2-row Springer Fibres and Khovanov Diagram Algebras for Type D

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Abstract. We study in detail two row Springer fibres of even orthogonal type from an algebraic as well as a topological point of view. We show that the irreducible components and their pairwise intersections are iterated $\mathbb{P}^1$-bundles. Using results of Kumar and Procesi we compute the cohomology ring with its action of the Weyl group. The main tool is a type D diagram calculus labelling the irreducible components in a convenient way that relates to a diagrammatical algebra describing the category of perverse sheaves on isotropic Grassmannians based on work of Braden. The diagram calculus generalizes Khovanov's arc algebra to the type D setting and should be seen as setting the framework for generalizing well-known connections of these algebras in type A to other types.

1 Introduction

This paper is part of a series of four quite different papers [10–12] dealing with type D generalizations of Khovanov's arc algebra. We develop in detail the geometric background of this algebra using the geometry of topological and algebraic Springer fibres and explain connections to category $\mathcal{O}$ for the orthogonal Lie algebra and to categories of finite dimensional representations of the associated $\mathcal{W}$-algebras.

The type D Khovanov algebra is defined in [10] using Braden's description of the category of perverse sheaves on isotropic Grassmannians; hence it gives an elementary description of parabolic category $\mathcal{O}$ of type (D_k, A_{k-1}) with all its nice properties, such as Koszulness, quasi-hereditary, cellularity, etc. Based on [8], we introduce in [11] suitable idempotent truncations of the type D Khovanov algebra and show in [12] that these algebras are indeed graded analogues of Brauer algebras $\mathrm{Br}_d(\delta)$ for arbitrary integral parameter δ. These idempotent truncations can also be viewed as a kind of limit, in which case this is very much in the spirit of a similar result obtained in [7] for the walled Brauer algebra using a limit version of generalized Khovanov algebras of type A. Although the flavour of results is very similar to the known type A results, the techniques and difficulties are very different, and include quite a few surprising new phenomena. In particular, the underlying geometry of Springer fibres is much more subtle and has not been studied so far. Let us now describe the content of this paper in more detail.

Let $V = \mathbb{C}^{2k}$ be an even dimensional vector space with a non-degenerate symmetric bilinear form β. Let $G = O(2k, \mathbb{C})$ be the group of automorphisms preserving β and let $\mathfrak{g} = \mathfrak{o}(2k, \mathbb{C})$ be the corresponding orthogonal Lie algebra. By a result of

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Williamson [54], the G-conjugacy classes of nilpotent elements in $\mathfrak{g}$ are classified by their Jordan normal form, *i.e.*, by partitions of $2k$ that are admissible, in the sense that even parts appear with even multiplicity.

Let $\mathcal{F}$ be the flag variety for G of full isotropic flags with respect to β,

$$\{F_0 \subset F_1 \subset \cdots \subset F_{2k} = \mathbb{C}^{2k} \mid \dim(F_i) = i, F_i^\perp = F_{2k-i}\}.$$

Given a nilpotent element $N \in \mathfrak{g}$, consider the Springer fibre $\mathcal{F}_N$ of all flags $F_\bullet \in \mathcal{F}$ fixed by N, *i.e.*, satisfying $N(F_i) \subseteq F_{i-1}$ for all $i > 0$. This is the fibre of the famous Springer resolution of the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ over N. The Springer fibres and their corresponding irreducible components are in general poorly understood. Even in type A the components are not smooth in general; the first case where singular components appear is the two columns case [14]. Recently, Fresse and Melnikov found a parametrisation of all smooth irreducible components in type A in terms of standard tableaux [13]. This includes the quite easy cases of two row partitions, where Fung showed much earlier ([16]), that irreducible components are iterated $\mathbb{P}(\mathbb{C})^1$-bundles and therefore smooth.

In this paper we are interested in generalizing the latter result to the orthogonal case and, moreover, in constructing an easy topological model for these Springer fibres, analogous to the type A model of Khovanov [23]. Our result in Section 6 is the following theorem.

Theorem 1.1 *Let $N \in \mathfrak{g}$ be a nilpotent element of Jordan type (λ_1, λ_2), where $\lambda_1 \geq \lambda_2$. Every irreducible component Y of $\mathcal{F}_N$ is an iterated fibre bundle of base type $(\mathbb{P}^1(\mathbb{C}), \ldots, \mathbb{P}^1(\mathbb{C}))$, where each $\mathbb{P}^1(\mathbb{C})$ corresponds canonically to a cup in the cup diagram associated with the signed admissible standard tableau labelling Y. In particular, $\mathcal{F}_N$ is equidimensional of dimension λ_2.*

Originally, irreducible components in the orthogonal case were classified by Spaltenstein [45] and van Leeuwen [53] using (signed) domino tableaux. Our results rely on a new parametrisation in terms of certain decorated cup diagrams on k vertices.

Fixing the points $P = \{(1, 0), (2, 0), \ldots, (k, 0)\}$ and the rectangle

$$R = \{(x, y) \mid 0 \leq x \leq k+1, -2 \leq y \leq 0\}$$

in the plane, an *undecorated cup diagram* is the isotopy class of a diagram consisting of k non-intersecting lines in R, each of them connecting two distinct vertices in P or one vertex in P with a point on the bottom face of R. A (decorated) cup diagram, is a cup diagram with possibly "$\bullet$"s (called *dots*) on the lines such that every dot is accessible from the left side of the rectangle, meaning for each dot there exists a line in the rectangle connecting the dot with the left face of R not intersecting the diagram. We allow at most one dot per line; here are some examples:

(1.1)
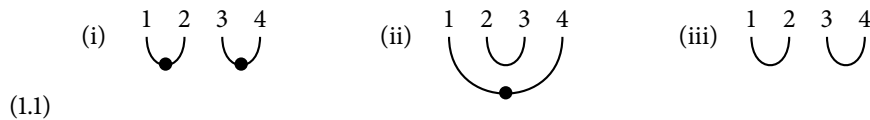
The set $\mathcal{C}_k$ of cup diagrams on k vertices describes combinatorially a cell partition of $\mathcal{F}_N$ where the set of the diagrams $\mathbb{B}_k$ with the maximal possible number of cups correspond to the irreducible components when taking the closure. In Section 7 we show that the $\mathcal{F}_N$ for N of Jordan type (k, k) have a filtration by subvarieties isomorphic to the Springer fibres of type $(2k - j, j)$ corresponding to cup diagrams in $\mathcal{C}_k$ with j cups. There is also a natural action of the Weyl group of type D_k on $\mathcal{C}_k$ in terms of type D-Temperley-Lieb diagrams [28]. This action preserves the parity of dots on the diagrams. Moreover, $\mathcal{C}_k$ labels a basis of the top degree part of the cohomology of the Springer fibre; see Theorem 7.5 and Lemma 4.17. In fact this illustrates nicely the Springer representation on the top degree part and on the whole cohomology, see Remark 5.21. It is the nontrivial component group which relates the subsets $\mathcal{C}_k^{\text{even}}$ and $\mathcal{C}_k^{\text{odd}}$ of diagrams with fixed parity; $\mathbb{Z}_2$ acts by adding or removing a dot on the line attached to vertex 1. Note that some sort of decorated diagrams also appear in [38], but the dots have a very different meaning, and the diagrams are used to describe the whole cohomology ring. We expect that an analogue of such a description also exists for our setup, but we do not pursue this direction here.

The cup diagrams should be seen as "foldings" or fixed point sets of an involution on the set of type A cup diagrams of doubled size; see [28]. This folding on the level of Springer fibres corresponds to a folding on the level of Nakajima quiver varieties recently studied in [20].

The cup diagrams are the only ingredients needed to construct the topological Springer fibre $\widetilde{S}$ following the main idea of [23]. For fixed k we consider the k-fold product of 2-spheres, $(\mathbb{S}^2)^k$ and assign to each $a \in \mathbb{B}_k$ the subspace S_a of all tuples $(x_1, x_2, \ldots, x_k) \in (\mathbb{S}^2)^k$ with the condition $x_i = x_j$ (resp. $x_i = -x_j$) if there is an undotted (resp. dotted) cup connecting vertices i and j, and x_i equals the north or south pole if there is a vertical ray at i, depending if the ray is dotted or not; see (4.1). The topological Springer fibre is then simply the union

$$\widetilde{S} = \bigcup_{a \in \mathbb{B}_k} S_a \subset (\mathbb{S}^2)^k.$$

We prove that the cohomology rings of the topological and algebraic Springer fibres agree and are described as follows.

Theorem 1.2 *Let $N \in \mathfrak{g}$ be nilpotent of Jordan type (k, k). Then we have isomorphisms of graded rings*

$$H(\widetilde{S}) \cong H(\mathcal{F}_N) \cong \bigoplus_{r=1}^{2} \mathbb{C}[x_1, \ldots, x_k] \Big/ \left\langle x_i^2, y_I \ \middle|\ \begin{matrix} 1 \le i \le k, \\ I \subset \{1, \ldots, k\}, |I| = \frac{k+\epsilon}{2} \end{matrix} \right\rangle,$$

where $\epsilon = 0$ if k is even and $\epsilon = 1$ if k is odd and

$$y_I = \begin{cases} x_I - x_{\{1, \ldots, k\} \setminus I} & \text{if } k \text{ is even,} \\ x_I & \text{if } k \text{ is odd,} \end{cases}$$

using the abbreviation $x_I = \prod_{i \in I} x_i$ for $I \subset \{1, \ldots, k\}$.

The two summands appear due to the natural decomposition of both the topological and the algebraic Springer fibres into "even" and "odd" components. The proof

of the second isomorphism is based on the Kumar–Procesi description of equivariant cohomology of Springer fibres [24]. This result requires N to be of standard Levi type and also requires the surjectivity of the canonical map $H(\mathcal{F}) \to H(\mathcal{F}_N)$, which fails in general. Using the filtration mentioned above, we verify this assumption in our special case (Proposition 7.4). The proof of the first isomorphism follows very closely the arguments in [23], although most of the small ingredients have to be reproved in a slightly different way. As for type A, and pointed out by Khovanov, the irreducible components of the topological Springer fibre are always trivial fibre bundles, whereas algebraically they are not. The non-triviality is nicely encoded in the degree of nestedness in the cup diagram and should be compared with the theory of embedded cobordisms and TQFT developed in [51].

Conjecture 1.3 *The topological and algebraic Springer fibres are homeomorphic in case N is of Jordan type (k, k).*

In the general two row case, such a homeomorphism cannot exist, since, for instance, the dimensions of pairwise intersections of components differ; see Section 6.5. For these cases a different construction of a topological Springer fibre is required.

The case of Jordan type (k, k) is, however, particularly nice. We show that the pairwise intersections $Y_1 \cap Y_2$ of irreducible components in $\mathcal{F}_N$ are again iterated $\mathbb{P}^1(\mathbb{C})$-bundles of dimension equal the number of possible orientations $a(Y_1)^* \lambda a(Y_2)$ of the circle diagram obtained by putting the cup diagram $a(Y_1)$ assigned to Y_1 upside down on top of $a(Y_2)$. Generalizing Khovanov's original construction [22] (see also [6]), we consider the vector space $\mathbb{K}_k$ of all such oriented circle diagrams and establish the following theorem.

Theorem 1.4 *Let $N \in \mathfrak{g}$ be nilpotent of Jordan type (k, k). There is an isomorphism of vector spaces*

$$(1.2) \qquad \mathbb{K}_k \cong \bigoplus_{(Y_1, Y_2) \in \mathrm{Irr}(\mathcal{F}_N) \times \mathrm{Irr}(\mathcal{F}_N)} H(Y_1 \cap Y_2).$$

This vector space can be equipped with an algebra structure using a geometric convolution product construction following the arguments in [51].

We finally explain partly conjectural connections with W-algebras and category $\mathcal{O}$. Let $e \in \mathfrak{g}$ be nilpotent and consider the associated finite W-algebra $\mathcal{W}(e) := \mathcal{U}(\mathfrak{g}, e)$ as originally introduced by Premet [36]; see [26] for an overview. In general, it is a hard problem to describe and understand the categories $\mathcal{F}in(e)$ of finite dimensional $\mathcal{W}(e)$-modules or its subcategory $\mathcal{F}in_0(e)$ of polynomial representations with generalized trivial central character; see [25, 27]. However, the original definition of $\mathcal{W}(e)$ and its realisation as deformation quantization of Slodowy slices, indicates a strong relationship between the representation of $\mathcal{W}(e)$ and the geometry of the Springer fibre. Our setup is related to the case where the nilpotent has Jordan type 2^k. This is an extremal case of a rectangular Jordan shape, where a complete purely combinatorial classification of finite dimensional $\mathcal{W}(e)$-modules is available; see [1, 5]. In the last section we give an easy bijection between Brown's parameter set of simple objects

and our set $\mathbb{B}_k$ of decorated cup diagrams labelling the irreducible components of $\mathcal{F}_N$, where N is of Jordan type (k, k). Based on this, we conjecture the following.

Conjecture 1.5   *Let $k \geq 4$ even and $e, N \in \mathfrak{g}$ nilpotent of Jordan type 2^k and (k, k) respectively. Let P be a minimal projective generator of $\mathcal{F}in_0(e)$. Under the identification* (1.2) *there exists an isomorphism of vector spaces*

$$\mathbb{K}_k = \bigoplus_{(Y_1, Y_2) \in \mathrm{Irr}(\mathcal{F}_N) \times \mathrm{Irr}(\mathcal{F}_N)} H(Y_1 \cap Y_2) \cong \mathrm{End}_{\mathcal{W}(e)}(P),$$

even of algebras with the diagrammatical multiplication in $\mathbb{K}_k$.

The conjecture implies in particular that with the appropriate grading shifts,

$$\bigoplus_{(Y_1, Y_2) \in \mathrm{Irr}(\mathcal{F}_N) \times \mathrm{Irr}(\mathcal{F}_N)} H(Y_1 \cap Y_2) \langle d(Y_1, Y_2) \rangle$$

induces a positive grading on $\mathrm{End}_{\mathcal{W}(e)}(P)$ which is the shadow of a Koszul grading. More precisely, by [10] the conjecture directly implies the following corollary.

Corollary 1.6   *Let $\mathfrak{p}, \mathfrak{p}' \subset \mathfrak{g}$ be the two maximal parabolic subalgebras of type A_{k-1}. Let $\mathcal{O}_0^{\mathfrak{p}}(D_k) \oplus \mathcal{O}_0^{\mathfrak{p}'}(D_k)$ be the direct sum of the principal blocks of the corresponding parabolic category $\mathcal{O}$ for $\mathfrak{g}$. Let $P_{\mathrm{inj}} = P_{\mathrm{inj}}^{\mathfrak{p}} \oplus P_{\mathrm{inj}}^{\mathfrak{p}'}$ be a minimal projective-injective generator. Then there is an isomorphism of algebras*

$$\mathrm{End}_{\mathfrak{g}}(P_{\mathrm{inj}}) \cong \mathrm{End}_{\mathcal{W}(e)}(P).$$

In particular, $\mathcal{F}in_0(e)$ is a quotient category of $\mathcal{O}_0^{\mathfrak{p}}(D_k) \oplus \mathcal{O}_0^{\mathfrak{p}'}(D_k)$. The latter is in fact a quasi-hereditary cover in the sense of Rouquier [37].

Note that this corollary is analogous to the results in [3] for arbitrary nilpotent elements $e \in \mathfrak{gl}_n$, which realizes the corresponding $\mathcal{F}in_0(e)$ as a quotient category of $\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{gl}_n)$, where $\mathfrak{p}$ has type transposed to the Jordan type of e. Such a general result is clearly false in general outside of type A. It already fails for Jordan types (λ_1, λ_2) with $\lambda_1 \neq \lambda_2$ or $\lambda_1 = \lambda_2$ odd. It is expected that the above holds more generally for all nilpotent e with an even good grading in the sense of [2] and trivial component group, but of course one cannot hope for such an explicit and elementary description of the endomorphism ring as in the case described above.

The paper is organized as follows: In Section 2 we introduce the algebraic Springer fibre and fix notation. Section 3 sets up the diagram combinatorics, which are crucial for the whole paper. Section 4 defines the topological Springer fibre with its cell partition and computes the cohomology rings of the components and their intersections and finally connects it with the diagram algebra $\mathbb{K}_k$ and its centre. Section 5 establishes several combinatorial bijections between signed admissible standard tableaux, standard tableaux, and cup diagrams and relates it to the combinatorics of representations of the Weyl group $W(D_k)$. This is necessary to connect our cup diagram combinatorics to the original work of Spaltenstein and van Leeuwen and Hotta-Springer. For this paper, the bijections are merely a tool, but we feel they are interesting on their own. Section 6 gives the description of the irreducible components for the algebraic

Springer fibre with an explicit algorithm for constructing the component attached to a cup diagram. Section 7 contains the proof of the main result, Theorem 1.2. In Section 8 we finally describe the connection and applications and prove Theorem 1.4.

2 Algebraic Springer fibre

General Assumptions

We fix the ground field $\mathbb{C}$ and denote $\sqrt{-1} \in \mathbb{C}$ by $\mathbf{i}$.

Fix an even natural number $n = 2k$ and consider a complex n-dimensional vector space W with fixed basis w_i, where $i \in \{\pm 1, \ldots, \pm k\}$. Equip W with a symmetric non-degenerate bilinear form β by setting $\beta(w_i, w_j) = \delta_{i,-j}$. Let $G = O(W, \beta) = O(2k, \mathbb{C})$ be the corresponding orthogonal group with Lie algebra $\mathfrak{g} = \mathfrak{o}(2k, \mathbb{C})$.

Definition 2.1 A *full isotropic flag* $F_\bullet$ in W is a sequence of subspaces in W, $\{0\} \subset F_0 \subset F_1 \subset \cdots \subset F_k$, such that $\dim F_i = i$ and β vanishes on $F_i \times F_i$ for all $0 \le i \le k$ (*i.e.*, F_i is isotropic). Let $\mathcal{F}$ be the set of all such flags.

Note that F_k is maximal isotropic and $\dim(F_i) = i$ for all i. Setting $F_{n-i} = (F_i)^\perp$ defines a full flag in $\mathbb{C}^n$ in the usual sense. We denote by B the standard Borel of G fixing the flag $F_\bullet$ where $F_i = \langle w_j \mid 1 \le j \le i \rangle$. The set $\mathcal{F}$ inherits the structure of a projective algebraic variety by identifying it with G/B.

Remark 2.2 In contrast to types A, B, C, this flag variety of type D is not connected, but decomposes into two components isomorphic to $\mathcal{F}' = G'/B'$, where $G' = SO(2n, \mathbb{C})$ and $B' = G' \cap B$. The component containing $F_\bullet$ is determined by F_k. Given $F_\bullet$, there is a unique flag $F'_\bullet$ (called the *companion flag*) that satisfies $F'_i = F_i$ for $0 \le i < k$ and $F'_k \ne F_k$. The two flags lie in different components.

A *partition* λ of $n = 2k$ is a weakly decreasing sequence $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_r$ of positive integers summing up to n. Such partitions label the nilpotent orbits in $\mathfrak{gl}(n, \mathbb{C})$ under the conjugation action of $GL(n, \mathbb{C})$, where the parts λ_i of λ correspond to the sizes of the Jordan blocks.

Definition 2.3 A partition λ of $n = 2k$ is called *admissible* if even parts occur with even multiplicity.

The classification of nilpotent orbits for $\mathfrak{gl}(n, \mathbb{C})$ restricts to $\mathfrak{g}$: the G-conjugation classes of nilpotent elements in $\mathfrak{g}$ are in bijection to admissible partitions of n ([18,54]). We now fix a nilpotent element $N \in \mathfrak{g}$.

Definition 2.4 The *Springer fibre* associated with (G, N) is the algebraic variety $\mathcal{F}_N := \mathcal{F}^{1+N}$ of $(1 + N)$-fixed points in $\mathcal{F}$, *i.e.*, of all flags $F_\bullet \in \mathcal{F}$ such that $NF_i \subset F_{i-1}$.

The Springer fibre again decomposes into two connected components isomorphic to $\mathcal{F}'_N := \mathcal{F}'^{1+N}$ and is, in contrast to $\mathcal{F}$, in general not smooth. By a result of Spaltenstein [44] it is equidimensional (see also [45, II.1.12]), *i.e.*, all irreducible components have the same dimension. Let $\mathrm{Irr}(\mathcal{F}_N)$ denote the set of irreducible components.

