_1 and v_2 be two linearly independent solutions of the differential equation*

$$v'' = a(t)v,$$

where $a(t)$ is continuous on $[0, \infty)$, and let $b(t)$ be a continuous function of t for $t \geqq 0$ satisfying

$$\int_0^\infty |b(t)|\, m(t)\, dt < \infty,$$

where $m(t) = \max\{|v_1(t)|^2, |v_2(t)|^2\}$. Then, if α_1 and α_2 are two arbitrary constants, there exists a solution u of

$$u'' = (a(t)+b(t))u$$

which can be written in the form

$$u = \alpha_1(t)v_1 + \alpha_2(t)v_2,$$

with

$$\lim_{t\to\infty} \alpha_i(t) = \alpha_i \qquad \text{for} \qquad i = 1, 2.$$

The purpose of this note is to extend the above theorem to general systems with non-linear perturbations. Here we consider

$$x' = A(t)x, \tag{1}$$

and

$$y' = A(t)y + f(t, y), \tag{2}$$

where $A(t)$ is a $n \times n$ matrix with real-valued continuous functions as elements, and $f(t, x)$ is an n-vector continuous for $0 \leqq t \leqq \infty$, $|y| < \infty$. (We define

$$|A| = \sup_{1 \leqq i \leqq n} \sum_{j=1}^{n} |a_{ij}|$$

for any matrix $A = \{a_{ij}\}$, and accordingly for any vector $v = \{v_i\}$, $|v| = \sup_{1 \leqq i \leqq n} |v_i|$.) Denote by $X(t) = \{x_{ij}(t)\}$ the fundamental solution matrix of (1). We envisage conditions on $f(t, y)$ such that every solution of (2) may be represented in the form $y(t) = X(t)c(t)$, where $\lim_{t\to\infty} c(t)$ exists and is finite, and that, for any given vector c, there exists a solution of (2) such that $y(t) = X(t)\, x(t)$ with $\lim_{t\to\infty} c(t) = c$. Several applications of our main theorem are also indicated.

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2. The main theorem. We assume throughout this discussion that $f(t, y)$ satisfies the condition

$$|f(t,y)| \leqq \lambda(t)\phi(|y|), \tag{3}$$

where $\lambda(t)$, $\phi(r)$ are non-negative continuous functions on $[0, \infty)$, with $\phi(r)$ non-decreasing in r and satisfying the condition

$$\phi(r_1 r_2) \leqq \phi(r_1)\phi(r_2). \tag{4}$$

Our extension of Trench's result is contained in the following theorem.

THEOREM 1. *Let $\phi(r)$ in addition satisfy the condition*

$$\int_{r_0}^{\infty} \frac{dr}{\phi(r)} = \infty, \tag{5}$$

for some $r_0 > 0$. If

$$\int_0^{\infty} \lambda(t)\exp\left(-\int_0^t \operatorname{trace} A(u)\,du\right)M^{n-1}(t)\phi(M(t))\,dt < \infty, \tag{6}$$

where

$$M(t) = \max_{1 \leqq i,\, j \leqq n} |x_{ij}(t)|,$$

then every solution of (2) *may be written in the form $y(t) = X(t)c(t)$, where $\lim_{t\to\infty} c(t)$ exists and is finite; moreover, for any given vector c there exists a solution of* (2) *such that $y(t) = X(t)c(t)$ with $\lim_{t\to\infty} c(t) = c$.*

The proof of the above theorem depends on the following two lemmas due to Wintner and Bihari.

LEMMA 1. (Wintner [7]) *Let $\mu(t)$ be a non-negative and continuous function on* $[0, \infty)$, *satisfying the condition*

$$\int_0^{\infty} \mu(t)\,dt < \infty, \tag{7}$$

and let $\phi(r)$ be given as above satisfying (4) *and* (5). *Then each solution of the system*

$$v' = g(t, v), \tag{8}$$

with $g(t, v)$ satisfying the condition

$$|g(t,v)| \leqq \mu(t)\phi(|v|), \tag{9}$$

tends to a finite vector as $t \to \infty$, and each finite vector is the limit as $t \to \infty$ of some solution of (8).

LEMMA 2. (Bihari [2], Brauer [3]) *Let $\mu(t)$, $\phi(r)$ be non-negative and continuous functions on* $[0, \infty)$. *Then every solution $v(t)$ of* (8) *with initial condition $|v(0)| \leqq v_0$ obeys the following*

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inequality

$$|v(t)| \leqq \Phi^{-1}\left(\Phi(v_0)+\int_0^t \mu(s)\,ds\right), \tag{10}$$

for all t whenever

$$\int_0^t \mu(s)\,ds \leqq \int_{v_0}^{\infty} \frac{du}{\phi(u)},$$

where

$$\Phi(r) = \int_{r_0}^{r} \frac{du}{\phi(u)}.$$

Proof of Theorem 1. Write $y(t) = X(t)c(t)$, where $c(t)$ is to be determined. Substituting this into (2), one easily obtains

$$c(t) = c(0)+\int_0^t X^{-1}(s)f(s, X(s)c(s))\,ds. \tag{11}$$

We now may estimate $X^{-1}(t)$ by

$$|X^{-1}(t)| \leqq C_1 M^{n-1}(t)\exp\left(-\int_0^t \operatorname{trace} A(u)\,du\right), \tag{12}$$