Let $u \in G$ be the unipotent element corresponding to N via the exponential map. The centraliser $C := C_G(u)$ acts on $\mathcal{F}$, $\mathcal{F}'$ and the set $\mathrm{Irr}(\mathcal{F}_N)$. In general, C is not connected, and we have the component group $A(u) = C_G(u)/C_G(u)^\circ$.

In the sequel, we are interested in nilpotent orbits corresponding to 2-row partitions. Hence N has Jordan type (k, k) or $(n - l, l)$ with l odd.

This situation is special from two perspectives. We will show in Theorem 6.5 that all components are smooth (in analogy to 2-block nilpotents of type A, [16, 51], and in huge contrast to the general case [14]), and secondly the component group is easy to calculate, [46, 2.26], as $A(u) = (\mathbb{Z}_2)^c$, where $c = 0$ for the partition (k, k) with k even, $c = 1$ if k is odd, and $c = 2$ otherwise.

3 Diagram Combinatorics

We start by introducing a diagram combinatorics which will later be used to describe the geometry of the topological Springer fibres. It generalizes the well-known approaches from [23], [39], [51] to the more involved type D setting and is motivated by its connections to the type D highest weight Lie theory [11] and [28].

Definition 3.1 We fix a rectangle R in the lower half plane $\mathbb{R} \times \mathbb{R}_{\leq 0}$ with the set of points $P = \{1, \ldots, k\}$ on the top face.

An *undecorated cup diagram* is a diagram consisting of non-intersecting lines, each of them either connecting two distinct vertices in P or one vertex in P with a point on the bottom face of R. The lines of the first type are called *cups*, the others *rays*. We consider two such diagrams as the same if they differ only by a planar isotopy of R fixing P pointwise and the bottom boundary line of the rectangle setwise. A *decorated cup diagram*, or short *cup diagram*, is a diagram with possibly "●"s (called *dots*) on the lines such that every dot is accessible from the left side of the rectangle, meaning for each dot there exists a line in the rectangle connecting the dot with the left face of R not intersecting the diagram. We allow at most one dot per line.

Example 3.2 The following first two diagrams are examples of cup diagrams, whereas the third is not, as the rightmost dot cannot be connected with the left face of R.
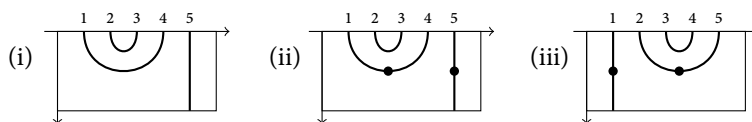
Remark 3.3 Let k be even. Then it is a well-known fact; see *e.g.*, [47], that the number of undecorated cup diagrams on k vertices equals $\frac{1}{k+1}\binom{k}{k/2}$ (the k-th Catalan number), which can be identified with the dimension of the irreducible S_k-module

indexed by the partition $(k/2, k/2)$; see *e.g.*, [49, Lemma 4.3.3]. On the other hand, the number of (decorated) cup diagrams on k vertices equals (see Lemma 5.20) twice the dimension of the irreducible representation of the Weyl group $S_k \ltimes (\mathbb{Z}/2\mathbb{Z})^{k-1}$ of type D_k indexed by the pair of partitions $((k/2), (k/2))$. So, there are $\binom{k}{k/2}$ such diagrams.

*Notation 3.4*   For a cup diagram c and $i, j \in \{1, \ldots, k\}$, we write

(i) i—j if c contains a cup connecting i and j,
(ii) i-•-j if c contains a dotted cup connecting i and j,
(iii) i—| if c contains a ray that ends at i,
(iv) i-•| if c contains a dotted ray that ends at i.

Example 3.5   In the cup diagram (i) from Example 3.2 we have 1—4, 2—3, and 5—|, while in the cup diagram (ii) we have: 1-•-4, 2—3, and 5-•|.

We define the followings sets of (undotted/dotted/all) cups respectively rays appearing in a cup diagram c:

$$\mathrm{cups}_\circ(c) = \{(i < j) \mid i—j\}, \qquad \mathrm{rays}_\circ(c) = \{i \mid i—|\},$$
$$\mathrm{cups}_\bullet(c) = \{(i < j) \mid i\text{-•-}j\}, \qquad \mathrm{rays}_\bullet(c) = \{i \mid i\text{-•|}\},$$
$$\mathrm{cups}(c) = \mathrm{cups}_\circ(c) \cup \mathrm{cups}_\bullet(c), \qquad \mathrm{rays}(c) = \mathrm{rays}_\circ(c) \cup \mathrm{rays}_\bullet(c).$$

Reflecting cup diagrams along the horizontal middle line of R gives us *cap diagrams*; *e.g.*, the cap diagrams arising from Example 3.2 are
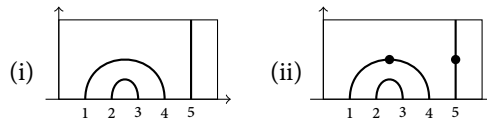
Definition 3.6   As above, we assume to have fixed the rectangle R and the set of points $P = \{1, \ldots, k\}$. A *cap diagram* is a diagram that is obtained from a cup diagram by reflecting it in the horizontal middle axis of R. If c is a cup diagram, we denote by c^* the corresponding cap diagram. We use the Notation 3.4 again for cap diagrams instead of cup diagrams.

To talk about oriented cup/cap diagrams we need the notion of a weight.

Definition 3.7   A *(combinatorial) weight* of length k is a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_i \in \{\wedge, \vee\}$.

A cup (resp. cap) diagram c together with a weight λ is called an *oriented cup (resp. cap) diagram* if the following holds:

(i)   if i—j, then $\lambda_i \neq \lambda_j$,
(ii)   if i-•-j, then $\lambda_i = \lambda_j$,
(iii)   if i—|, then $\lambda_i = \vee$,
(iv)   if i-•|, then $\lambda_i = \wedge$.

We denote the resulting diagram λc (resp. $c\lambda$).

For instance, out of the following three only the first two diagrams are oriented. Note also that λc is oriented if and only if $c^* \lambda$ is oriented.

(3.1)
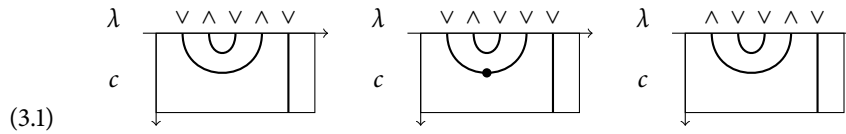
Gluing a cap diagram on top of a cup diagram yields a *circle diagram*.

Definition 3.8 A *circle diagram* (bc) consists of a cap diagram b and a cup diagram c where we identify the lower border of the rectangle of b with the upper border of the rectangle of c (identifying the marked points from left to right). An *oriented circle diagram* $b\lambda c$ consists of a cap diagram b, a cup diagram c and a weight λ such that $b\lambda$ and λc are oriented diagrams:
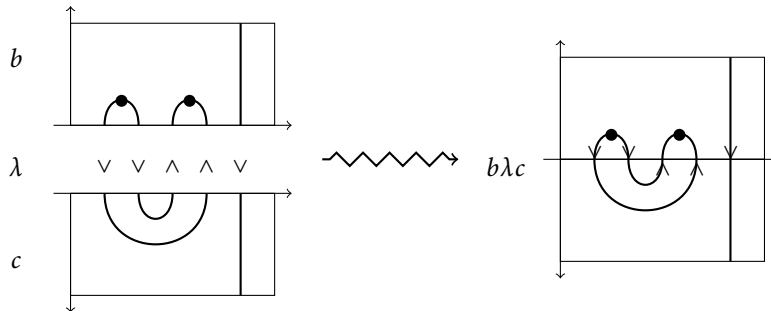

A circle diagram thus consists of circles (=closed connected components) and lines. Note that each line has at most one allowed orientation, whereas each circle has precisely two [10, Lemma 4].

Definition 3.9 A circle in an oriented circle diagram is called *anticlockwise* if its rightmost label is $\wedge$ and *clockwise* if its rightmost label is $\vee$.

In the following we will usually omit drawing the rectangle R.

Fix $k \in \mathbb{N}$. Then denote by $\mathbb{B}_k$ the set of all cup diagrams on k points with the maximal possible number of cups. We want to further divide this set into those cup diagrams with an even (resp. odd) number of dots, $\mathbb{B}_k = \mathbb{B}_k^{\text{even}} \coprod \mathbb{B}_k^{\text{odd}}$.

Example 3.10 For $k = 3$ the set $\mathbb{B}_3$ contains the following six diagrams
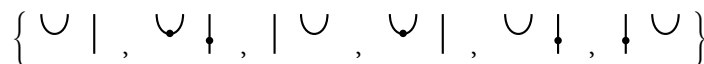
with $\mathbb{B}_3^{\text{even}}$ and $\mathbb{B}_3^{\text{odd}}$ the first three, respectively, last three diagrams.

4 Topological Springer Fibre

In this section we construct the topological Springer fibre and establish basic properties and a cell decomposition.

4.1 The Definition of the Topological Springer Fibre

In the following denote by S the 2-sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ and by $X = S^k$ its k-th power. By $H(Y)$ we denote the cohomology of a topological space Y with complex coefficients, in particular, $H(S) \cong \mathbb{C}[x]/\langle x^2 \rangle$ with $\deg(x) = 2$. The isomorphism sends x to the top class and 1 to the class of a point that we fix and call p. We denote by $x \mapsto -x$ the antipodal map (*i.e.*, the involution on S that maps every point to its antipodal point). Corresponding to $a \in \mathbb{B}_k$, we define a subspace $S_a \subset X$ as follows:

$$(4.1) \qquad (z_1, \ldots, z_k) \in S_a :\Leftrightarrow \begin{cases} z_i = z_j & \text{if } i\!-\!j, \\ z_i = -z_j & \text{if } i\!-\!\bullet\, j, \\ z_i = \text{p} & \text{if } i\!\dashv\,, \\ z_i = -\text{p} & \text{if } i\!-\!\bullet\!\dashv\,. \end{cases}$$

Obviously, S_a is homeomorphic to $S^{\lfloor \frac{k}{2} \rfloor} = S^{\text{cups}(a)}$. Hence, we have

$$H(S_a) \cong \left(\mathbb{C}[x]/\langle x^2 \rangle \right)^{\otimes \lfloor \frac{k}{2} \rfloor}.$$

We will later use an explicit homeomorphism. For $a \in \mathbb{B}_k$, $1 \leq i \leq k$ set

$$\sigma(i) = (-1)^{\,\#\left\{(k < l) \in \text{cups}_\bullet(a) \mid i \leq k\right\} + \#\left\{k \in \text{rays}_\bullet(a) \mid i \leq k\right\}};$$

i.e., σ encodes the parity of the number of dotted cups and rays to the right of the position i, including cups with left endpoint at i.
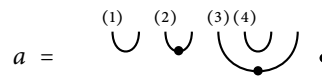
Lemma 4.1 *Let $\{(i_r < j_r)\} = \text{cups}(a)$ with $i_1 < i_2 < \cdots < i_{\lfloor \frac{k}{2} \rfloor}$. The map*

$$(4.2) \qquad \qquad \Psi_a \colon S_a \longrightarrow S^{\text{cups}(a)} = S^{\lfloor \frac{k}{2} \rfloor}$$

$$(z_1, \ldots, z_k) \longmapsto (y_1, \ldots, y_{\lfloor \frac{k}{2} \rfloor}),$$

where $y_r = \sigma(r)z_{i_r}$, $1 \leq r \leq \lfloor \frac{k}{2} \rfloor$ defines a homeomorphism.

Proof The inverse map $(\Psi_a)^{-1}$ is given by $z_t = \sigma(i_r)y_r$ if $t = i_r$, $z_t = \sigma(i_r)y_r$ if $(i_r < j_r = t) \in \text{cups}_\circ(a)$, $z_t = -\sigma(i_r)y_r$ if $(i_r < j_r = t) \in \text{cups}_\bullet(a)$ and $z_t = -p$ or $z_t = p$ otherwise, depending on if the ray is dotted or not. ∎

Example 4.2 Consider the following cup diagram

$$a \;=\;$$

and number the cups according to their left endpoints from left to right (as indicated by the bracketed numbers). Then

$$(\Psi_a)^{-1}(x_1, x_2, x_3, x_4) = (-x_1, -x_1, -x_2, x_2, x_3, -x_4, -x_4, -x_3, -\text{p}).$$

The following lemma follows directly from the definitions.

Lemma 4.3 *The subset $S_a \cap S_b \neq \varnothing$ if and only if the corresponding circle diagram $(a^* b)$ can be oriented. Furthermore, if $(a^* b)$ can be oriented, then $S_a \cap S_b \cong S^r$, where r is the number of closed circles in $(a^* b)$. For the cohomology this implies*

$$H(S_a \cap S_b) \cong \begin{cases} \left(\mathbb{C}[x]/\langle x^2 \rangle \right)^{\otimes r} & \text{if } S_a \cap S_b \neq \varnothing \text{ and } r \text{ as above,} \\ 0 & \text{otherwise.} \end{cases}$$

Each orientation of $(a^* b)$ then corresponds exactly to one standard basis vector $b_1 \otimes \cdots \otimes b_l$, $b_l \in \{1, x\}$ in cohomology, by setting $b_i = 1$ if the corresponding circle is oriented anticlockwise and $b_i = x$ if it is oriented clockwise.

Remark 4.4 Note the difference to [23, Section 3], where the subsets $S_a \cap S_b$ are never empty.

Let $\widetilde{S} = \bigcup_{a \in \mathbb{B}_k} S_a \subset X$. If $a \in \mathbb{B}_k^{\text{even}}$ and $b \in \mathbb{B}_k^{\text{odd}}$, then $S_a \cap S_b = \varnothing$ by Lemma 4.3; thus, we can decompose $\widetilde{S}$ into the disjoint subsets

$$(4.3) \qquad \widetilde{S}^{\text{even}} = \bigcup_{a \in \mathbb{B}_k^{\text{even}}} S_a \text{ and } \widetilde{S}^{\text{odd}} = \bigcup_{a \in \mathbb{B}_k^{\text{odd}}} S_a.$$

We call $\widetilde{S}$ the *topological Springer fibre*, as it is the topological substitute for the Springer fibre $\mathcal{F}_N$ (see Remark 1.3) and (4.3) corresponds to the decomposition of $\mathcal{F}_N$ into the two copies of $\mathcal{F}'_N$. Our goal is to show that the cohomology $H(\widetilde{S})$ of $\widetilde{S}$ is isomorphic to the cohomology of the Springer fibre. We do this by identifying $H(\widetilde{S})$ with the centre of a diagrammatically defined algebra introduced in [10, 28] of which we know that it has the desired centre.

4.2 The Cohomology Rings $H(S_a)$

We first realize all the involved cohomology rings as quotients of

$$H(X) = \mathbb{C}[x]/\langle x^2 \rangle^{\otimes k} = \mathbb{C}[x_1, \dots, x_k]/\langle x_i^2 \mid 1 \le i \le k \rangle,$$

where x_i denotes the top class of the i-th copy of S.

Notation 4.5 For $a, b \in \mathbb{B}_k$ and $1 \le i, j \le k$, we write $i \sim_{a;b} j$ if there exists a sequence $i = i_1, i_2, \dots, i_r = j$ such that for all l the points i_l and i_{l+1} are connected by a cup in $\text{cups}(a) \cup \text{cups}(b)$ (i.e., $(i_l, i_{l+1}) \in \text{cups}(a) \cup \text{cups}(b)$ or $(i_{l+1}, i_l) \in \text{cups}(a) \cup \text{cups}(b)$). We call such a sequence a *path* connecting i and j. This is obviously an equivalence relation on the set $\{1, \dots, k\}$. For a chosen path from i to j, we denote by $\alpha(i, j)$ the number of undotted cups used in the path. The parity of $\alpha(i, j)$ is independent of the chosen path and so $(-1)^{\alpha(i,j)}$ is well defined. For $i, j \in \{1, \dots, k\}$, we define

$$\epsilon(i, j) = \begin{cases} 0 & \text{if } i \nsim_{a;b} j, \\ (-1)^{\alpha(i,j)} & \text{if } i \sim_{a;b} j. \end{cases}$$

We denote the set of equivalence classes by $\mathrm{Conn}(a, b)$ (they are in canonical bijection with the connected components of the corresponding circle diagram) and by $\bar{i}$ the equivalence class corresponding to the point i. We denote by $\mathrm{Conn}_\circ(a, b)$ those classes that correspond to closed circles in the diagram. We will later use the map

$$\mathrm{mx}_{a;b} \colon \{1, \ldots, k\} \longrightarrow \{1, \ldots, k\}$$
$$i \longmapsto \max\{j \mid j \sim_{a;b} i\},$$

which maps each element to the maximal element in its equivalence class.

Lemma 4.6   (i)   *Let $a \in \mathbb{B}_k$. The canonical map induces a surjection*

$$\pi_a \colon \quad H(X) \twoheadrightarrow H(S_a)$$

with kernel

$$J_a = \begin{cases} x_i + x_j & \text{if } (i, j) \in \mathrm{cups}_\circ(a), \\ x_i - x_j & \text{if } (i, j) \in \mathrm{cups}_\bullet(a), \\ x_i & \text{if } i \in \mathrm{rays}(a). \end{cases}$$

(ii)   *Let $a, b \in \mathbb{B}_k$; then the canonical map induces a surjection*

$$\pi_{a;b} \colon \quad H(X) \twoheadrightarrow H(S_a \cap S_b),$$

which in the case where $S_a \cap S_b \neq \varnothing$ (i.e., (a^*b) can be oriented) has kernel

$$J_{a;b} = \begin{cases} x_i - \epsilon(i, \mathrm{mx}_{a;b}(i)) x_{\mathrm{mx}_{a;b}(i)} & \text{if } \bar{i} \in \mathrm{Conn}_\circ(a, b), \\ x_i & \text{if } \bar{i} \notin \mathrm{Conn}_\circ(a, b). \end{cases}$$

Proof   In part (i), the map is induced from the embedding of S_a into X. In the case of an undotted cup, this is the induced map of embedding a sphere diagonally into a product of two spheres, hence gives the first type of relations in the ideal. If there is a dotted cup, then the embedding is twisted by the antipodal map on one side, which produces a sign change and we obtain the second line of generators for the ideal. The last line is the induced map of embedding a point into a sphere. The generators of $H(S_a)$ are obviously in the image of the map. By comparing the dimensions the statement follows.

For part (ii) it is again clear that the kernel $\ker(\pi_{a;b})$ contains both J_a and J_b, and thus we have a well-defined map

$$H(X)/(J_a + J_b) \longrightarrow H(S_a \cap S_b),$$

which contains the generators of $H(S_a \cap S_b)$ in the image. By part (i), $J_a + J_b$ has the asserted generators. Comparing the dimensions using Lemma 4.3 implies the claim.   ∎

For $a, b \in \mathbb{B}_k$, the canonical maps

$$\psi_{a;a,b} \colon H(S_a) \longrightarrow H(S_a \cap S_b) \quad \text{and} \quad \psi_{b;a,b} \colon H(S_b) \longrightarrow H(S_a \cap S_b),$$

induced by the inclusions of the corresponding spaces, satisfy $\pi_{a;b} = \psi_{a;a,b}\pi_a$ and $\pi_{b;a} = \psi_{b;a,b}\pi_b$, and they are compatible with the $(H(S_a))$-module structure (resp. $(H(S_b))$-module structure) given by the cup product.

4.3 Connection with the Type D Khovanov Arc Algebra

We shortly recall the main features of the type D Khovanov arc algebra.

Definition 4.7 For k still fixed, let $\mathbb{K}_k$ be the vector space with basis the set

$$\mathcal{B}_k := \left\{ b^* \lambda c \mid b, c \in \mathbb{B}_k \text{ such that } b\lambda \text{ and } \lambda c \text{ are oriented} \right\}$$

of oriented circle diagrams for k points, where the involved cup and cap diagrams have the maximal possible number of cups. The *degree*, $\deg(b\lambda c)$, of a basis vector $b\lambda c$ is defined to be the number of *clockwise* oriented cups and caps in the sense of [28] (*i.e.*, oriented cups and caps labelled $\wedge\vee$ or $\vee\vee$ in this order from left to right; *e.g.*, the first two diagrams in (3.1) have 1 respectively 2 clockwise cups).

We have $\mathbb{K}_k = \bigoplus_{(a,b) \in \mathbb{B}_k \times \mathbb{B}_k} {}_a(\mathbb{K}_k)_b$, where ${}_a(\mathbb{K}_k)_b$ is spanned by all basis vectors $a\lambda b \in \mathcal{B}_k$ with a, b fixed. By [10], $\mathbb{K}_k$ can be equipped with a multiplication turning it into a non-negatively graded associative algebra, $\mathbb{K}_k$, where the grading is given by the degree function deg. The degree zero part is semisimple and spanned by the idempotents ${}_b\mathbb{1}_b := b\lambda b$ for $b \in \mathbb{B}_k$, where λ is the unique weight such that $b\lambda b$ is oriented of degree zero.

Definition 4.8 More generally, if $a, b \in \mathbb{B}_k$ such that $a\lambda b \in \mathcal{B}_k$ for some λ, we denote by ${}_a\mathbb{1}_b$ the unique basis vector of ${}_a(\mathbb{K}_k)_b$ of minimal degree, $d(a, b)$ (for an explicit formula see Lemma 4.21).

By [10] and Lemma 4.6, we have for $a, b \in \mathbb{B}_k$ canonical isomorphisms of graded vector spaces

$$(4.4) \qquad {}_a(\mathbb{K}_k)_a \cong H(S_a), \quad {}_a(\mathbb{K}_k)_b \cong H(S_a \cap S_b)\langle d(a,b)\rangle$$

sending a basis vector $a\lambda b$ to $b_1 \otimes \cdots \otimes b_l$, $b_l \in \{1, x\}$, where $b_i = 1$ if the circle at vertex i is anticlockwise and $b_i = x$ if it is clockwise. Here $\langle d \rangle$ denotes the grading shift upwards by d. By the multiplication rules of the algebra $\mathbb{K}_k$, a product $(a\lambda b)(c\mu d)$ of two basis vectors in $\mathbb{K}_k$ can only be nonzero if $b^* = c$, moreover, there are induced linear maps

$$\gamma_{a;a,b} \colon {}_a(\mathbb{K}_k)_a \longrightarrow {}_a(\mathbb{K}_k)_b \quad \text{and} \quad \gamma_{b;a,b} \colon {}_b(\mathbb{K}_k)_b \longrightarrow {}_a(\mathbb{K}_k)_b,$$

defined as $d \mapsto d \, {}_a\mathbb{1}_b$ and $d \mapsto {}_a\mathbb{1}_b \, d$, and turns them into left ${}_a(\mathbb{K}_k)_a$-module (resp. right ${}_b(\mathbb{K}_k)_b$-module) homomorphisms. With the vertical isomorphisms (4.4) we obtain the following proposition.

Proposition 4.9 *The following diagram commutes:*

$$(4.5) \qquad
\begin{array}{ccccc}
H(S_a) & \xrightarrow{\;\psi_{a;a,b}\;} & H(S_a \cap S_b) & \xleftarrow{\;\psi_{b;a,b}\;} & H(S_b) \\
\Big\downarrow{\scriptstyle\cong} & & \Big\downarrow{\scriptstyle\cong} & & \Big\downarrow{\scriptstyle\cong} \\
{}_a(\mathbb{K}_k)_a & \xrightarrow{\;\gamma_{a;a,b}\;} & {}_a(\mathbb{K}_k)_b & \xleftarrow{\;\gamma_{b;a,b}\;} & {}_b(\mathbb{K}_k)_b.
\end{array}$$

Proof   Using the isomorphisms from Lemma 4.6, it is enough to show that

$$
\begin{array}{ccc}
H(X)/J_a & \longrightarrow & H(X)/J_{a;b} \\
\Big\downarrow{\cong} & & \Big\downarrow{\cong} \\
{}_a(\mathbb{K}_k)_a & \longrightarrow & {}_a(\mathbb{K}_k)_b
\end{array}
$$

commutes for any $a \neq b$. By definition of the maps, $1 \in H(X)/J_a$ is sent to ${}_a\mathbb{1}_b \in {}_a(\mathbb{K}_k)_b$ in both ways. Using first the vertical map, x_i is sent to the basis vector which differs from ${}_a\mathbb{1}_b$ by making the circle (if it exists) pass through vertex i clockwise and multiplying by $(-1)^{\alpha(i,\mathrm{mx}_{a;b}(i))}$, and it is sent to zero otherwise; see [10]. On the other hand, the horizontal map sends x_i to x_i, which, by Lemma 4.6, equals $(-1)^{\alpha(i,\mathrm{mx}_{a;b}(i))}x_{\mathrm{mx}_{a;b}(i)}$, which is then sent to the same basis vector as above, since i and $\mathrm{mx}_{a;b}(i)$ lie on the same component. Since the x_i generate $H(X)/J_a$ as ring, the claim follows.   ∎

The modules $P(a) := \bigoplus_{b \in \mathbb{B}_k} {}_b(\mathbb{K}_k)_a$ for $a \in \mathbb{B}_k$ form the indecomposable left $\mathbb{K}_k$-modules with endomorphism rings ${}_a(\mathbb{K}_k)_a$. The centre $Z(\mathbb{K}_k)$ is therefore the subalgebra of $D := \bigoplus_{a \in \mathbb{B}_k} {}_a(\mathbb{K}_k)_a = \prod_{a \in \mathbb{B}_k} {}_a(\mathbb{K}_k)_a$ given by all elements $(z_a)_{a \in \mathbb{B}_k}$ such that $z_a f = f z_b$ for all $f \in \mathrm{Hom}_{\mathbb{K}_k}(P(a), P(b)) = {}_a(\mathbb{K}_k)_b$, $b \in \mathbb{B}_k$. Since ${}_a(\mathbb{K}_k)_b$ is a cyclic ${}_a(\mathbb{K}_k)_a$-module generated by ${}_a\mathbb{1}_b$, and ${}_a(\mathbb{K}_k)_a$ is commutative [10], it follows that $Z(\mathbb{K}_k) = \mathrm{Eq}(\gamma)$, where

$$
(4.6) \qquad \mathrm{Eq}(\gamma) = \left\{ (z_a)_{a \in \mathbb{B}_k} \in D \ \left| \ \begin{array}{c} \gamma_{a;a,b}({}_a\mathbb{1}_a z_a \mathbb{1}_a) = \gamma_{b;a,b}({}_b\mathbb{1}_b z_b \mathbb{1}_b), \\ \text{for } a, b \in \mathbb{B}_k, a \neq b \end{array} \right. \right\},
$$

i.e., the equalizer of $\gamma = \sum_{a \neq b} \gamma_{a;a,b} + \gamma_{b;a,b} : \prod_a {}_a(\mathbb{K}_k)_a \to \prod_{a \neq b} {}_a(\mathbb{K}_k)_b$. From (4.5) we obtain therefore an algebra isomorphism $Z(\mathbb{K}_k) \cong \mathrm{Eq}(\psi)$, where $\mathrm{Eq}(\psi)$ is defined analogously to (4.6). Since the inclusion of $S_a \cap S_b$ into $\widetilde{S}$ is the same regardless of whether it is viewed as a subset of S_a or of S_b, the canonical map $\tau' : H(\widetilde{S}) \to \prod_a H(S_a)$ factors through $\mathrm{Eq}(\psi)$ and yields a homomorphism of algebras

$$
\tau : H(\widetilde{S}) \longrightarrow \mathrm{Eq}(\psi).
$$

Our goal in Section 4.5 will be to prove the following theorem.

Theorem 4.10   *The map τ is an isomorphism.*

To show Theorem 4.10, we follow [23] closely. However, we have to reprove most of the involved steps, since there are a few difficulties when transferring the arguments to type D. We start by introducing a partial order on $\mathbb{B}_k$ and a cell decomposition of $\widetilde{S}$ and show that it behaves nicely with respect to the partial order.

4.4   Partial Order on Diagrams

Define a partial order on $\mathbb{B}_k$ as chains of arrows as follows. For $a, b \in \mathbb{B}_k$ we write $a \to b$ if a and b only differ locally by one of the moves in Figure 1 (The diagrams might have many more cups even separating the involved cups; we only depict the ones that change in the move from a, depicted in the first column, to b, depicted in
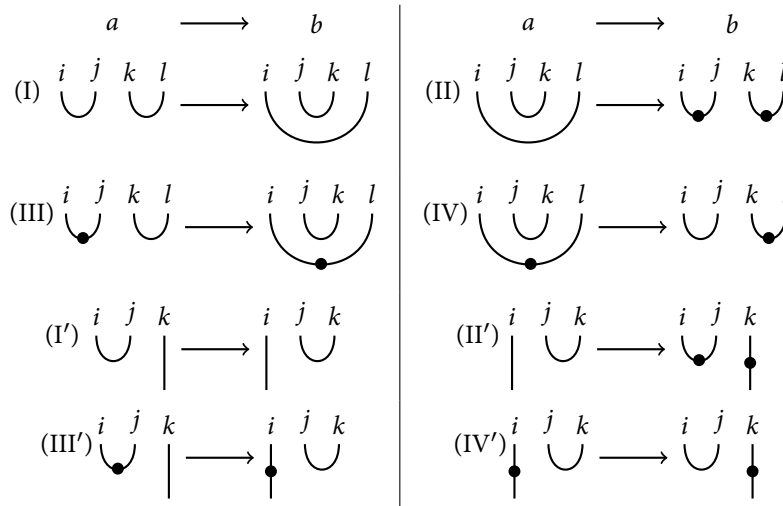
Figure 1

the second column. The move always exists whenever both sides of the diagram give an allowed cup diagram. For instance, there cannot be a dotted cup between the two cups of a in case (I), since then b would have a nested dotted cup.

We extend this to a partial order on $\mathbb{B}_k$ by setting $a \prec b$ if there exists a finite chain of arrows $a \to c_1 \to \cdots \to c_r \to b$. Since the local moves listed above always preserve the parity of the number of dots, an element from $\mathbb{B}_k^{\text{even}}$ is never $\prec$-comparable with an element from $\mathbb{B}_k^{\text{odd}}$. On the other hand, it is easy to see that the arrows induce connected graphs on $\mathbb{B}_k^{\text{even}}$ and $\mathbb{B}_k^{\text{odd}}$. From now on, we will work with the two sets separately (each of them could be seen as the analog of [23]), and fix a refinement of these two partial orders to a total order, denoted by $<$.

Remark 4.11 The graphs on $\mathbb{B}_k^{\text{even}}$ (resp. $\mathbb{B}_k^{\text{odd}}$) appear after doubling the arrows as the Ext-quiver of the category of perverse sheaves on isotropic Grassmannians, constructible with respect to the Schubert stratification and describe the microlocal geometry there; see [4,10].

Definition 4.12 Given a cup diagram a cup is called *outer* if it is not nested in any other cup and does not contain any dotted cup to the right. We call it *inner* if it is not outer. The *degree of nesting* of a cup is defined inductively: it is zero if it is outer and d if it is outer after removing all cups with degree of nesting strictly smaller than d. (Note that the local moves $a \to b$ never decrease the degree of nesting.)

Example 4.13 Every cup diagram has at least one outer cup, *i.e.*, a cup with degree of nesting equal to zero. For instance, the outer cups for (1.1) are the cup $(3,4)$ in (i), the cup $(1,4)$ in (ii), and both cups $(1,2)$ and $(3,4)$ in (iii). In addition we have here
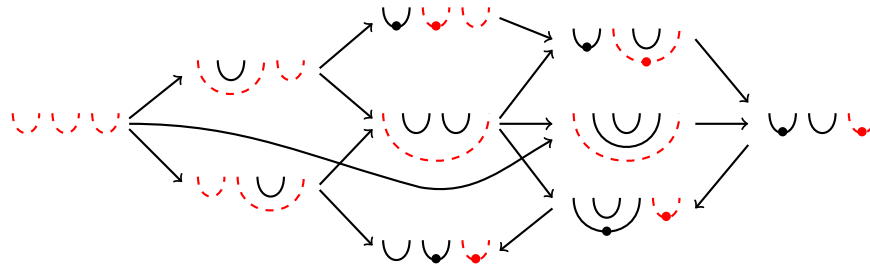
Figure 2

cups with degree of nesting equal to 1, namely the cup $(1, 2)$ in (i) and the cup $(2, 3)$ in (ii).

Example 4.14   The partial ordering on $\mathbb{B}_3^{\text{even}}$, with outer cups in (dashed) red, is displayed in Figure 2. For $\mathbb{B}_3^{\text{odd}}$ just add or remove (whatever is possible) a dot on the cup passing through the leftmost point.

4.5 The Cell Decomposition

We will construct a cell decomposition of the topological Springer fibre by defining a paving of each S_a, $a \in \mathbb{B}_k$, by even dimensional real spaces that we then show to behave well under intersections. To define a cell decomposition of S_a for $a \in \mathbb{B}_k$, we first attach to each cup diagram $a \in \mathbb{B}_k$ a directed graph Γ_a as follows. The set of vertices

$$\mathcal{V}(\Gamma_a) = \text{cups}(a)$$

is the set of cups in a; while the set of edges $\mathcal{E}(\Gamma_a)$ is defined as follows. For two vertices $(i_1, j_1), (i_2, j_2) \in \mathcal{V}(\Gamma_a)$, we put an arrow $(i_1, j_1) \to (i_2, j_2)$ if there exists a cup diagram $b \in \mathbb{B}_k$ with $b \to a$ such that a is obtained from b by a local move of type (I)–(IV) at the positions i_1, i_2, j_1, j_2 and furthermore we demand that the degree of nesting of (i_2, j_2) is greater than the one of (i_1, j_1).

Example 4.15   Figure 3 gives an example of a cup diagram a with associated graph Γ_a.

By construction Γ_a is a forest, *i.e.*, a union of trees. The roots $\mathcal{R}(\Gamma_a)$ of Γ_a are precisely the outer cups. Any cup is either a root or it is the target of a unique arrow, hence $\#\mathcal{R}(\Gamma_a) + \#\mathcal{E}(\Gamma_a) = \lfloor k/2 \rfloor$, and we have a natural bijection between $\mathcal{R} \cup \mathcal{E}$ and cups(a).

Now assign to each subset $J \subset \mathcal{R}(\Gamma_a) \cup \mathcal{E}(\Gamma_a)$ the subspace C'_J of $S^{\text{cups}(a)}$ given by all elements $(x_c)_{c \in \text{cups}(a)}$ satisfying the following:
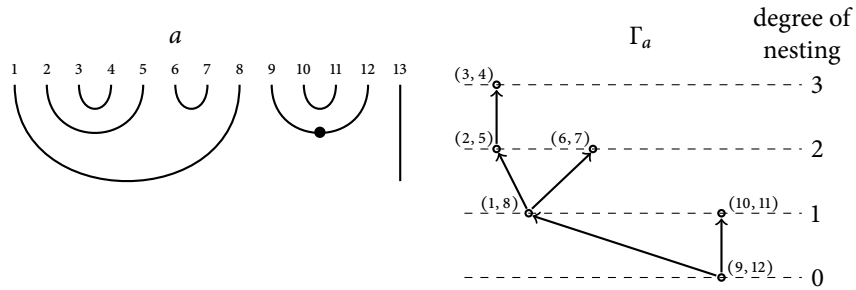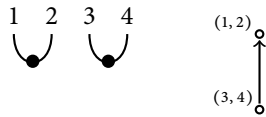
Figure 3

(C1) If $c \in \mathcal{R}(\Gamma_a) \cap J$, then $x_c = \mathrm{p}$,
(C2) if $c \in \mathcal{R}(\Gamma_a)$ but $c \notin J$, then $x_c \neq \mathrm{p}$;
(C3) if $(c \to d) \in \mathcal{E}(\Gamma_a) \cap J$, then $x_c = x_d$;
(C4) if $(c \to d) \notin \mathcal{E}(\Gamma_a) \cap J$, then $x_c \neq x_d$.

Example 4.16   Consider the following cup diagram a with its graph Γ_a:

We have one root α and one edge e. The possible subsets J are thus $\varnothing$, $\{\alpha\}$, $\{e\}$, and $\{\alpha, e\}$. The corresponding cells in $S \times S \cong S_a$ are then

J	$C_J' \subset S \times S$	$C_j = (\Psi_a)^{-1}(C_J')$
$\varnothing$	$\{(x, y) \mid x \neq y, y \neq \mathrm{p}\}$	$\{(x, -x, -y, y) \mid y \neq \mathrm{p}, x \neq y\},$
$\{\alpha\}$	$\{(x, \mathrm{p}) \mid y \neq \mathrm{p}\}$	$\{(x, -x, -\mathrm{p}, \mathrm{p}) \mid x \neq \mathrm{p}\},$
$\{e\}$	$\{(x, x) \mid x \neq p\}$	$\{(x, -x, -x, x) \mid x \neq \mathrm{p}\},$
$\{\alpha, e\}$	$\{(\mathrm{p}, \mathrm{p})\}$	$\{(\mathrm{p}, -\mathrm{p}, -\mathrm{p}, \mathrm{p})\}.$

Lemma 4.17   *With the notation above there is a decomposition*

$$S^{\mathrm{cups}(a)} = \bigsqcup_{J \subset \mathcal{R}(\Gamma_a) \cup \mathcal{E}(\Gamma_a)} C_J'$$

into disjoint affine cells C_J' homeomorphic to $\mathbb{R}^{2(n-|J|)}$.

Proof   The above decomposition is by construction disjoint. To see that it is exhaustive, we construct to each point $P \in S^{\mathrm{cups}(a)}$ a set J such that $P \in C_J'$. First note that conditions (C1) and (C2) precisely define $J \cap \mathcal{R}(\Gamma_a)$. Now we move upwards along the edges that uniquely determines $J \cap \mathcal{E}(\Gamma)$ via conditions (C3) and (C4) by comparing the values at the two ends of each edge. The construction gives spaces homeomorphic

to $\mathbb{R}^{2(n-|J|)}$ and it verifies that

$$\overline{C'_J} = \bigcup_{J \subset I} C'_I.$$

The lemma follows.   ∎

Pushing forward along (4.2) gives a cell decomposition

$$S_a = \bigsqcup_{J \subset \mathcal{R}(\Gamma_a) \cup \mathcal{E}(\Gamma_a)} C_J,$$

with $C_J = (\Psi_a)^{-1}(C'_J)$.

## 4.6 Compatibility with Intersections

Definition 4.18   For $a, b \in \mathbb{B}_k$, we define the *distance* $d(a, b)$ between a and b as the minimal length of a non-directed chain of arrows connecting a and b. If no such chain exists, we put $d(a, b) = \infty$.

Remark 4.19   Since all local moves preserve parity, we obviously have $d(a, b) = \infty$ if a and b have different parity. Otherwise $d(a, b)$ is always finite, since the graphs for $\mathbb{B}_k^{\mathrm{even}}$ and $\mathbb{B}_k^{\mathrm{odd}}$ are finite and connected.

Lemma 4.20   *Let $a, b \in \mathbb{B}_k$ with $d(a, b) < \infty$. Then there exists c such that $d(a, b) = d(a, c) + d(c, b)$ and $a > c < b$.*

Proof   Let $(a = a_0, a_1, \ldots, a_r = b)$ be a minimal non-directed chain from a to b. If $a_0 > a_1 > \cdots > a_i < a_{i+1} < \cdots < a_r$ for some i, then we set $c = a_i$, and there is nothing to do. Otherwise, there exists some subsequence $a_{i-1} \to a_i \leftarrow a_{i+1}$, and the idea of the proof is to successively eliminate all such subsequences without changing the length of the chain. From the set of all such subsequences, choose one (not necessarily uniquely defined) such that its middle term a_i is maximal in the order.

Due to the minimality of the chain we have $a_{i-1} \neq a_{i+1}$, and there exists by [4, Lemma 1.8.3] with the identification from [10] a diamond
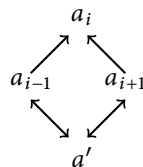

with the property that either $a' > a_{i-1}$ or $a' > a_{i+1}$ (indicated by the double arrows). We now substitute a_i with a' in our chain of arrows. If this procedure creates a subsequence such that $a_{i-2} \to a_{i-1} \leftarrow a'$ or $a' \to a_{i+1} \leftarrow a_{i+2}$, then both a_{i-1} and a_{i+1} are lower in the order and we proceed by induction. In the end this will produce a sequence of the form $a \leftarrow a'_1 \leftarrow \cdots \leftarrow a'_i \to \cdots \to a'_{r-1} \to b$, which is still minimal, and a'_i satisfies the conditions for the element c. (Note that a'_i can be equal to a or b.)   ∎

As in type A, the distance function has a nice diagrammatical interpretation also showing the consistency of our notation $d(a, b)$.