where C_1 is some convenient constant which may be taken as $(n-1)^{\frac{1}{2}(n-1)}$, for example. Using (3), (4) and (12), we obtain for (10) the following estimate:

$$|c(t)| \leqq |c(0)|+C_2\int_0^t \lambda(s)EA(s)M^{n-1}(s)\phi(M(s))\phi(|c(s)|)\,ds, \tag{13}$$

where C_2 is another constant and $EA(t) = \exp\left(-\int_0^t \operatorname{trace} A(u)\,du\right)$. Since $\phi(r)$ satisfies (5), we may apply Lemma 2 to (13) (In this case, the upper bound given in (10) holds unrestrictedly.), and obtain

$$|c(t)| \leqq \Phi^{-1}\left(\Phi(|c(0)|)+C_2\int_0^t \lambda(s)EA(s)M^{n-1}(s)\phi(M(s))\,ds\right). \tag{14}$$

Hence we conclude from (6) that $|c(t)|$ is bounded. Now, since the integral in (11) is dominated by a constant multiple of the integral in (6), it follows that $\lim_{t\to\infty} c(t)$ exists and is finite. On the other hand, $c(t)$ satisfies the differential equation

$$c'(t) = X^{-1}(t)f(t, X(t)c(t)), \tag{15}$$

and from (3), (4) and (12) we have

$$|X^{-1}(t)f(t, X(t)c(t))| \leqq C_1\,\lambda(t)EA(t)M^{n-1}(t)\phi(M(t))\phi(|c(t)|).$$

By considering Lemma 1 with $g(t, c) = X^{-1}(t)f(t, X(t)c)$ and $\mu(t) = C_1\,\lambda(t)EA(t)M^{n-1}(t)\phi(M(t))$ we may conclude that, for each finite vector c there exists a solution $c(t)$ of (15) such that $\lim_{t\to\infty} c(t) = c$, which completes the proof.

Remark 1. The above theorem easily reduces to the result of Trench by taking for $n = 2$,

$$A(t) = \begin{pmatrix} 0 & 1 \\ a(t) & 0 \end{pmatrix}$$

and

$$f(t, y) = \begin{pmatrix} 0 \\ b(t)y_1 \end{pmatrix}.$$

Here $f(t, y)$ satisfies the condition $|f(t, y)| \leqq |b(t)||y|$.

Remark 2. As a typical example of a function $\phi(r)$ satisfying the assumptions given in Theorem 1, we may take $\phi(r) = r^p$ for $0 \leqq p \leqq 1$.

Remark 3. A similar approach to that given in **[5]** will yield the above result with $f(t, y) = B(t)y$ under the additional hypothesis that trace $B(t) = 0$. On the other hand, Trench's result may also be deduced from well-known results on linear systems; in particular, Coddington and Levinson **[4]**, p. 99, Problems 5, 6.

3. Applications and remarks.

THEOREM 2. *In addition to the above assumptions on $\lambda(t)$ and $\phi(r)$, we assume further that $\phi(r)$ satisfies* (5), *and*

$$\int_0^\infty \lambda(t)\, dt < \infty, \tag{16}$$

$$\lim_{t\to\infty} \int_0^t \operatorname{trace} A(u)\, du > -\infty. \tag{17}$$

If all solutions of (1) *are bounded, then* (*a*) *all solutions of* (2) *are bounded,* (*b*) *all solutions of* (1) *are asymptotically stable if and only if all solutions of* (2) *are asymptotically stable.*

Proof. Since (16), (17) and the hypothesis that all solutions of (1) are bounded imply (6), the above assertion follows immediately from Theorem 1.

Remark 4. In case $f(t, y) = B(t)y$ and $\phi(r) = r$, the above theorem reduces to a standard result on stability of linear systems (e.g. Bellman **[1]**, p. 43, Theorem 6; Coddington and Levinson **[4]**, pp. 98–99, Problems 3, 4).

THEOREM 3. *Under the same assumptions as those of Theorem* 1, *if we assume instead of* (5) *that*

$$\lim_{\delta\to 0} \int_\delta^{r_0} \frac{du}{\phi(u)} = \infty, \tag{18}$$

then there exist solutions $y(t)$ of (2) *which can be written in the form $y(t) = X(t)c(t)$, where $\lim_{t\to\infty} c(t)$ exists and is finite.*

Proof. Write $y(t) = X(t)c(t)$ with $|c(0)| = c_0$ so chosen such that

$$\int_{c_0}^{\infty} \frac{du}{\phi(u)} \geqq \int_0^{\infty} n\mu(t)\,dt,$$

where $\mu(t)$ is defined in the proof of Theorem 1. It is easy to see from Lemma 2 that the same argument carries over almost verbatim.

Remark 5. In case $n = 2$, with $A(t)$ as given in Remark 1 and

$$f(t, y) = \begin{pmatrix} 0 \\ b(t)y_2^p \end{pmatrix},$$

where $p \geqq 1$, the above result reduces to a theorem of Waltman [6], where the result is stated for $p = 2m-1$, with m a positive integer.

Added in Proof. Further results and related references may be found in F. Brauer and J. S. W. Wong, "On asymptotic behaviour of perturbed linear systems," to appear in *J. Differential Equations.*

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