Lemma 4.21 *Let $a, b \in \mathbb{B}_k$ with $d(a, b) < \infty$ and let $m = \lfloor \frac{k}{2} \rfloor$. If $b^* a$ can be oriented, then the two definitions of $d(a, b)$ (Definition 4.8 and 4.18) agree, and $m - d(a, b)$ equals the numbers of circles in $b^* a$.*

Proof We first claim that the minimal degree equals $m - \gamma = \lfloor \frac{k}{2} \rfloor - \gamma$, where γ is the number of circles in $b^* a$. We argue by induction on k. The base cases $k = 1$ or $k = 2$ are easy to check. If $a = b$, then $a^* a$ has m circles each containing exactly one cup and cap. Hence, orienting each of them anti-clockwise gives the minimal degree 0 and the claim follows. If $a \neq b$, then the diagram $a^* b$ contains at least one kink. If we can find an undecorated kink, we remove it by the following straightening rules:
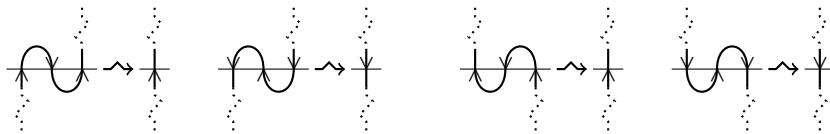

This decreases m and the degree function $d(a, b)$ by one, but keeps the number of circles, and we are done by induction. If there is no such undecorated kink, then there is at least one decorated kink. The kink can be followed on both sides by rays which means it is part of a subdiagram from the following list (recalling that dotted cups can't be to the right of rays):
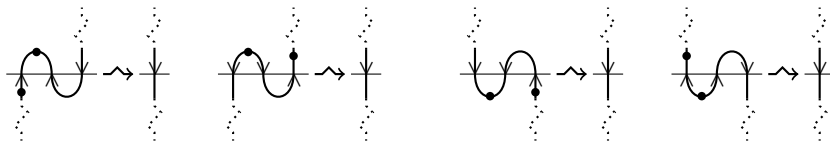
Then we apply the "dot removal trick", which means we remove the two dots and adjust the orientation in the unique possible way. Observe that the degree of the diagram is not changed. The result contains an undecorated kink, and we can argue by induction as above.

In all other cases, the kink is part of a subdiagram from the following list (recalling the assumption that there is no undecorated kink and also the rules for placing the dots):
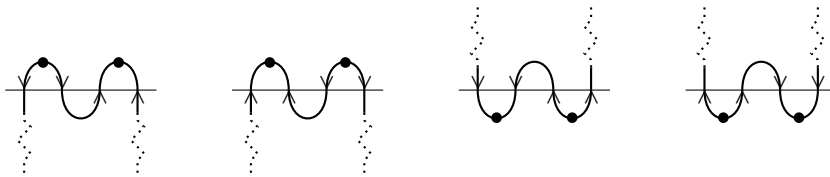
Apply the dot removal trick again, so as to remove the two dots, and adjust the orientation such that the two outer labels, thus the type (clockwise or anticlockwise)

as well as the degree of the diagram, is preserved. Again, the result contains an un-decorated kink, and we can argue by induction as above. This proves the claim.

Starting from a^*a we apply the minimal path to a to obtain b. In each step the number of circles decreases at most by one, hence the distance $d(a,b)$ is at least m minus the number of circles. To show equality consider a^*b. If $a = b$, then there is nothing to do, so assume $a \neq b$. If there is a circle containing two outer cups, then we can apply a move $b \to b'$ or $b \leftarrow b'$, which splits the circle into two. Otherwise, we find two cups on the same circle paired by an edge such that one of the cups is outer. Again we can apply some move to split the original circle. Repeating the procedure we finally give a path $d(a,b)$ of the required length starting at a^*b and changing b successively.. It ends in a^*a, since in each step one new circle was created (resulting in m circles) without changing the cap diagram a^*.   ∎

Lemma 4.22   *Let c be such that $d(a,b) = d(a,c) + d(c,b)$; then*

$$S_a \cap S_b = S_a \cap S_c \cap S_b.$$

Proof   If $S_a \cap S_b = \varnothing$, there is nothing to show, so we assume $S_a \cap S_b \neq \varnothing$. In the case where $a = b$, we have $c = a = b$ and the claim is again trivial. Now we argue by some twofold induction, namely on $d(a,b)$ starting with $d(a,b) = 1$ and on $d(a,c)$ starting with $d(a,c) = 1$.

In the case $d(a,b) = 1$ where, there is again nothing to show, since either $c = a$ or $c = b$, so we assume now $d(a,c) = 1$ (and $a \neq b$ arbitrary). Since $S_a \cap S_b \neq \varnothing$, the circle diagram a^*b can be oriented and contains $\lfloor \frac{k}{2} \rfloor - d(a,b)$ circles by Lemma 4.21. Since c lies on a minimal length path between b and a there is a sequence of moves from a^*b to a^*c and then to a^*a that creates a new circle in each step. In particular, every orientation of a^*b gives rise to an orientation of a^*c and then obviously also of a^*a, and therefore $S_a \cap S_b \subset (S_a \cap S_c) \cap S_b$.

It remains to show the statement for $d(a,b) = r$, $d(a,c) = s > 1$ with $c \neq b$. We assume that it holds for all a', b', c' such that $d(a',b') = d(a',c') + d(c',b') < r$ and all x such that $d(a,b) = d(a,x) + d(x,b)$ and $d(a,x) < s$.

Choose $x \neq a, c$ on a minimal path connecting a and c thus satisfying $d(x,b) = d(x,c) + d(c,b) < r$. Then by assumption,

$$S_x \cap S_b = S_x \cap S_c \cap S_b.$$

Since $d(a,x) < s$ and $d(a,b) = d(a,x) + d(x,b)$, we also have

$$S_a \cap S_b = S_a \cap S_b \cap S_x.$$

Altogether this gives

$$S_a \cap S_b = S_a \cap (S_b \cap S_x) = S_a \cap S_b \cap S_x \cap S_c = S_a \cap S_b \cap S_c,$$

and the lemma follows.   ∎

Let $S_{<a} = \bigcup_{b<a} S_b$ and $S_{\leq a} = \bigcup_{b \leq a} S_b$.

Lemma 4.23   *For any $a \in \mathbb{B}_k$, we have*

(4.7)
$$S_{<a} \cap S_a = \bigcup_{b \to a} (S_b \cap S_a).$$

Proof The right-hand side, say R, of (4.7) is by definition contained in the left-hand side L. For the other inclusion, $L \subset R$, we have to show that $S_b \cap S_a \in R$ for $b < a$, $b \neq a$. Using Lemma 4.20 (and its proof) we can chose c such that $d(a, c) + d(c, b) = d(a, b)$ and $c \rightarrow a$, and thus $(S_a \cap S_b) \subset (S_a \cap S_b \cap S_c)$ by Lemma 4.22. Hence $(S_a \cap S_b) \subset (S_a \cap S_c)$, and the claim follows. ∎

The cell decompositions behave well under pairwise intersections.

Proposition 4.24 *Let $a, b \in \mathbb{B}_k$ and $b \rightarrow a$. Then the following hold:*

(i) *If $b \rightarrow a$ is of type* (I)–(IV), *then, by definition of Γ_a, the local move determines a unique edge $e \in \mathcal{E}(\Gamma_a)$, and we have*

$$S_a \cap S_b = \bigcup_{J \subset \mathcal{R}(\Gamma_a) \cup \mathcal{E}(\Gamma_a), e \in J} C_J.$$

(ii) *If $b \rightarrow a$ is of type* (I′)–(IV′), *there exists a unique cup $\alpha \in \mathrm{cups}(a)$ such that $\alpha \notin \mathrm{cups}(b)$. Furthermore, $\alpha \in \mathcal{R}(\Gamma_a)$ and*

$$S_a \cap S_b = \bigcup_{J \subset \mathcal{R}(\Gamma_a) \cup \mathcal{E}(\Gamma_a), \alpha \in J} C_J.$$

Proof This follows directly from the moves in Figure 1: in (i), the cups involved in the move determine a new edge in Γ_a; in case (ii), the move creates a new cup that is outer, hence contained in $\mathcal{R}(\Gamma_a)$. Now any point in $S_a \cap S_b$ is contained in one of the cells on the right by construction and (C1)–(C4). Conversely, points from other cells are not contained in the intersection. ∎

We call the unique cup in $\mathcal{R}(\Gamma_a)$ obtained from some fixed $b \rightarrow a$ via the construction from Proposition 4.24(ii) *special*. Let $\mathcal{R}(\Gamma_a)^{\mathrm{sp}}$ be the set of special cups in a.

Remark 4.25 For $a \in \mathbb{B}_k$ with k odd, the moves I′)-IV′) in Figure 1 imply that all outer cups are special if the ray involved in a is dotted, whereas only outer cups to the right of the ray are special if the ray is undotted.

Corollary 4.26 *Let $a \in \mathbb{B}_k$.*

(i) *If k is even, then*

$$S_{<a} \cap S_a = \bigcup_{\substack{J \subset \mathcal{R}(\Gamma_a) \cup \mathcal{E}(\Gamma_a) \\ J \cap \mathcal{E}(\Gamma_a) \neq \varnothing}} C_J.$$

(ii) *If k is odd, then*

$$S_{<a} \cap S_a = \bigcup_{\substack{J \subset \mathcal{R}(\Gamma_a) \cup \mathcal{E}(\Gamma_a) \\ J \cap (\mathcal{E}(\Gamma_a) \cup \mathcal{R}(\Gamma_a)^{\mathrm{sp}}) \neq \varnothing}} C_J.$$

Proof This follows directly from the definition of $S_{<a}$ from Proposition 4.24 using the definition of special cups from the moves in Figure 1. ∎

4.7 Proof of Theorem 4.10 and Combinatorial Dimension Formula

The following cohomology vanishing is completely analogous to [23].

Proposition 4.27 *Let $a, b \in \mathbb{B}_k$.*

(i) *The subset $S_{<a} \cap S_a$ has cohomology in even degrees only. The map $H(S_a) \to H(S_{<a} \cap S_a)$, induced by the inclusion, is surjective.*

(ii) *The homomorphism*

$$H(S_{<a} \cap S_a) \longrightarrow \bigoplus_{b<a} H(S_b \cap S_a)$$

induced by the inclusion is injective.

(iii) *The subset $S_{\leq a}$ has cohomology in even degrees only. The Mayer-Vietoris sequence for $(S_{<a}, S_a)$ breaks down into short exact sequences*

$$0 \to H^{2l}(S_{\leq a}) \to \bigoplus_{H^{2l}(S_a)}^{H^{2l}(S_{<a})} \to H^{2l}(S_{<a} \cap S_a) \to 0,$$

for $0 \leq l \leq k$.

Proof For the proof of part (i) we refer to [23, Proposition 4]; for part (ii), see [23, Lemma 5], and for part (iii), see [23, Proposition 3]. In all cases the arguments apply directly to our situation. ∎

Now we are prepared to give the proof of Theorem 4.10.

Proof of Theorem 4.10 From Proposition 4.27 we deduce as in [23, Proposition 3] that the following sequence is exact:

$$0 \longrightarrow H(S_{\leq a}) \overset{\phi}{\longrightarrow} \bigoplus_{b \leq a} H(S_b) \overset{\psi^-}{\longrightarrow} \bigoplus_{b<c\leq a} H(S_b \cap S_c),$$

where ϕ is induced by the inclusion $S_b \subset S_{\leq a}$, and $\phi^- := \sum_{b<c\leq a}(\phi_{b,c} - \phi_{c,b})$, where $\phi_{b,c} \colon H(S_b) \longrightarrow H(S_b \cap S_c)$, again induced by the corresponding inclusion. When a is chosen maximal in $\mathbb{B}_k^{\mathrm{even}}$, we obtain that

$$0 \longrightarrow H(\widetilde{S}^{\mathrm{even}}) \longrightarrow \bigoplus_{b \in \mathbb{B}_k^{\mathrm{even}}} H(S_b) \longrightarrow \bigoplus_{b,c \in \mathbb{B}_k^{\mathrm{even}}, b<c} H(S_b \cap S_c)$$

is exact, and analogously for $\widetilde{S}^{\mathrm{odd}}$. This is equivalent to Theorem 4.10. ∎

The dimension of $H(\widetilde{S})$ can be computed now purely combinatorially.

Proposition 4.28 *The following hold*

$$\dim H(\widetilde{S}) = 2\dim H(\widetilde{S}^{\mathrm{even}}) = 2\dim H(\widetilde{S}^{\mathrm{odd}}) = 2^k.$$

Proof We have the cell decomposition on $S_a \cap S_{<a}$ by Corollary 4.24. To determine the dimension of the cohomology, it is enough to count for each $a \in \mathbb{B}_k$ the cells contained in S_a but not in any intersection $S_a \cap S_b$ with $b < a$ by considering the induced cell partition (see [23, Lemma 6]), and add all of these numbers up.

We start with the case of even k. For fixed $a \in \mathbb{B}_k$, the number of cells that do not lie in an intersection of the form $S_a \cap S_b$ for $b < a$ is equal to $2^{\mathrm{outer}(a)}$ by Lemma 4.24, where $\mathrm{outer}(a)$ denotes the number of outer cups of a. To count these we proceed by induction. The case $k = 2$ is obvious. For arbitrary even k, consider the cup C in a that contains the last vertex. Say it connects vertex k with vertex $k - (2j - 1)$ (with $1 \le j \le m = k/2$). If this cup is dotted, then it is the only outer cup of the diagram, and the number of cells to be counted for such a is just twice the number of diagrams with a cup of this form, which gives

$$2\left[\binom{2(m-j)}{m-j} \cdot \frac{1}{j}\binom{2(j-1)}{j-1} \right],$$

where the second factor is the Catalan number $\mathcal{C}_{j-1}$ counting the, necessarily undecorated, diagrams inside our fixed cup C of width $2(j-1)$, while the first factor is the number of diagrams to the left of the fixed cup; see Remark 3.3. We claim that the sum over $j = 0, \ldots, m$ of these terms gives $\binom{2m}{m}$. Indeed, using induction hypothesis in (4.9), the equality

$$\binom{2(m-j)}{m-j} = \left(4 - \frac{2}{m-j}\right)\binom{2(m-1-j)}{m-1-j}$$

in (4.8), and the Segner recurrence $\sum_{j=0}^{m-2} \mathcal{C}_j \mathcal{C}_{m-2-j} = \mathcal{C}_{m-1}$ in (4.10) we obtain

$$\sum_{j=1}^{m} 2\left[\binom{2(m-j)}{m-j} \cdot \frac{1}{j}\binom{2(j-1)}{j-1} \right]$$

$$= \frac{2}{m}\binom{2(m-1)}{m-1} + \sum_{j=1}^{m-1} 2\left[\binom{2(m-j)}{m-j} \cdot \frac{1}{j}\binom{2(j-1)}{j-1} \right]$$

(4.8)
$$= \frac{2}{m}\binom{2(m-1)}{m-1} + 4\sum_{j=1}^{m-1} 2\left[\binom{2(m-1-j)}{m-1-j} \cdot \frac{1}{j}\binom{2(j-1)}{j-1} \right]$$

$$- 4\sum_{j=0}^{m-2} \left[\frac{1}{j+1}\binom{2j}{j}\frac{1}{m-1-j}\binom{2(m-2-j)}{m-2-j} \right]$$

(4.9)
$$= \frac{2}{m}\binom{2(m-1)}{m-1} + 4\binom{2(m-1)}{m-1} - 4\sum_{j=0}^{m-2} \left[\frac{1}{j+1}\binom{2j}{j}\frac{1}{m-1-j}\binom{2(m-2-j)}{m-2-j} \right]$$

(4.10)
$$= \frac{2}{m}\binom{2(m-1)}{m-1} + 4\binom{2(m-1)}{m-1} - \frac{4}{m}\binom{2(m-1)}{m-1}$$

$$= \frac{4m-2}{m}\binom{2(m-1)}{m-1} = \binom{2m}{m}.$$

If the cup C is undotted, then it is an outer cup; but the outer cups to the left still contribute to the number of cells, giving us

$$2\left[2^{k-2j} \cdot \frac{1}{j}\binom{2(j-1)}{j-1} \right],$$

where we have again the number of diagrams inside our fixed cup C times the contribution from the part to the left of C, which is 2^{k-2j} by induction. Altogether, we

get

$$\sum_{j=1}^{m} 2\left[\binom{2(m-j)}{m-j} \cdot \frac{1}{j}\binom{2(j-1)}{j-1}\right] + \sum_{j=1}^{m} 2\left[2^{2m-2j}\frac{1}{j}\binom{2(j-1)}{j-1}\right]$$

$$= \binom{2m}{m} + \frac{2}{m}\binom{2(m-1)}{m-1} + 4\sum_{j=1}^{m-1} 2\left[2^{2(m-1)-2j}\frac{1}{j}\binom{2(j-1)}{j-1}\right]$$

$$\underset{\text{(Induction)}}{=} \binom{2m}{m} + \frac{2}{m}\binom{2(m-1)}{m-1} + 4\left[2^{2(m-1)} - \binom{2(m-1)}{m-1}\right] = 2^{2m} = 2^{k}.$$

Now let $k = 2m + 1$ be odd. We distinguish the case where the ray has a dot from the case where it does not. In the first case the diagram only contributes a single cell, since all outer cups are special, thus we only have to count the possible diagrams when the dotted ray is at position $k - 2j$ for a fixed $0 \leq j \leq m$ giving us

$$\binom{2(m-j)}{m-j} \cdot \frac{1}{j+1}\binom{2j}{j},$$

with the first factor being the number of possible decorated cup diagrams to the left of the ray, and the second factor the number of undecorated cup diagrams to the right. If instead the ray is undotted, then only outer cups to the right of the ray are special, and we have to count the cells arising from the outer cups to the left, which gives us $2^{2(m-j)} \cdot \frac{1}{j+1}\binom{2j}{j}$.

Hence it is enough to verify $\sum_{j=0}^{m}\left[\binom{2(m-j)}{m-j} + 2^{2(m-j)}\right] \cdot \frac{1}{j+1}\binom{2j}{j} = 2^{2m+1}$, which is clear for $m = 0$. The left-hand side, however, equals

$$\frac{2}{m+1}\binom{2m}{m} + \sum_{j=0}^{m-1}\left[\binom{2(m-j)}{m-j} + 2^{2(m-j)}\right] \cdot \frac{1}{j+1}\binom{2j}{j}$$

$$= \frac{2}{m+1}\binom{2m}{m} + \sum_{j=0}^{m-1}\left[\binom{2(m-j-1)}{m-j-1} + 2^{2(m-j-1)}\right] \cdot \frac{1}{j+1}\binom{2j}{j}$$

$$+ \sum_{j=0}^{m-1}\left[\binom{2(m-j)}{m-j} - \binom{2(m-j-1)}{m-j-1} + 3 \cdot 2^{2(m-j-1)}\right] \cdot \frac{1}{j+1}\binom{2j}{j}$$

$$\underset{\text{(Induction)}}{=} \frac{2}{m+1}\binom{2m}{m} + 2^{2m-1}$$

$$+ \sum_{j=0}^{m-1}\left[\frac{3(m-j)-2}{m-j}\binom{2(m-j-1)}{m-j-1} + 3 \cdot 2^{2(m-j-1)}\right] \cdot \frac{1}{j+1}\binom{2j}{j}$$

$$= \frac{2}{m+1}\binom{2m}{m} + 2^{2m-1} + 3\sum_{j=0}^{m-1}\left[\binom{2(m-j-1)}{m-j-1} + 2^{2(m-j-1)}\right] \cdot \frac{1}{j+1}\binom{2j}{j}$$

$$- 2\sum_{j=0}^{m-1}\frac{1}{m-j}\binom{2(m-j-1)}{m-j-1} \cdot \frac{1}{j+1}\binom{2j}{j}$$

$$\underset{\text{(Induction)}}{=} \frac{2}{m+1}\binom{2m}{m} + 2^{2m-1} + 3 \cdot 2^{2m-1} - 2\sum_{j=0}^{m-1}\frac{1}{m-j}\binom{2(m-j-1)}{m-j-1} \cdot \frac{1}{j+1}\binom{2j}{j}$$

$$\underset{\text{(Segner rec.)}}{=} \frac{2}{m+1}\binom{2m}{m} + 2^{2m+1} - \frac{2}{m+1}\binom{2m}{m} = 2^{2m+1},$$

and so the claim holds. Hence the lemma is proved. ∎

5 Domino Tableaux and Combinatorial Bijections

In the following, let λ be a partition of $2k$, viewed as a Young diagram of shape λ (*i.e.*, the parts of λ are the length of the rows of the diagram). We will picture them using "English notation".

5.1 (Signed Admissible) Domino Tableaux and Clusters

We now introduce the combinatorics of domino diagrams and tableaux, following [53], and connect them afterwards with the geometry of Springer fibres and our cup diagram combinatorics. The cup diagrams then provide an elementary description of the geometry, in particular pairwise intersections of components.

Definition 5.1 A *domino diagram T* with k dominoes of shape λ is a Young diagram $\overline{T}$ of shape λ together with a partitioning of the boxes into subsets of order 2, called *dominoes*, such that the paired boxes share a common edge (vertically or horizontally). The set of all domino diagrams with shape λ will be denoted by $DY(\lambda)$.

When picturing a domino diagram, we omit the common edge of a pair.

Example 5.2 Here is an example of a domino diagram of shape $\lambda = (4, 3, 1)$, the left diagram shows the 2-subsets, which are then turned into dominoes in the right diagram:

Recall from Section 2 the notion of admissible partition that was used to label conjugacy classes of nilpotent elements. We now define admissible domino tableaux.

Definition 5.3 An *admissible domino tableau t* is a domino diagram from $DY(\lambda)$ together with a filling of the $2k$ boxes in the diagram with the integers $1, \ldots, k$ such that
(ADT1) Each integer occurs exactly twice and the two boxes it appears in form a domino.
(ADT2) The entries in each row and column are weakly increasing from left to right and top to bottom.
(ADT3) The sequence of partitions starting with t and then obtained by successively eliminating the domino labelled with the largest integer consists only of admissible partitions as in Definition 2.3.

We denote the *set of all admissible domino tableaux* of shape λ by $ADT(\lambda)$. If we drop the last condition (ADT3), then we get more generally the set $DT(\lambda)$ of *standard domino tableaux* of shape λ.

We again abbreviate admissible domino tableaux by drawing dominoes (instead of pairs of boxes) with only one number per domino (instead of twice the same number if the corresponding two boxes form a pair).

Example 5.4   There are two admissible domino tableaux of shape $(3,3)$,

$$ADT((3,3)) = \left\{ \quad t_1 = \boxed{\begin{array}{ccc} 1 & 2 & 3 \end{array}} \quad , \quad t_2 = \boxed{\begin{array}{cc} 1 & \begin{array}{c} 2 \\ \hline 3 \end{array} \end{array}} \quad \right\}.$$

Indeed, we first fill the Young diagram of shape $(3,3)$ with the integers $1, 2, 3$, each occurring twice such that the fillings are weakly increasing in rows and columns:

$$t_1 = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline \end{array} \qquad t_2 = \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 1 & 3 & 3 \\ \hline \end{array} \qquad t_3 = \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 & 3 \\ \hline \end{array} \qquad t_4 = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & 3 \\ \hline \end{array}$$

To satisfy (ADT1) we have to remove t_4, since the two boxes labelled 2 do not form a domino. Since t_3 violates (ADT3) we are left with t_1 and t_2, which indeed satisfy (ADT1)–(ADT3).

From now on we restrict ourselves to partitions $\lambda = (\lambda_1, \lambda_2)$ with only 2 parts. Note that the definition of an admissible domino tableau forces horizontal dominoes to occur with the left box in an even column. In particular, we must have a vertical domino in the first column (*e.g.*, t_3 is not possible).

Definition 5.5   Let $T \in ADT((\lambda_1, \lambda_2))$ be an admissible domino tableau and let p be a domino in T. We define the *type* of p as

$$\text{type}(p) = \begin{cases} V^0 & \text{if } p \text{ is vertical and in an even column,} \\ V^1 & \text{if } p \text{ is vertical and in an odd column,} \\ H & \text{if } p \text{ is horizontal.} \end{cases}$$

Remark 5.6   In [53] dominoes of type V^0, V^1, H are called of type $I-, I+, N$, respectively.

To get a labelling set for the irreducible components of the Springer fibre we need to introduce an additional data in form of a sign for the dominoes of type V^1.

Definition 5.7   A *signed domino tableau* is an admissible domino tableau with signs (*i.e.*, an element in $\{+, -\}$) attached to each domino of type V^1.

The set of signed domino tableaux of shape λ is denoted $ADT_{\text{sgn}}(\lambda)$.

Example 5.8   There are six signed domino tableaux of shape $(3,3)$ (with the underlying domino tableaux from Example 5.4):

$$t_{1,++} = \boxed{\begin{array}{ccc} \begin{array}{c}1\\+\end{array} & 2 & \begin{array}{c}3\\+\end{array} \end{array}} \quad t_{1,+-} = \boxed{\begin{array}{ccc} \begin{array}{c}1\\+\end{array} & 2 & \begin{array}{c}3\\-\end{array} \end{array}} \quad t_{1,-+} = \boxed{\begin{array}{ccc} \begin{array}{c}1\\-\end{array} & 2 & \begin{array}{c}3\\+\end{array} \end{array}} \quad t_{1,--} = \boxed{\begin{array}{ccc} \begin{array}{c}1\\-\end{array} & 2 & \begin{array}{c}3\\-\end{array} \end{array}}$$

$$t_{2,+} = \boxed{\begin{array}{cc} \begin{array}{c}1\\+\end{array} & \begin{array}{c}2\\\hline 3\end{array} \end{array}} \qquad t_{2,-} = \boxed{\begin{array}{cc} \begin{array}{c}1\\-\end{array} & \begin{array}{c}2\\\hline 3\end{array} \end{array}}$$

The admissibility condition (ADT3) implies that dominoes of type H have their left part in an even column, which means that our tableaux are built up from two basic building blocks, called clusters.

A *closed cluster* is a connected part of an admissible domino tableau containing all dominoes between a domino of type V^1 and the next domino to its right of type V^0, including the two vertical dominoes. Hence after removing the filling, it is a part of the underlying Young diagram of the following form (starting at an odd column with an even number of dominoes of type H arranged in two rows of the same length between the two vertical dominoes)
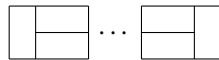
An *open cluster* is a connected part of an admissible domino tableau given by a domino D of type V^1 with no domino of type V^0 to its right and all dominoes of type H to the right, *i.e.*, a diagram of the form
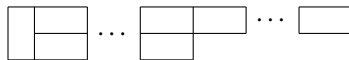
Clearly, every (signed) admissible standard domino tableau decomposes uniquely into a finite disjoint union of closed clusters and possibly one extra open cluster. Such an open cluster exists if and only if the shape is (λ_1, λ_2) with λ_1 (and then also λ_2) odd.

Example 5.9 The admissible domino tableau

has three closed clusters

$$C_1(t) = \{1, 2, 3, 4, 5, 6\}, \quad C_2(t) = \{7, 8\}, \quad \text{and} \quad C_3(t) = \{9, 10, 11, 12\}$$

and one open cluster $C_4(t) = \{13, 14, 15, 16, 17, 18, 19\}$.

Since by definition each of the clusters includes exactly one domino of type V^1, we can speak of the *sign of a cluster* for a signed domino tableau.

Definition 5.10 Let $t, t' \in ADT_{\mathrm{sgn}}(\lambda)$ be two signed domino tableaux that coincide if we forget the signs. Then $t \sim_{\mathrm{cl}} t'$ if the signs of all closed clusters of t and t' coincide. This is obviously an equivalence relation. We denote the equivalence classes for $\sim_{\mathrm{cl}}$ by $ADT_{\mathrm{sgn,cl}}(\lambda)$.

Since in our case there is at most one open cluster for a given admissible domino tableau, the equivalence classes consist of either one element, if there is no open cluster, or of two elements, if there is an open cluster.

5.2 Combinatorial Bijections

Given a cluster $\mathcal{C}$ in a (signed) domino tableau we call the part consisting of the dominoes of type H the *standard tableau part* of $\mathcal{C}$, since viewing the dominoes as ordinary

boxes of a tableau we have in fact a standard tableau $T(\mathcal{C})$. Recall the well-known bijection

(5.1) $\{$(ordinary) standard tableaux of shape $(r, s)\}$

$$\overset{1:1}{\longleftrightarrow}$$

$\{$undecorated cup diagrams on $r + s$ vertices with s cups$\}$

sending a standard tableau T to the unique cup diagram $C(T)$ where the s numbers appearing in the bottom row mark the right endpoints of the cups and the number in the top row are the left endpoint of cups or endpoints of rays; see *e.g.*, [51, Proposition 3].

Example 5.11 The following five standard tableaux T

1	2	5		1	3	5		1	3	4		1	2	4		1	2	3
---	---	---		---	---	---		---	---	---		---	---	---		---	---	---
3	4			2	4			2	5			3	5			4	5	
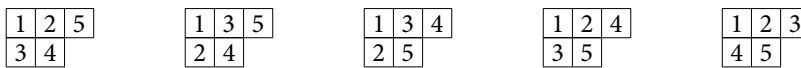
correspond to the undecorated cup diagrams $C(T)$:

We extend the bijection (5.1) and assign to a signed closed cluster $\mathcal{C}$ with standard tableau part containing $2m$ dominoes of type H a cup diagram on $2m + 2$ vertices with $m + 1$ cups by adding an additional vertex to the left and right of $C(T(\mathcal{C}))$ connected by a cup. This cup is dotted if the sign of $\mathcal{C}$ is − and undotted if the sign is +. If the cluster $\mathcal{C}$ is open, we again take the cup diagram $C(T(\mathcal{C}))$ associated with its standard tableau part and add an additional vertex with a ray to the left of $C(T(\mathcal{C}))$. This ray is dotted if the sign of $\mathcal{C}$ is − and undotted if the sign is +. Since every signed standard tableau is a concatenation of closed and possibly one open cluster, our assignment
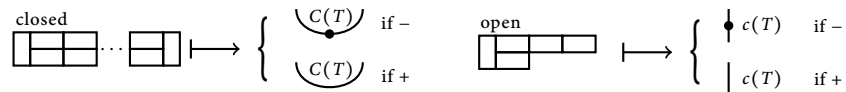
extends to a map F that assigns to each admissible tableau a cup diagram obtained by putting the cup diagrams associated with the single clusters next to each other (in the same order as the clusters).

Lemma 5.12 *Let r and s be natural numbers such that (r, s) is an admissible shape. The assignment F defines a bijection*

$$ADT_{\mathrm{sgn}}((r, s)) \overset{1:1}{\longleftrightarrow} \left\{\textit{cup diagrams on } \tfrac{r+s}{2} \textit{ vertices with } \left\lfloor \tfrac{s}{2} \right\rfloor \textit{ cups}\right\}.$$

Proof The map is indeed well defined, since no closed cluster appears to the right of an open cluster, thus no dotted cup to the right of a ray and of course no dotted cup nested inside another cup. The inverse map is given as follows: take a cup diagram a of the above form and first ignore all dots. Then create a domino tableau with vertical dominoes exactly in the rows labelling the endpoints of outer cups to

the left of all the rays in a and the leftmost ray in a. Fill them with their column number and add additionally a sign in case the column number is odd. This sign is + if the corresponding cup/ray was originally undotted and − in case it was dotted. Then forget these outer cups and the leftmost ray and add dominoes of type H in between and to the right of these vertical dominoes to get the correct shape. Turn them into standard tableaux parts of the clusters using the bijection (5.1). For the filling, the numbers labelling the rows between the corresponding two vertical dominoes or to the right of the rightmost vertical dominoes should be used (so the bijection (5.1) has to be slightly adjusted by shifting the numbers for the filling by the column number of the next vertical domino to the left). Since inside an outer cup or to the right of the leftmost ray we have only undotted cups, the assignment makes sense. We omit the straightforward calculations to verify that this is indeed the inverse map. ∎

Corollary 5.13 *The bijection from Lemma 5.12 restricts to bijections*

$$\{ T \in ADT_{sgn}((r,s)) \mid \text{the number of} - \text{signs is even (resp. odd)} \}$$

$$\overset{1:1}{\longleftrightarrow}$$

$$\left\{ \begin{array}{c} \text{cup diagrams on } \frac{r+s}{2} \text{ vertices with } \lfloor \frac{s}{2} \rfloor \text{ cups} \\ \text{and an even (resp. odd) number of dots} \end{array} \right\}$$

and gives the following two bijections in the special case $r = s = k$:

$$\{ T \in ADT_{sgn}((k,k)) \mid \text{the number of} - \text{signs is even} \} \overset{1:1}{\longleftrightarrow} \mathbb{B}_k^{\text{even}}$$

$$\{ T \in ADT_{sgn}((k,k)) \mid \text{the number of} - \text{signs is odd} \} \overset{1:1}{\longleftrightarrow} \mathbb{B}_k^{\text{odd}}$$

Proof This follows directly from the definitions and Lemma 5.12. ∎

Remark 5.14 Lemma 5.12 transfers the notion of closed and open clusters to cup diagrams: a *closed cluster* consists there of a cup c together with all cups contained in c, with the property that c is not contained in any other cup and located to the left of all rays, whereas an *open cluster* consists of the leftmost ray and all cups and rays to the right of it.

A *cycle move* on a closed cluster is the move illustrated in the following diagram changing a closed cluster C into a (not admissible) standard domino tableau $\text{Cyc}(C)$ containing only horizontal dominoes:
(5.2)

Cycle :

a	b	c	d	e	j
	f	g	h	i	

$\longmapsto$

a	b	c	d	e
f	g	h	i	j

Note that neither the shape nor the set of numbers used for the filling is changed. The numbers are consecutive from a set of $2m$ numbers for the shape $(2m, 2m)$. In contrast, it is important to observe that the filling in the subdiagrams of shape $(2m', 2m')$ using the first $2m'$ dominoes for $m' < m$ is not a set of consecutive numbers.

The concept of cycle moves goes back to [17]. We use them here to establish a bijection between signed domino tableaux of a fixed shape and ordinary domino tableaux of the same shape.

Lemma 5.15 *Applying the cycle move* (5.2) *to each closed cluster with sign +, and forgetting afterwards all signs determines a bijection*

$$\mathrm{Cyc}\colon ADT_{\mathrm{sgn},cl}((r,s)) \overset{1:1}{\longleftrightarrow} DT((r,s))$$

for any admissible shape (r,s).

Proof Obviously, the map is well defined and injective. For surjectivity take a standard domino tableau $S \in DT((r,s))$. If there is no horizontal domino starting at an odd column number, then just place minus signs at all vertical dominoes in odd columns to obtain a preimage of S. Otherwise, take the first such horizontal domino D in the first row. We claim it is the top left domino of a unique connected subdiagram S' of S of the form shown on the right-hand side of (5.2); in particular, it is the smallest such rectangular shaped diagram containing consecutive numbers as filling. Hence, the uniqueness is clear and also the existence in case S has two rows of the same length. Otherwise, S has shape (r,s) with r,s odd in which case there is at least one vertical domino to the right of D and the existence is again clear. Now use (5.2) and apply Cycle^{-1} to S' and assign a + sign to the resulting cluster. Repeat this procedure with the part of the diagram to the right of S' observing that this is again of admissible shape. The result is an admissible signed domino tableau except that probably some signs are missing. Finally insert $-$'s for all missing signs to get $\mathrm{Cyc}^{-1}(S)$. ∎

Example 5.16 The bijection from Corollary 5.13 assigns to the cup diagrams from Example 4.14 (read from left to right columnwise from top to bottom) the following signed domino tableaux:

They correspond via Lemma 5.15 to the following standard domino tableaux:

Remark 5.17 Another bijection between signed domino tableaux of a fixed shape and certain standard domino tableaux using cycle moves was established in [34]. This bijection differs slightly from ours, since we never apply a cycle move to open clusters, thus the shape stays unchanged; we fix instead the parity of the total number of minus signs.

Definition 5.18 A 2-*row character of type* D_k is an unordered pair $\{\lambda, \mu\}$ of 1-row partitions λ, μ with a total number of k boxes. A *special bitableau of type* D_k is a 2-row character of type D_k filled with the numbers $\{1, \ldots, k\}$ increasing in the two rows. Let $\Omega_{a,b} = \Omega(D_k)_{a,b}$ be the set of such special bitableaux where λ has a boxes and μ has b boxes.

The irreducible representations of the Weyl group $W(B_k)$ are indexed by ordered pairs of partitions (λ, μ); see *e.g.*, [30] or [41, 8.2]. Their restrictions $S^{(\lambda,\mu)}$ to the index 2 subgroup $W(D_k)$ stay either irreducible, but two irreducible representations become isomorphic (namely the one indexed by (λ, μ) and (μ, λ) in case $\lambda \neq \mu$) or split into two non-isomorphic irreducible summands $S^{\lambda,\mu,+}, S^{\lambda,\mu,-}$ (if $\lambda = \mu$). Let s_i, $0 \leq i \leq k-1$ be the Coxeter generators of $W(D_k)$ labelled by the vertices of the Dynkin diagram. We choose the labelling such that there is an edge between 0 and 2 and between i and $i+1$ for $1 \leq i \leq k-2$. Then $\{s_i \mid 1 \leq i \leq k-1\}$ generate a subgroup isomorphic to S_k. The 2-row characters appear naturally as summands in the following induced module.

Lemma 5.19 *The induced trivial representation* $\mathrm{Ind}_{S_k}^{W(D_k)} \mathbb{1}$ *of* S_k *decomposes into a multiplicity-free sum of* 2-*row characters.*

Proof Let $A = (\mathbb{Z}/2\mathbb{Z})^{k-1}$. Since $W(D_k) \cong A \rtimes S_k$ we can apply the Wigner–Mackey little group argument ([41, 8.2]) to construct the irreducible characters of $W(D_k)$. The S_k-action on the group of characters of A has exactly $\lfloor k/2 \rfloor + 1$ orbits, and we can chose representatives $\chi_0, \ldots, \chi_{\lfloor k/2 \rfloor}$ such that the stabiliser H_i of χ_i equals $S_i \times S_{k-i}$ if $i < k/2$ and $\mathbb{Z}/2\mathbb{Z} \ltimes (S_{k/2} \times S_{k/2})$ if $i = k/2$. An irreducible character of $W(D_k)$ is of the form $\mathrm{Ind}_{A \rtimes H_i}^{W(D_k)}(\chi_i \otimes \rho)$ for some i and ρ an irreducible character of H_i. Using Frobenius reciprocity and the Mackey formula we obtain

$$\mathrm{Hom}_{W(D_k)}\big(\mathrm{Ind}_{A \rtimes H_i}^{W(D_k)}(\chi_i \otimes \rho), \mathrm{Ind}_{S_k}^{W(D_k)} \mathbb{1}\big)$$
$$\cong \mathrm{Hom}_{S_k}\big(\mathrm{Res}_{S_k}^{W(D_k)}\mathrm{Ind}_{A \rtimes H_i}^{W(D_k)}(\chi_i \otimes \rho), \mathbb{1}\big)$$
$$\cong \mathrm{Hom}_{S_k}\big(\mathrm{Ind}_{H_i}^{S_k}\mathrm{Res}_{H_i}^{A \rtimes H_i}(\chi_i \otimes \rho), \mathbb{1}\big) = \mathrm{Hom}_{S_k}\big(\mathrm{Ind}_{H_i}^{S_k}\rho, \mathbb{1}\big).$$

The latter is non-trivial if and only if ρ is the trivial representation of H_i; *i.e.*, the induced representation is of 2-row type and in case of $i = k/2$ equal to $S^{(k/2),(k/2),+}$ or $S^{(k/2),(k/2),-}$, depending on our choice $W(D_k) \cong A \rtimes S_k$ of isomorphism. As the dimension is at most 1, the claim follows. ∎

We finish this section with a bijection between cup diagrams and a labelling set of the basis elements of our irreducible representations of $W(D_k)$. Given a cup diagram a on k vertices consider all the cups c that have no ray to the left and are not contained in any other cup. If c is undotted (resp. dotted), mark its left (right) endpoint and also the left (right) endpoints of all the cups contained in c. Then mark the vertices at rays (if they exist) and all left endpoints of cups that are to the right of at least one ray. Let x be the number of marked vertices. The special bitableau $\Omega(c)$ of type D_k associated with c is then the pair (λ, μ), where λ has x boxes and μ has $k - x$ boxes filled in increasing order with the number of the marked (resp. not marked) vertices.

The following bijection identifies cup diagrams with special bitableaux, explaining the counting formula from Remark 3.3.

Lemma 5.20 *The assignment $c \mapsto \Omega(c)$ defines for odd r, s bijections*

$$\left\{ \begin{array}{c} \text{cup diagrams on } k = \frac{r+s}{2} \text{ vertices with } \lfloor \frac{s}{2} \rfloor \\ \text{cups and an even (resp. odd) number of dots} \end{array} \right\} \overset{1:1}{\longleftrightarrow} \Omega_{\frac{r+1}{2}, \frac{s-1}{2}}$$

and a bijection $\mathbb{B}_k \overset{1:1}{\leftrightarrow} \Omega_{\frac{k}{2}, \frac{k}{2}}$ for even $r = s = k$.

Proof The number of marked points equals the number of cups plus the number of rays. ∎

Remark 5.21 (i) The induced sign representation was identified in [28, Theorem 5.17] with a vector space with basis all cup diagrams on k vertices together with a diagrammatical action of $W(D_k)$, and the decomposition into irreducible modules given by the span of cup diagrams with a fixed number of cups, [28, Remark 5.24]. We get the refined decompositions of Lemma 5.19:

$$(5.3) \qquad \operatorname{Ind}_{S_k}^{W(D_k)} \mathbb{1} \cong \bigoplus_{l=0}^{\frac{k-\epsilon}{2}} S^{((k-l),(l))},$$

where $\epsilon = 1$ for k odd and $\epsilon = 0$ for k even with $S^{((k/2),(k/2))}$ then denoting one out of the two $S^{((k/2),(k/2)),\pm}$ depending on the choices made, as in the proof of Lemma 5.19. The summands can be identified with Kazhdan–Lusztig cell modules. Lemma 5.20 does not induce a $W(D_k)$-equivariant isomorphism (even after twisting with the sign representation) of the corresponding irreducible representations of $W(D_k)$, but only compares the dimensions. To establish an explicit isomorphism between the two representations, one has to link the canonical basis for the cell module (the cup diagrams) with the analog of a Specht basis (the bitableau) using, for instance, [32], as was done successfully for type A in [33]. Note, however, that outside of the class of 2-row characters the cell modules are not necessarily irreducible.

(ii) The bijection from Lemma 5.20 induces via Lemmas 5.12 and 5.15 bijections

$$DT(r, s) \overset{1:1}{\longleftrightarrow} \Omega_{\frac{r+1}{2}, \frac{s-1}{2}} \quad \text{if } r, s \text{ odd,}$$

$$DT(r, s) \overset{1:1}{\longleftrightarrow} \Omega_{\frac{k}{2}, \frac{k}{2}} \qquad \text{if } r = s = k \text{ even.}$$

They are in fact special cases of known bijections between standard domino tableaux and pairs of partitions with standard fillings established in [50]. These Stanton–White bijections were used, for instance, in the study of decompositions of tensor products of representations of $\mathfrak{gl}_n$ and generalized Littlewood–Richardson theory, see *e.g.,* [9] and in the representation theory of the Weyl group $W(B_k)$; see *e.g.,* [35].

Example 5.22 Figure 4 gives an example summarising our bijections (we only display half of the bitableaux, since the second half is then determined):

Note once more the different behaviour for the shape (r, s) depending on $r \neq s$ or $r = s = k$.
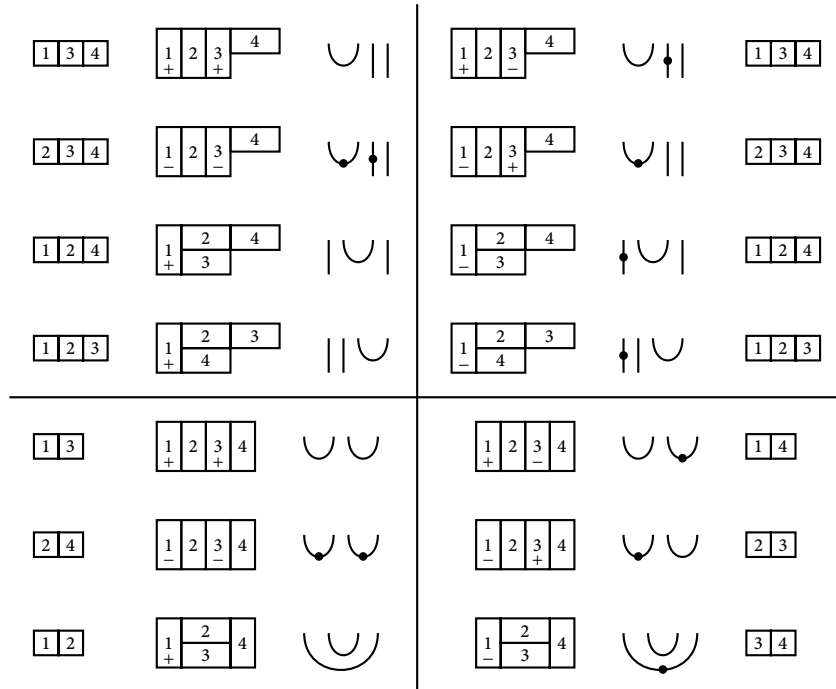
Figure 4

6 Irreducible Components of the Algebraic Springer Fibre

We consider again the algebraic Springer fibre of type D_k from Definition 2.4. The irreducible components of $\mathcal{F}_N$ for classical G were parameterized and partly described by Spaltenstein in [45]. We briefly recall the constructions needed in our case.

6.1 Classification of Irreducible Components

Let $\lambda = \lambda_N$ be the partition corresponding to the nilpotent element $N \in \mathfrak{g}$ and let $F_\bullet \in \mathcal{F}_N$. Then N defines a sequence of induced endomorphisms on $F_{n-i}/F_i = (F_i)^\perp/F_i$, for $k \geq i \geq 0$. Their Jordan types give us a sequence of domino diagrams where successive shapes differ by exactly one domino. Putting the number i into the i-th domino added in this way defines an admissible domino tableau $\mathcal{S}_N(F_\bullet)$ of shape λ_N, thus an assignment

$$(6.1) \qquad\qquad \mathcal{S}_N : \mathcal{F}_N \longrightarrow ADT(\lambda_N)$$

For details we refer the reader to [45] or [53].

The analogous construction in type A (to each ordinary full flag $F_\bullet$ one assigns an ordinary standard tableau $\mathcal{S}_N^A(F_\bullet)$ encoding the Jordan types of the restriction of N to

the i-th part of the flag) gives rise to the well-known Spaltenstein–Vargas classification of irreducible components of Springer fibres for $SL(n)$.

Theorem 6.1 ([43, 52]) *Let $N \in \mathfrak{sl}(n, \mathbb{C})$ be nilpotent and let $\mathcal{F}^A$ be the flag variety for $\mathrm{SL}(n, \mathbb{C})$ containing the Springer fibre $\mathcal{F}_N^A$. The map $\mathcal{S}_N^A$ defines a surjection onto the set $oSYT(\lambda_N)$ of ordinary standard Young tableaux which separates the irreducible components $\mathrm{Irr}(\mathcal{F}_N^A)$ of $\mathcal{F}_N^A$. That is, it defines a bijection*

$$\mathcal{S}_N^A : Irr(\mathcal{F}_N^A) \longrightarrow oSYT(\lambda_N).$$

Given $T \in oSYT(\lambda_N)$, the irreducible component $X_T^A = (\mathcal{S}_N^A)^{-1}(T)$ equals the closure of $(X_T^A)^0 = \{F_\bullet \mid \mathcal{S}_N^A(F_\bullet) = T\}$.

In type D, admissible tableaux do not fully separate the components of $\mathcal{F}_N$. It is still true that $F_\bullet, G_\bullet \in \mathcal{F}_N$ with $\mathcal{S}_N(F_\bullet) \neq \mathcal{S}_N(F_\bullet)$ lie in different irreducible components, but the converse fails, since of course $(X_T)^0 = \{F_\bullet \mid \mathcal{S}_N^A(F_\bullet) = T\}$ and its closure X_T need not be connected. The signed domino tableaux were introduced to account for this disconnectedness in form of the following classification of Van Leeuwen [53, Lemmas 3.2.3 and 3.3.3].

Theorem 6.2 *Let $N \in \mathfrak{g}$ be nilpotent of admissible Jordan type $\lambda_N = (\lambda_1, \lambda_2)$ and $T \in ADT(\lambda_N)$.*

(i) *X_T is a union of irreducible components of $\mathcal{F}_N$, labelled by all signed domino tableaux that equal T after forgetting the signs.*
(ii) *All irreducible components of $\mathcal{F}_N$ are obtained in this way.*
(iii) *There is a bijection between the set of admissible signed standard domino tableaux of shape λ_N and the set of irreducible components of $\mathcal{F}_N$,*

$$ADT_{\mathrm{sgn}}(\lambda_N) \overset{1:1}{\longleftrightarrow} \mathrm{Irr}(\mathcal{F}_N).$$

Remark 6.3 For general Jordan type, the labelling set of irreducible components is given by certain equivalence classes of signed domino tableaux ([53, Definition 3.3.2]), but in the special case of 2-row shapes the equivalence relation becomes trivial, since every cluster contains only one domino of type V^1.

If N has Jordan type (r, s), irreducible components are of dimension $\lfloor \frac{s}{2} \rfloor$, [53, Lemma 2.3.4 and p. 14]. Hence, the dimension equals the number of cups in the cup diagram associated with the admissible signed domino tableau from Theorem 6.2 via the bijection from Lemma 5.12.

6.2 Components as Iterated $\mathbb{P}^1(\mathbb{C})$-bundles

The rest of this section gives an inductive construction of the irreducible components and proves the following refinement of Theorem 6.2.

Definition 6.4 A topological space X_1 is an iterated fibre bundle of base type $(B_1, \ldots, B_l)$ if there exist spaces $X_1, B_1, X_2, B_2, \ldots, X_l, B_l, X_{l+1} = \mathrm{pt}$ and maps $p_1, \ldots, p_l$ such that $p_j : X_j \to B_j$ is a fibre bundle with typical fibre X_{j+1}.

Theorem 6.5   *Let $N \in \mathfrak{g}$ nilpotent of type $\lambda_N = (\lambda_1, \lambda_2)$ and $T \in ADT(\lambda_N)$.*

(i)   *The assignment $\mathcal{S}_N$ from (6.1) induces a bijection*

$$S_N^{\mathrm{sgn}} \colon ADT(\lambda_N) \overset{1:1}{\longleftrightarrow} \mathrm{Irr}(\mathcal{F}_N),$$

where $(S_N^{\mathrm{sgn}})^{-1}(T)$ is given by the algorithm in Section 6.4.

(ii)   *Every irreducible component Y is an iterated fibre bundle of base type*

$$(\mathbb{P}^1(\mathbb{C}), \mathbb{P}^1(\mathbb{C}), \ldots, \mathbb{P}^1(\mathbb{C})),$$

where the number of components equals the number of cups in the cup diagram corresponding to $(S_N^{\mathrm{sgn}})^{-1}(T)$ via Lemma 5.12.

The proof of the theorem will be given in the next subsections.

6.3 Explicit Linear Algebra Conventions

Let $W = W_{a,a}$ be a complex vector space of dimension $n = 2a$. We fix a basis $w_1, \ldots, w_a, w_{-1}, \ldots, w_{-a}$ and set $w_0 = 0 = w_{-a-1}$. Define a bilinear form $\beta = \beta_a$ on W by $\beta(w_i, w_j) = \delta_{i,-j}$, hence the non-trivial pairings can be displayed as
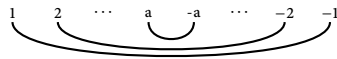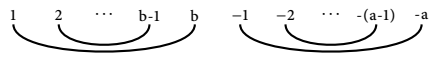

Setting $N(w_i) = \sigma(i)w_{i-1}$ for $-a \le i \le a$ with $\sigma(i) = -1$ if $i < 0$ and $\sigma(i) = 1$ if $i > 0$, defines a linear endomorphism of W. It satisfies $\beta(Nw, w') + \beta(w, Nw') = 0$ for $w, w' \in W$, hence $N \in \mathfrak{o}(W, \beta)$.

Let $V_{a,b} = \mathbb{C}^{a+b}$ be a complex vector space of dimension $n = a + b$ with a, b odd. We fix a basis $e_1, \ldots, e_b, e_{-1}, \ldots, e_{-a}$, set $e_0 = 0 = e_{-a-1}$ and fix the symmetric bilinear form $\gamma = \gamma_{a,b}$ on V defined by

$$\gamma(e_i, e_j) = (-1)^i \left(\delta_{i+j, b+1} - \delta_{i+j, -(a+1)} \right)$$

i.e., graphically

Setting $N(e_i) = e_{i-1}$ for $-a \le i \le -1, 1 \le i \le b$ defines a linear endomorphism N of V satisfying $\gamma(Nv, v') + \gamma(v, Nv') = 0$ for $v, v' \in V$. Hence, $N \in \mathfrak{o}(V, \gamma)$.

We identify $V_{a,a}$ with $W_{a,a}$ via the following isomorphism.

**Lemma 6.6**   *For odd a, the following defines an isomorphism*

$$\begin{aligned}
\varphi \colon V_{a,a} &\longrightarrow W_{a,a} \\
e_i &\longmapsto \begin{cases} \frac{1}{\sqrt{2}} \left((-1)^{i+1} w_i + w_{i+a+1} \right) & \text{if } i < 0, \\ \frac{1}{\sqrt{2}} \left((-1)^{i+1} w_{i-a-1} - w_i \right) & \text{if } i > 0. \end{cases}
\end{aligned}$$

of vector spaces satisfying $\beta(\varphi(v), \varphi(v')) = \gamma(v, v')$ for any $v, v' \in V$.

Proof The claim follows by a straightforward direct calculation. (The inverse map is given by $\varphi^{-1}(w_i) = \frac{1}{\sqrt{2}}(-1)^{i+1}(e_i + e_{a+i+1})$ for $i < 0$ and $\varphi^{-1}(w_i) = \frac{1}{\sqrt{2}}(e_{i-a-1} - e_i)$ for $i > 0$.) ∎

6.4 Inductive Construction of the Irreducible Components

In this section we prove Theorem 6.5 by constructing the irreducible component $(S_N^{\mathrm{sgn}})^{-1}(T)$ corresponding to a signed domino tableau T. This will be done by induction on the number of dominoes. We distinguish the following basic cases (displaying the rightmost cluster and indicating the position of the domino D with the largest entry):

(6.3)
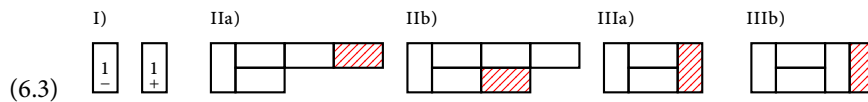
Proof of Theorem 6.5 To each admissible signed domino tableau T with, say, d dominoes, we construct now the corresponding irreducible component.

Our starting point is $d = 1$. Then N has two Jordan blocks of size 1 and we consider $W = W_{1,1}$. The possible signed domino tableaux are displayed in (6.3)(I). The 1-dimensional isotropic flags in W are $F_1 = \langle w_{-1} \rangle$, and $F_1 = \langle w_1 \rangle$ which are automatically in the kernel of N. Each of them defines an irreducible component in $\mathcal{F}_N$. We assign $F_1 = \langle w_{-1} \rangle$ to the first (negative) signed tableau and $F_1 = \langle w_1 \rangle$ to the second (positive) signed tableau. The signs indicate the components of $\mathcal{F}$ with respect to our chosen basis. (Alternatively we could work with $V = V_{1,1}$ and take $F_1 = \langle e_1 + e_{-1} \rangle$ and $F_1 = \langle e_{-1} - e_1 \rangle$ respectively.)

Now let $d > 1$ and assume the theorem holds for less than d dominoes. Let D be the domino in T filled with the highest number.

Case 1 T is of shape (a, b) and D is horizontal: Consider $V = V_{a,b}$.

A line in the kernel of N is of the form $F_1 = \langle \alpha e_{-a} + \beta e_1 \rangle$, for some $[\alpha{:}\beta] \in \mathbb{P}^1(\mathbb{C})$, and is automatically isotropic. First focus on a generic point given by $[\alpha{:}\beta] \notin \{[0{:}1], [1{:}0]\}$. As a new basis B_{new} of V pick

$$\left\{ \begin{array}{l} \beta e_{-a}, \beta e_{-a+1}, \ldots, \beta e_{-b-1}, \beta e_{-b} + \alpha e_1, \ldots, \beta e_{-1} + \alpha e_b, \\ \hspace{4cm} \alpha e_{-a} + \beta e_1, \ldots, \alpha e_{-a+b-1} + \beta e_b \end{array} \right\}$$

Note that N maps an element of B_{new} to its left neighbour, the two leftmost to zero. Here $F_1^\perp$ is spanned by all vectors in B_{new} except $\alpha e_{-a+b-1} + \beta e_b$. Hence the endomorphism induced by N on $F_1^\perp / F_1$ is of Jordan type $(a, b-2)$.

For the point $[0{:}1]$, i.e., $F_1 = \langle e_1 \rangle$, we can use the original basis of V and it holds $F_1^\perp = \langle e_j \mid -a \le j \le -1, 1 \le j < b \rangle$. Again, the induced endomorphism is of Jordan type $(a, b-2)$. We identify $F_1^\perp / F_1$ and $V_{a,b-2}$ by mapping $e_i \in V_{a,b-2}$ to $e_i + F_1$ for $i < 0$ and to $\mathbf{i} e_{i+1} + F_1$ for $i > 0$. By taking the closure we thus obtain for Case (IIb) a $\mathbb{P}^1(\mathbb{C})$-bundle over the space of flags given by the tableaux T with the domino D removed. Under the identification of domino tableaux and cup diagrams, the label of D is exactly the position of the right end of a cup; see Subsection 6.5 for an example.

For the point $[1:0]$, *i.e.*, $F_1 = \langle e_{-a} \rangle$, we see that $F_1^\perp = \langle e_j \mid -a \le j < -1, 1 \le j \le b \rangle$ and $F_1^\perp / F_1$ is identified with $V_{a-2,b}$ by mapping the basis element e_i of $V_{a-2,b}$ to $e_i + F_1$ for $i > 0$ and to $\mathbf{i} e_{i-1} + F_1$ for $i < 0$. The Jordan type of the endomorphism induced by N on $F_1^\perp / F_1$ is $(a-2, b)$ in this case. Hence, for Case (IIa) we get a unique choice of a subspace. Under the bijection with cup diagrams, the label of D will be a ray or left end of a cup.

For both cases the statement follows then by induction by looking at the quotient $F_1^\perp / F_1$ with the respective identifications.

Case 2 T of shape (a, a) and D **vertical:** Consider the space $W_{a,a}$. Now, a line in the kernel of N is of the form $F_1 = \langle \alpha w_{-a} + \beta w_1 \rangle$ for $[\alpha:\beta] \in \mathbb{P}^1(\mathbb{C})$ and again automatically isotropic. For a generic point given by $[\alpha:\beta] \notin \{[0:1], [1, :0]\}$, we choose the new basis B_{new} of W as

$$\left\{ \begin{array}{llll} \alpha w_{-a} - \beta w_1, & \alpha w_{-a+1} + \beta w_2, & \dots & \alpha w_{-1} + (-1)^a \beta w_a, \\ \alpha w_{-a} + \beta w_1, & -\alpha w_{-a+1} + \beta w_2, & \dots & (-1)^{a+1} \alpha w_{-1} + \beta w_a \end{array} \right\}$$

Note again that N sends each vector to its left neighbour (but multiplied by -1 in the first row), the leftmost ones to zero. Now $F_1^\perp$ is spanned by all elements in B_{new} except $\alpha w_{-1} + \beta w_a$. In the case where a is even, this is the rightmost vector in the first row, and thus N induces on $F_1^\perp / F_1$ an endomorphism of Jordan type $(a-1, a-1)$, and, moreover, $F_1^\perp / F_1 \cong W_{a-1,a-1}$ with the induced bilinear form. In the case where A is odd, $\alpha w_{-1} + \beta w_a$ is a multiple of the rightmost vector in the second row, so the induced endomorphism is of Jordan type $(a, a-2)$.

Hence for Case (IIIa), *i.e.*, a is even, we obtain, as the closure, a $\mathbb{P}^1(\mathbb{C})$-bundle over the space of flags for the tableaux where D is removed. Again the label of D is the position for the right end of a cup.

Looking at the two points $[1:0]$ and $[0:1]$, *i.e.*, $F_1 = \langle w_{-a} \rangle$ and $F_1 = \langle w_1 \rangle$, it holds $F_1^\perp = \langle w_i \mid -a \le i \le -1, 1 \le i < a \rangle$ and $F_1^\perp = \langle w_i \mid -a \le i < -1, 1 \le i \le a \rangle$, respectively. For both choices $F_1^\perp / F_1 \cong W_{a-1,a-1}$ with the induced bilinear form, and N induces an endomorphism of Jordan type $(a-1, a-1)$.

For Case (IIIb), *i.e.*, a is odd, we obtain two choices corresponding to the sign of the domino D in this case. We assign the choice $F_1 = \langle w_{-a} \rangle$ to the negative sign in D and $F_1 = \langle w_1 \rangle$ to the positive sign in D (In this case corresponding to a dotted or undotted ray at the label of D or the left end of a cup). Again the statement follows by induction looking at $F_1^\perp / F_1$.

By construction, the first part of Theorem 6.5 holds. Moreover, we have shown that the components are iterated $\mathbb{P}^1(\mathbb{C})$-bundles of the correct base type and therefore smooth (see *e.g.*, [40, Section 8] for a detailed argument). ∎

6.5 An Explicit Example

We illustrate the inductive construction of the irreducible components for the admissible shape $(5, 3)$ and refer to Example 5.22 for the corresponding cup diagrams. We identify $\mathbb{C}^{5+3}$ with the vector space $V_{5,3}$ with basis $e_{-5}, e_{-4}, e_{-3}, e_{-2}, e_{-1}, e_1, e_2, e_3$ and form γ.

The irreducible components attached to the four diagrams :

The domino labelled 4 puts us in Case (IIa). This implies $F_1 = \langle e_{-5} \rangle$ and removing the domino means passing to $\langle e_i \mid -5 \leq i \leq -2, 1 \leq i \leq 3 \rangle / F_1$, which we identify with $W_{3,3}$ via Lemma 6.6 with basis

$$\left\{ \begin{array}{lll} w_{-3} = \tfrac{1}{\sqrt{2}}(\mathbf{i}e_{-4} + e_1), & w_{-2} = \tfrac{1}{\sqrt{2}}(-\mathbf{i}e_{-3} - e_2), & w_{-1} = \tfrac{1}{\sqrt{2}}(\mathbf{i}e_{-2} + e_3), \\ w_1 = \tfrac{1}{\sqrt{2}}(\mathbf{i}e_{-4} - e_1), & w_2 = \tfrac{1}{\sqrt{2}}(\mathbf{i}e_{-3} - e_2), & w_3 = \tfrac{1}{\sqrt{2}}(\mathbf{i}e_{-2} - e_3) \end{array} \right\}.$$

The domino labelled 3 belongs to Case (IIIb), thus $F_2 = F_1 + \langle w_{-3} \rangle$ if the sign is $-$ and $F_2 = F_1 + \langle w_1 \rangle$ if it is $+$. We identify the span of w_{-2}, w_{-1}, w_1, w_2 (resp. w_{-3}, w_{-2}, w_2, w_3) with $F_2^{\perp}/F_2$. The domino labelled 2 belongs to Case (IIIa), hence we consider the generic bases (i.e., $[\alpha{:}\beta] \notin \{[1{:}0], [0{:}1]\}$) of $F_2^{\perp}/F_2$:

$$\left\{ \begin{array}{ll} \alpha w_{-2} - \beta w_1, & \alpha w_{-1} + \beta w_2, \\ \alpha w_{-2} + \beta w_1, & -\alpha w_{-1} + \beta w_2 \end{array} \right\} \text{ and } \left\{ \begin{array}{ll} \alpha w_{-3} - \beta w_2, & \alpha w_{-2} + \beta w_3, \\ \alpha w_{-3} + \beta w_2, & -\alpha w_{-2} + \beta w_3 \end{array} \right\}$$

depending on the two choices made previously, together with $F_3 = \langle \alpha w_{-2} + \beta w_1 \rangle$ and $F_3 = \langle \alpha w_{-3} + \beta w_2 \rangle$, respectively.

The final domino labelled 1 puts us in Case (I) with the choice $F_4 = F_3 + \langle -\alpha w_{-1} + \beta w_2 \rangle$ corresponding to $+$ and $F_4 = F_3 + \langle \alpha w_{-2} - \beta w_1 \rangle$ to $-$; this is similar for the second choice. Altogether, we obtain dense subsets of the four components are given by the following generic choices for each step in the complete full flags:
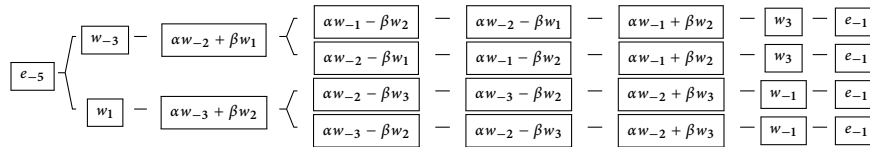
Expressing these as flags in the original basis we get the isotropic flags $F_\bullet$ with $F_j = F_{j-1} + \langle x_j \rangle$, $1 \leq j \leq 4$, where the tuple (x_4, x_3, x_2, x_1) is given as follows (depending on the sign configuration):

$$\begin{array}{l|llll} -- & \alpha(\mathbf{i}e_{-2} + e_3) - \beta(\mathbf{i}e_{-3} - e_2), & \alpha(\mathbf{i}e_{-3} + e_2) - \beta(\mathbf{i}e_{-4} - e_1), & \mathbf{i}e_{-4} + e_1, & e_{-5} \\ +- & \alpha(\mathbf{i}e_{-3} + e_2) + \beta(\mathbf{i}e_{-4} - e_1), & \alpha(\mathbf{i}e_{-3} + e_2) - \beta(\mathbf{i}e_{-4} - e_1), & \mathbf{i}e_{-4} + e_1, & e_{-5} \\ -+ & \alpha(\mathbf{i}e_{-3} + e_2) + \beta(\mathbf{i}e_{-2} - e_3), & \alpha(\mathbf{i}e_{-4} + e_1) + \beta(\mathbf{i}e_{-3} - e_2), & \mathbf{i}e_{-4} - e_1, & e_{-5} \\ ++ & \alpha(\mathbf{i}e_{-4} + e_1) - \beta(\mathbf{i}e_{-3} - e_2), & \alpha(\mathbf{i}e_{-4} + e_1) + \beta(\mathbf{i}e_{-3} - e_2), & \mathbf{i}e_{-4} - e_1, & e_{-5}. \end{array}$$

(We denoted the flags in reversed order, with F_1 on the right, so they fit with the cup diagrams and domino shapes.)

The irreducible components attached to the two diagrams :

We have again $F_1 = \langle e_{-5} \rangle$ and we pass to $\langle e_{-5}, \mathbf{i}e_{-4}, \mathbf{i}e_{-3}, \mathbf{i}e_{-2}, e_1, e_2, e_3 \rangle / F_1$, which we identify with $V_{3,3}$. Case (IIb) uses the basis B_{new}, for the generic choice, given by

$$\left\{ \begin{array}{lll} \beta \mathbf{i}e_{-4} + \alpha e_1, & \beta \mathbf{i}e_{-3} + \alpha e_2, & \beta \mathbf{i}e_{-2} + \alpha e_3, \\ \alpha \mathbf{i}e_{-4} + \beta e_1, & \alpha \mathbf{i}e_{-3} + \beta e_2, & \alpha \mathbf{i}e_{-2} + \beta e_3. \end{array} \right\}$$

After applying Cases (IIa), (I) we obtain dense subsets of the components are given by the following generic choices of flags:

$$\begin{array}{l|llll} - & -e_{-3} + \mathbf{i}e_2, & \mathbf{i}e_{-4}, & \alpha \mathbf{i}e_{-4} + \beta e_1, & e_{-5} \\ + & -e_{-3} - \mathbf{i}e_2, & \mathbf{i}e_{-4}, & \alpha \mathbf{i}e_{-4} + \beta e_1, & e_{-5}. \end{array}$$

The irreducible components attached to the two diagrams $\boxed{\begin{array}{c|c|c} 1 & 2 & 3 \\ \pm & 4 \end{array}}$:

The dense subsets are given by the following generic flags:

$$
\begin{array}{c|cccc}
- & -e_{-3} + \mathbf{i}e_2, & e_{-4}, & e_{-5}, & \alpha e_{-5} + \beta e_1 \\
+ & -e_{-3} - \mathbf{i}e_2, & e_{-4}, & e_{-5}, & \alpha e_{-5} + \beta e_1.
\end{array}
$$

We now compare the intersection behaviour of the irreducible components in both the algebraic and the topological Springer fibre for the case of an odd number of decorations. We look at the following diagrams respectively the corresponding cup diagrams (the components will be labelled by A, B, C and D).
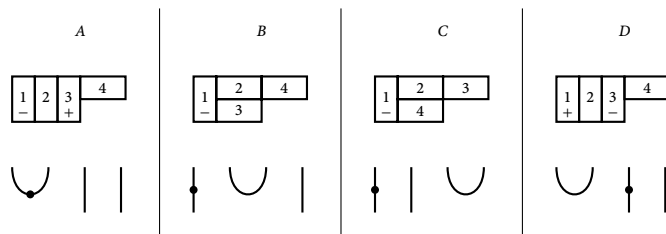
Starting with the topological Springer fibre, we note that the definition of the components from (4.1) generalises and one easily checks that the only non-empty intersections (a single point in each case) are: $S_A \cap S_B$, $S_B \cap S_C$, $S_B \cap S_D$, and finally $S_C \cap S_D$.

For the algebraic Springer fibres, we summarise what the irreducible components look like for the cases above by giving the flags $F_\bullet$ in each case:

$$
X_A = \left\{ \begin{array}{l}
F_1 = \langle e_{-5} \rangle, F_2 = F_1 \oplus \langle \mathbf{i}e_4 - e_{-1} \rangle, \\
F_3 = F_2 \oplus \langle \alpha(\mathbf{i}e_{-4} + e_1) + \beta(\mathbf{i}e_{-3} - e_2) \rangle, \\
F_4 = F_3 \oplus \langle \alpha(\mathbf{i}e_{-3} + e_2) + \beta(\mathbf{i}e_{-2} - e_3) \rangle
\end{array} \middle| [\alpha{:}\beta] \in \mathbb{P}^1(\mathbb{C}) \right\}
$$

$$
X_B = \left\{ \begin{array}{l}
F_1 = \langle e_{-5} \rangle, F_2 = F_1 \oplus \langle \alpha\mathbf{i}e_{-4} + \beta e_1 \rangle, \\
F_3 = \langle e_{-5}, e_{-4}, e_1 \rangle, F_4 = F_3 \oplus \langle -e_{-3} + \mathbf{i}e_2 \rangle
\end{array} \middle| [\alpha{:}\beta] \in \mathbb{P}^1(\mathbb{C}) \right\}
$$

$$
X_C = \left\{ \begin{array}{l}
F_1 = \langle \alpha e_{-5} + \beta e_1 \rangle, F_2 = \langle e_{-5}, e_1 \rangle, \\
F_3 = \langle e_{-5}, e_{-4}, e_1 \rangle, \\
F_4 = F_3 \oplus \langle -e_{-3} + \mathbf{i}e_2 \rangle
\end{array} \middle| [\alpha{:}\beta] \in \mathbb{P}^1(\mathbb{C}) \right\}
$$

$$
X_D = \left\{ \begin{array}{l}
F_1 = \langle e_{-5} \rangle, F_2 = F_1 \oplus \langle \mathbf{i}e_{-4} + e_1 \rangle, \\
F_3 = F_2 \oplus \langle \alpha(\mathbf{i}e_{-3} + e_2) + \beta(\mathbf{i}e_{-4} - e_1) \rangle, \\
F_4 = \langle e_{-5}, e_{-4}, \mathbf{i}e_{-3} + e_2, e_1 \rangle
\end{array} \middle| [\alpha{:}\beta] \in \mathbb{P}^1(\mathbb{C}) \right\}.
$$

This description directly implies that $X_C \cap X_D = \varnothing$, while the other components have the same intersection behaviour as the topological version. In the algebraic case we thus have a Kleinian singularity of type D_4; see [42].

7 Cohomology of Algebraic and Topological Springer Fibre

We give a concrete description of the cohomology ring the algebraic Springer fibres and prove Theorem 1.2. Let $n = 2k$ and $G' = SO(2k, \mathbb{C})$ and $\mathfrak{g}'$ its Lie algebra with Cartan subalgebra $\mathfrak{h}$. Let P be a maximal parabolic of type A with Lie algebra $\mathfrak{p}$ and Levi component L. The connected centre S of L is one-dimensional. Let $\mathfrak{s}$ be its Lie

algebra. Explicitly, we realize $\mathfrak{g}'$ as a Lie algebra of complex $n \times n$-matrices A that are skew-symmetric with respect to the anti-diagonal, *i.e.*, $A_{i,j} = -A_{n-j+1,n-i+1}$. It has basis

$$
\begin{aligned}
L_{i,j} &= E_{i,j} - E_{n-j+1,n-i+1}, & 1 \leq i, j \leq k, \\
X_{i,j} &= E_{i,n-j+1} - E_{j,n-i+1}, & 1 \leq i < j \leq k, \\
Y_{i,j} &= E_{n-j+1,i} - E_{n-i+1,j}, & 1 \leq i < j \leq k,
\end{aligned}
$$

where $E_{i,j}$ denotes the elementary matrix with 1 at position (i,j) and zeros elsewhere. We choose $\mathfrak{h}$ to be the Lie subalgebra given by diagonal matrices on which the Weyl group $W = W(D_k)$ acts by $w.L_{i,i} = L_{w(i),w(i)}$ for $w \in S_k$ and $s_0.L_{1,1} = -L_{2,2}, s_0.L_{2,2} = -L_{1,1}$, and $s_0.L_{i,i} = L_{i,i}$ for $i \neq 1, 2$. The $L_{i,j}$'s form a Levi subalgebra isomorphic to $\mathfrak{gl}(k, \mathbb{C})$ inside the subalgebra $\mathfrak{p}$ generated by the $L_{i,j}$'s and $X_{i,j}$. In this realisation, $\mathfrak{s}$ is the one-dimensional subspace spanned by $x_L = \sum_{i=1}^{k} L_{i,i}$. (Note $\mathfrak{g}' = \mathfrak{so}(W, \beta)$).

7.1 Algebraic Springer Fibre

We consider the closed affine subvariety of $\mathfrak{h} \times \mathfrak{h}$ defined as

$$(7.1) \qquad \mathcal{Z}_k = \left\{ (wx, x) \in \mathfrak{h} \times \mathfrak{h} \mid x \in \mathfrak{s}, w \in W \right\}.$$

Its coordinate ring $\mathbb{C}[\mathcal{Z}_k]$ has, by [24], an $S(\mathfrak{s}^*)$-algebra structure coming from the projection onto $\mathfrak{s}$, where $S(\mathfrak{s}^*) = \mathbb{C}[\mathfrak{s}]$ denotes the algebra of regular functions on $\mathfrak{s}$. To determine $\mathbb{C}[\mathcal{Z}_k]$, and then compute $H(\mathcal{F}'_N)$, we view $\mathcal{Z}_k$ as a closed subvariety of $\mathfrak{h} \times \mathfrak{s}$ and choose explicit coordinates. We identify $\mathbb{C}[\mathfrak{s}]$ with $\mathbb{C}[T]$ such that T evaluated on x_L is equal to 1, and $\mathbb{C}[\mathfrak{h}]$ with $\mathbb{C}[X_1, \ldots, X_k]$ by identifying X_i with the basis element dual to $L_{i,i}$.

For any $I \subset \{1, 2, \ldots, k\}$ we denote $X_I = \prod_{i \in I} X_i \in \mathbb{C}[X_1, X_2, \ldots, X_k]$.

Theorem 7.1   *Let $N \in \mathfrak{g}'$ be nilpotent of Jordan type (k, k).*

(i)   *The S-equivariant cohomology ring of $\mathcal{F}'_N$ equals*

$$(7.2) \qquad H_S(\mathcal{F}'_N) \cong \begin{cases} \frac{\mathbb{C}[X_1, X_2, \ldots, X_k, T]}{\langle \{X_i^2 - T^2, X_I - X_J\} \rangle} & \text{if } k \text{ is even,} \\[2mm] \frac{\mathbb{C}[X_1, X_2, \ldots, X_k, T]}{\langle \{X_i^2 - T^2, X_I - X_J T\} \rangle} & \text{if } k \text{ is odd,} \end{cases}$$

where $1 \leq i \leq k$ and $I, J \subset \{1, 2, \ldots, k\}$ with $I \cap J = \varnothing$ such that $|I| = \frac{k}{2} = |J|$ for even k and $|I| = \frac{k+1}{2} = |J| + 1$ for odd k.

(ii)   *In particular,*

$$H(\mathcal{F}'_N) \cong \begin{cases} \mathbb{C}[X_1, X_2, \ldots, X_k]/\langle \{X_i^2, X_I - X_J\} \rangle & \text{if } k \text{ is even,} \\ \mathbb{C}[X_1, X_2, \ldots, X_k]/\langle \{X_i^2, X_I\} \rangle & \text{if } k \text{ is odd.} \end{cases}$$

Explicit bases are given by the images of the X_I, $|I| \leq \frac{k-1}{2}$, in the case where k is odd; by the X_I for all I, $|I| < \frac{k}{2}$, in the case where k is even and $k \notin I$; and by the X_I for all I, $|I| \leq \frac{k}{2}$, in the case where k is even and $k \in I$.

Proof   We first show that $H_S(\mathcal{F}'_N) \cong A_S(k)$, where $A_S(k)$ denotes the quotient rings on the right-hand side of (7.2), but assuming [24, Theorem 1.2]. The homomorphism

of the inclusion $\mathcal{Z}_k \subset \mathfrak{h} \times \mathfrak{s}$ as closed subvariety induces a surjection

$$(7.3) \qquad \mathbb{C}[X_1, \dots, X_k, T] \longrightarrow \mathbb{C}[\mathcal{Z}_k].$$

By [24, Theorem 1.2], its kernel is contained in the kernel of the corresponding map to $H_S(\mathcal{F}'_N)$ given by the equivariant Borel morphism on the first factor and the induced map $\mathbb{C}[\mathcal{Z}_k] \to H_S(\mathcal{F}'_N)$ is an isomorphism.

Since $\mathcal{Z}_k$ is a cone, to determine the vanishing ideal of $\mathcal{Z}_k$, it is enough to look at homogeneous polynomials vanishing on all points of the form $(w.x_L, x_L)$. The W-orbit of $x_L = \sum_{i=1}^{k} L_{i,i}$ is easy to determine, namely

$$W.x_L = \left\{ \sum_{i=1}^{k} \epsilon_i L_{i,i} \mid \epsilon_i = \pm 1, \prod_{i=1}^{k} \epsilon_i = 1 \right\}.$$

It is a trivial calculation to check that the polynomials $X_i^2 - T^2$ and $X_I - X_J$ (resp. $X_I - X_J T$), as defined above vanish on $W.x_L \times \{x_L\} \subset \mathfrak{s} \times \mathfrak{h}$. Thus, the map (7.3) factors through $A_S(k)$.

Since $\mathcal{F}'_N$ is equivariantly formal in the sense of [19], $H_S(\mathcal{F}'_N)$ is naturally a free $S(\mathfrak{s}^*)$-module of rank $|W/S_k| = 2^{k-1}$, which is the number of S-fixed points, [24, Lemma 2.1]. On the other hand, $A_S(k)$ is also a free module over $\mathbb{C}[T] = S(\mathfrak{s}^*)$ with basis given by all X_I for all I such that $|I| \leq \frac{k-1}{2}$ in the case where k is odd, and $|I| < \frac{k}{2}$ or $|I| = \frac{k}{2}$ with $k \in I$ in the case where k is even. The rank equals

$$\begin{cases} \sum_{l=0}^{\frac{k-1}{2}} \binom{k}{l} = 2^{k-1} & \text{if } k \text{ is odd,} \\ \frac{1}{2}\binom{k}{k/2} + \sum_{l=0}^{\frac{k}{2}-1} \binom{k}{l} = 2^{k-1} & \text{if } k \text{ is even.} \end{cases}$$

Thus, (7.3) induces an isomorphism between $A_S(k)$ and $H_S(\mathcal{F}'_N)$. Again, using equivariant formality, we get $H(\mathcal{F}'_N) \cong H_S(\mathcal{F}'_N) \otimes_{\mathbb{C}[T]} \mathbb{C}$, and part (ii) follows. It remains to verify the surjectivity assumption of [24], which is Proposition 7.2. ∎

Proposition 7.2 *Let $N \in \mathfrak{g}$ be a nilpotent endomorphism of $\mathbb{C}^{2k}$ with Jordan type (k, k). Then the canonical map $H(\mathcal{F}) \to H(\mathcal{F}_N)$ is surjective. The same holds for the canonical map $H(\mathcal{F}') \to H(\mathcal{F}'_N)$.*

Proof We first claim that the top degree is completely contained in the image of the canonical map for any nilpotent of admissible Jordan type (r, s) of type D_k. Recall from Section 2 that for $r = s = k$ even, the component group is trivial. Then the claim is a special case of a general result of Hotta and Springer [21, Theorem 1.1, Corollary 1.3]. In case of odd $r = s = k$, the component group is $\mathbb{Z}/2\mathbb{Z}$ and swaps between the two connected components of $\mathcal{F}_N$, see [53, Section 3]. By Theorem 6.5 and Lemma 5.20, the total number of irreducible components equals twice the dimension of the corresponding irreducible representation of $W(D_k)$ attached to N via the Springer correspondence. Hence, when we replace $\mathcal{F}$ by the set $\mathcal{B}$ of Borel subgroups and consider the top degree of $H(\mathcal{B}_N)$ for the corresponding fixed point variety $\mathcal{B}_N$, it carries the structure of an irreducible $W(D_k)$-module. Hence, the statement follows again by [21, Corollary 1.3]. The remaining case $r \neq s$ is similar.

Let now N be of type (k, k). We claim there is a filtration by subvarieties

$$(7.4) \qquad \mathcal{Y}_0 \subset \mathcal{Y}_1 \subset \cdots \subset \mathcal{Y}_{\frac{k-\epsilon}{2}} = \mathcal{F}_N$$

(with $\epsilon = 0$ or $\epsilon = 1$ depending on the parity of k) such that $\mathcal{Y}_j = F_{N_j}$, where the Jordan type of N is (k, k) if $j = k/2$ and $(2k - 2j - 1, 2j + 1)$ otherwise. To see this consider the labelling set, $\mathcal{C}_j$, of $\operatorname{Irr}(\mathcal{F}_{N_j})$ by cup diagrams. The diagrams in $\mathcal{C}_j$ have exactly one cup less than the ones in $\mathcal{C}_{j+1}$ and can all be obtained (in a nonunique way) by replacing a cup by two rays in some diagram from $\mathcal{C}_{j+1}$. Now recall the inductive construction of the components. Each cup corresponded to a $\mathbb{P}^1(\mathbb{C})$-choice. If we interpret replacing a cup by two rays as picking the nongeneric point in $\mathbb{P}^1(\mathbb{C})$, then this gives exactly the construction of the components of the smaller dimensional Springer fibres and thus constructs the required chain of subvarieties. By construction, it is independent of the choices and the claim follows. Now each $\mathcal{Y}_i$ is equidimensional of complex dimension i, so that it contributes only to $H^{2i}(\mathcal{F}_N)$, the degree $2i$ in cohomology. Moreover, by the beginning of the proof the subspace spanned by the classes of the irreducible components is in the image of the canonical map $H(\mathcal{F}) \to H(\mathcal{F}_N)$. Hence, it remains to show that all of these (linearly independent) classes span $H(\mathcal{F}_N)$. We have $\dim H(\mathcal{F}_N) = 2^k$, since $\mathcal{F}_N$ has a cell partition with $2 \cdot 2^{k-1}$ strata by [24, Lemma 2.1]. On the other hand, the parametrization of irreducible components from Theorem 6.2 together with Lemmas 5.12 and 5.20 gives

$$\sum_{l=0}^{\frac{k-\epsilon}{2}} |\operatorname{Irr}(\mathcal{Y}_l)| = \begin{cases} 2\sum_{l=0}^{\frac{k-1}{2}} \binom{k}{l} = \sum_{l=0}^{k} \binom{k}{l} = 2^k & \text{if } k \text{ is odd,} \\ \binom{k}{k/2} + 2\sum_{l=0}^{\frac{k}{2}-1} \binom{k}{l} = \sum_{l=0}^{k} \binom{k}{l} = 2^k & \text{if } k \text{ is even.} \end{cases}$$

Hence, the canonical map is surjective.   ∎

Remark 7.3   The last steps of this proof follow more directly from a general result of Lusztig [29], providing an isomorphism $H(\mathcal{F}'_N) \cong \operatorname{Ind}_{S_k}^{W(D_k)} \mathbf{1}$ as $W(D_k)$-modules. The filtration (7.4) then corresponds to the decomposition (5.3) into irreducible summands. The two possible choices of embedding S_k into $W(D_k)$ (using $s_i, i \neq 0$ or $s_i, i \neq 1$) give for even k the two different representations labelled $(k/2), (k/2)$ in top degree.

Remark 7.4   The action of $s_i, i > 0$, on $H(\mathcal{F}'_N)$ is given by permuting the variables, and s_0 acts by swapping X_1 and $-X_2$. It is induced from the natural left action on the first factor in (7.1). The different representations $S^{(k/2),(k/2),\pm}$ arise in top degree depending on the two possible choices of $\mathfrak{p}$ of which we chose one.

7.2   Cohomology of Topological Springer Fibre

We are now ready to prove Theorem 1.2.

Theorem 7.5   *Let $N \in \mathfrak{g}$ be a nilpotent endomorphism of $\mathbb{C}^{2k}$ with Jordan type (k, k). Then we have isomorphisms of graded rings $H(\widetilde{S}) \cong H(\mathcal{F}_N)$.*

Proof   Recall that via the isomorphism τ from Theorem 4.10 the ring $H(\widetilde{S})$ is identified with the centre of the algebra $\mathbb{K}_k$, which in turn is isomorphic to $H(\mathcal{F}_N)$ by [10].   ∎

8 Jordan Type (k, k): Category $\mathcal{O}$ and $\mathcal{W}$-algebras

Let now $N \in \mathfrak{g}$ be nilpotent of Jordan type (k, k). In this case the S-fixed points $\mathcal{F}_N^S$ of $\mathcal{F}_N$ are easy to determine, namely all flags $F_\bullet = F_\bullet(I)$ where for each $1 \le j \le k$ we have $F_j = \langle w_l \mid l \in I \rangle$ where I is a union $I = I_- \cup I_+$ of two sets of consecutive indices of the form

$$I_- = \{-k, -k+1, -k+2, \ldots, -k+r+1\}, \quad I_+ = \{1, 2, \ldots, s\}$$

such that $|I| = r + s = j$. (These are obviously fixed points and there are altogether $2 \cdot 2^{k-1} = 2^k$ such choices, which is the desired number by [24, Lemma 2.1].)

In the sequel, we will identify S-fixed points with combinatorial weights via the following easy observation.

Lemma 8.1 *There is a bijection between fixed points and combinatorial weights of length k sending $F_\bullet(I) \in \mathcal{F}_N$ to the combinatorial weight $\lambda = \lambda(I)$ which has at vertex i an $\wedge$ if $-i \in I$ and a $\vee$ if $i \in I$.*

The following proposition generalizes [51, Lemma 12].

Proposition 8.2 *Let N be of Jordan type (k, k). Then the following hold.*

(i) *The fixed points $F_\bullet(I) \in \mathcal{F}_N^S$ contained in an irreducible component Y are precisely the combinatorial weights $\lambda = \lambda(I)$ such that λc is an oriented cup diagram, where c denotes the cup diagram associated with Y via Theorem 6.5 and Corollary 5.13.*

(ii) *The fixed points $P(I)$ contained in the intersection $Y \cap Y'$ of two irreducible components are precisely the combinatorial weights $\lambda = \lambda(I)$ such that $c^* \lambda d$ is an oriented circle diagram, where c (resp. d) denotes the cup diagram associated with Y and Y'.*

From the constructions we directly obtain a refinement of Theorem 6.5.

Corollary 8.3 *Pairwise intersections of components are either empty or again iterated $\mathbb{P}^1(\mathbb{C})$-bundles, namely of base type $(\mathbb{P}^1(\mathbb{C}))^c$, where c is the number of circles in the corresponding circle diagram.*

Remark 8.4 Each $P \in \mathcal{F}_N^S$ defines a vector of $H(\mathcal{F}_N)$ as follows. Take the weight λ of length k identified with P. This defines the unique cup diagram $c = c(\lambda)$ on k vertices such that λc is oriented and of degree 0, see [28]. Using the filtration (7.4) it defines a subvariety of $\mathcal{F}_N$ whose class defines a vector in cohomology. By the proof of Theorem 7.2, these vectors form indeed a basis.

Example 8.5 Consider $\mathcal{F}_N$, where N has Jordan type $(4, 4)$. The S-fixed points in the pairwise intersections of irreducible components labelled by $\mathbb{B}_4^{\text{odd}}$ are displayed

in the following table (the case $\mathbb{B}_4^{\mathrm{even}}$ is similar).

	(cup diagram)	(cup diagram)	(cup diagram)
(cup diagram)	$\wedge\wedge\vee\wedge \quad \vee\vee\vee\wedge$ $\wedge\wedge\wedge\vee \quad \vee\vee\wedge\vee$	$\wedge\wedge\vee\wedge$ $\vee\vee\wedge\vee$	empty intersection
(cup diagram)	$\wedge\wedge\vee\wedge$ $\vee\vee\wedge\vee$	$\wedge\vee\wedge\wedge \quad \vee\wedge\vee\vee$ $\wedge\vee\wedge\wedge \quad \vee\wedge\vee\vee$	$\wedge\vee\wedge\wedge$ $\vee\wedge\vee\vee$
(cup diagram)	empty intersection	$\wedge\vee\wedge\wedge$ $\vee\wedge\vee\vee$	$\vee\wedge\wedge\wedge \quad \wedge\vee\wedge\wedge$ $\vee\wedge\vee\vee \quad \wedge\vee\vee\vee$

Proof of Proposition 8.2   For part (i) we assume first that k is even. In addition, assume that the signed domino tableau attached to Y has only one cluster. By construction of the component Y (Case IIIa) in (6.3)) we choose the basis $y_{-k}, \ldots, y_{-1}, y_1, \ldots, y_k$, where

$$y_{-j} = \alpha w_{-j} + (-1)^{j+1}\beta w_{k+1-j} \quad \text{and} \quad y_j = (-1)^{j+1}\alpha w_{j-k-1} + \beta w_j$$

for $1 \le j \le k$ and have $F_1 = \langle y_1 \rangle$. To describe F_2, we apply Case IIb) of (6.3) and choose a basis of $F_1^\perp/F_1$ of the form $z_{k-1}, \ldots, z_{-1}, z_1, \ldots, z_{k-1}$, where $z_{-j} = (-1)^{j+1}y_j + y_{k-j}$ and $z_j = (-1)^{j+1}y_{j-k} - y_j$ for $1 \le j \le k-1$ such that $F_2 = F_1 + \langle \delta z_{-(k-1)} + \gamma z_1 \rangle$ for $\delta, \gamma \in \mathbb{C}$. Now $\delta z_{-(k-1)} + \gamma z_1 = (\gamma + \delta)y_{-(k-1)} + (\delta - \gamma)y_1$, thus choosing $\gamma = \delta$ gives us $y_{-(k-1)}$ and $\gamma = -\delta$ gives us y_1. Now, choosing $(\alpha, \beta) \in \{(0,1),(1,0)\}$ finally gives us the required w's. These arguments can be repeated to give the claim for the standard tableau part. For the remaining vertical domino we have $F_k = F_{k-1} + \langle b + b' \rangle$ if the sign is $-$ and $F_k = F_{k-1} + \langle b - b' \rangle$ if the sign is $+$, where for $b = \gamma z_{-k/2} + \delta z_{k/2}$ and $b' = \delta z_{-k/2} + \gamma z_{k/2}$, hence $b + b' = (\gamma + \delta)z_{-k/2} + (\gamma - \delta)z_{k/2}$ and $b - b' = (\gamma - \delta)z_{-k/2} + (\gamma + \delta)z_{k/2}$. Hence, choosing $\gamma = \delta$ or $\gamma = -\delta$ and α, β as above shows part (i) in case there is only one cluster. In the case where there is more than one cluster, we start as above with the first cluster and then repeat the same arguments for the space W/F_k with the basis given by the remaining w's (*i.e.*, in case the first cluster had shape (s, s) these are the w_i's where $-k + s/2 \le i \le -1$ and $s/2 + 1 \le i \le k$ if the sign was $+$, and $-k + s/2 + 1 \le i \le -1$ and $s/2 \le i \le k$ if the sign was $-$) etc. This settles the case for even k. For k odd we argue as for $k + 1$ in the even case, but remove afterwards the rightmost vertical domino that destroys half of the fixed points and the claim follows again.

    Part (ii) follows directly from part (i) and the definition of oriented circle diagrams.   ■

Remark 8.6   Note that Proposition 8.2 is, in general, wrong for Jordan types (r, s) where $r \ne s$. Consider Section 6.5 for a counterexample where the intersections of the algebraic Springer are not given by the cup diagram combinatorics.

8.1   Connections with the Algebra $\mathbb{K}_k$ and Category $\mathcal{O}$

As a consequence we obtain Theorem 1.4.

Theorem 8.7   Let $N \in \mathfrak{g}$ be nilpotent of Jordan type (k, k). Then there exists an isomorphism of graded vector spaces

$$\mathbb{K}_k = \bigoplus_{(Y_1, Y_2) \in \mathrm{Irr}(\mathcal{F}_N) \times \mathrm{Irr}(\mathcal{F}_N)} H(Y_1 \cap Y_2) \big\langle d\big(a(Y_1), a(Y_2) \big) \big\rangle$$

where $a(Y_i)$, $i = 1, 2$ denotes the cup diagram associated with Y_i.

Proof   By Theorem 7.1 we have an isomorphism of vector spaces $H(\mathcal{F}_N) \cong \mathbb{C}^{|\mathcal{F}_N^S|}$ with basis naturally labelled by the S-fixed points. Hence, there is at least an isomorphism of vector spaces by Proposition 8.2 and the definition of $\mathbb{K}_k$. The filtration (7.4) determines the degree of each basis vector, and the theorem follows easily from the definition of the grading on $\mathbb{K}_k$.   ∎

Remark 8.8   Since circle diagrams encode the (D_k, A_{k-1})-Kazhdan–Lusztig combinatorics, Theorem 8.7 implies that the combinatorics of intersection of components in $\mathcal{F}_N$ for the equal length two row case is controlled by Kazhdan–Lusztig polynomials.

8.2 Connections with $\mathcal{W}$-algebras and Category $\mathcal{O}$

Let $e \in \mathfrak{g}$ be nilpotent of Jordan type 2^k for even k and consider the associated $\mathcal{W}$-algebra $\mathcal{W}(e)$ of finite type as introduced originally by Premet, [36]. Let $\mathcal{F}in_0(e)$ be the abelian category of finite dimensional $\mathcal{W}(e)$-modules with trivial central character. By a result of Brown [5], the simple objects in $\mathcal{F}in_0(e)$ are naturally indexed by the set $\mathcal{P}_k$ of s-tables of shape 2^k.

An *s-table* is a Young diagram of shape (2^k) with a filling with the numbers $\pm 1, \ldots, \pm k$ such that the columns are strictly decreasing from top to bottom and the rows are strictly increasing from left to right. Moreover the filling needs to be skew-symmetric in the sense that the i-th entry in the first column equals the negative of the $(k - i + 1)$-th entry in the second column.

For an s-table P we denote by $\mathfrak{I} = \mathfrak{I}(P)$ the set of numbers in the second column and encode the associated type D_k weight as in [28] as the combinatorial weight with the i-th symbol an $\wedge$ if $i \in \mathfrak{I}$ and a $\vee$ if $-i \in \mathfrak{I}$, that is the weight $\lambda = \lambda(-\mathfrak{I}(P))$ in the notation of Lemma 8.1. To this weight λ we can assign the associated cup diagram $c(\lambda)$ as in Remark 8.4. Hence, we get an assignment $\mathfrak{I} \mapsto \mathrm{Cup}(\mathfrak{I})$ sending an s-table to a cup diagram. For instance, $\mathfrak{I} = \{1, 2, 3, 4\}$, $\mathfrak{I} = \{1, -2, 3, 4\}$, respectively $\mathfrak{I} = \{-1, 2, -3, 4\}$ define the cup diagrams in (1.1). It is easy to check for a weight λ that $c(\lambda) \in \mathbb{B}_k$ if and only if strictly to the left of each $\vee$ in λ there is at least one more $\wedge$ than $\vee$.

Lemma 8.9   The assignment $\mathcal{P}(\mathfrak{I}) \mapsto \mathrm{Cup}(\mathfrak{I})$ defines a bijection $\mathcal{P}_k \overset{1:1}{\leftrightarrow} \mathbb{B}_k$.

Proof   Let $P \in \mathcal{P}_k$. If there is a $\vee$ in $\mathrm{Cup}(P)$ at position i, then P has i in column 1 and fewer numbers bigger than i in column 1 than in 2. Hence the map is well-defined and obviously injective. To see that it is surjective, we have to show that for any $D \in \mathbb{B}_k$ the associated filling P defines an s-table. This is clear if D has only dotted cups, as then $I = \{1, 2, \ldots, k\}$. Otherwise, choose an undotted cup c connecting vertices i and $i + 1$. Increase the numbers below $-(i + 1)$ in the first column by 2 and decrease

the numbers above $i + 1$ in the second column by 2 and then remove $\pm i$ and $\pm(i + 1)$ from the filling. The result is the filling associated with D with c removed. The claim follows by induction observing that the base holds for even k.   ∎

Let $\mathfrak{p}$ and $\mathfrak{p}'$ be the two parabolic subalgebras in $\mathfrak{g}$ with Weyl group generated by s_i, $i \neq 0$ and $i \neq 1$, respectively. Let $\mathcal{O}_0^{\mathfrak{p}}(\mathfrak{g})$ and $\mathcal{O}_0^{\mathfrak{p}'}(\mathfrak{g})$ be the corresponding principal blocks of parabolic category $\mathcal{O}$. By [10], the (isomorphism classes) of indecomposable projective-injective modules are naturally labelled by $\mathbb{B}_k^{\mathrm{even}}$ respectively $\mathbb{B}_k^{\mathrm{odd}}$ and have all the same Loewy-length namely $2k + 1$. (Cup diagrams on k vertices label all indecomposable projective modules. The number of cups encodes, as in [15, Proposition 58], the GK-dimension of the corresponding simple quotient, and induces a filtration categorifying the decomposition (5.3) by taking successive subquotient categories.) Now Conjecture 1.5 is implied by [48, Theorem 10.1]

Corollary 8.10 *Let $\mathcal{O} := \mathcal{O}^{\mathfrak{p}}(\mathfrak{g})_0 \oplus \mathcal{O}^{\mathfrak{p}'}(\mathfrak{g})_0$ and consider $P := \oplus P(\mathrm{Cup}(\mathcal{I}))$, where I runs over all s-tables. Then $\mathrm{Hom}_{\mathcal{O}}(P, ?) \colon \mathcal{O} \to \ \mathrm{mod} - \mathrm{End}_{\mathcal{O}}(P)$ is fully faithful on projective objects. Assuming Conjecture 1.5 this defines via the identification from Lemma 8.9 a functor $\mathrm{Hom}_{\mathcal{O}}(P, ?) \colon \mathcal{O} \to \mathcal{F}in_0(e)$, again fully faithful on projective objects. In particular, $\mathcal{F}in_0(e)$ is a quotient category of $\mathcal{O}$.*

Remark 8.11 The indecomposable projective objects in $\mathcal{F}in_0(e)$ are also in bijection with orbital varieties of G, since k is even. Identifying the two equivalent summands of $\mathcal{O}$ corresponds then to passing to orbital varieties for the adjoint group. We normalized our setup using cup diagrams with an even and odd number of dots such that they describe precisely the fibres of the Spaltenstein-Steinberg map; see [31, Theorem 3].

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