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THE UNIVERSAL ENVELOPING TERNARY RING OF OPERATORS OF A JB*-TRIPLE SYSTEM

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Abstract We associate to every JB*-triple system a so-called universal enveloping ternary ring of operators (TRO). We compute the universal enveloping TROs of the finite dimensional Cartan factors.

Keywords: JC*-triple system; J^*-algebra; Cartan factor; ternary ring of operators; ternary algebra; universal enveloping TRO

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1. Introduction

This paper is part of a project by the authors that aims to show that Cartan's classification of the (Hermitian) symmetric spaces has a K-theoretic background. This project will be concluded in the follow-up paper [**3**].

The symmetric spaces that are discussed here consist of the open unit balls of so-called JB*-triples, an important generalization of the concept of a C^*-algebra. If the dimension is finite, their open unit balls coincide exactly with the Hermitian symmetric spaces of non-compact type so that all of these spaces are obtained through duality.

In the present paper we overcome a difficulty that is one of the main obstacles for a direct generalization of the K-theory of C^*-algebras: the impossibility, in general, of defining tensor products of a JB*-triple with $n \times n$ matrices over the complex numbers. The JB*-triples that do have this property are precisely the ternary rings of operators that coincide as spaces with the class of (full) Hilbert C^*-modules.

We will study a construction which allows the passage from an arbitrary JB*-triple to such a ternary ring in a way that behaves so nicely that it will pave the way for the programme ahead. In § 2 we will collect some definitions and preliminary results, § 3 contains the actual construction of the enveloping ternary ring of operators, and in § 4 we calculate the enveloping ternary rings of all finite-dimensional Cartan factors. We do this quite differently from the approach in [**5**] (the results of which were roughly obtained around the same time) in that we use *grids*. These objects will be helpful in the sequel paper [**3**], and are reminiscent of the root systems that are central to the classical approach. Finally, in § 5, we slightly improve a result from [**5**] on the structure of the enveloping ternary ring in some special cases.

All the results in the present paper are taken from [**2**].

2. Preliminaries

We will first provide some notation, definitions and well-known facts of triple theory. Our general references for the theory of JB*-triple systems are [**11**] and [**19**]. For $n \in \mathbb{N}$, we denote by $\mathbb{M}_n$ the $n \times n$ matrices over the complex numbers, and if Z is a Banach space, then $B(Z)$ is the Banach algebra of bounded linear operators on Z. A Banach space Z together with a sesquilinear mapping

$$Z \times Z \ni (x, y) \mapsto x \,\square\, y \in B(Z)$$

is called a JB*-*triple system* if, for the triple product

$$\{x, y, z\} := (x \,\square\, y)(z)$$

and all $a, b, x, y, z \in Z$, the following conditions are satisfied.

The triple product $\{x, y, z\}$ is continuous in (x, y, z), it is symmetric in the outer variables and the C^*-condition $\|\{x, x, x\}\| = \|x\|^3$ is satisfied. Moreover, the *Jordan triple identity*

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$$

holds, the operator $x \,\square\, x$ has non-negative spectrum in the Banach algebra $B(Z)$, and it is Hermitian (i.e. $\exp(\mathrm{i}t(x \,\square\, x))$ is isometric for all $t \in \mathbb{R}$).

A closed subspace W of a JB*-triple system Z which is invariant under the triple product, and therefore is a JB*-triple system itself, is called a JB*-*subtriple* (or subtriple for short) of Z.

A closed subspace I of a JB*-triple system Z is called a JB*-*triple ideal* if $\{Z, I, Z\} + \{I, Z, Z\} \subseteq I$. JB*-triple ideals of Z are JB*-subtriples of Z, and the kernel of a JB*-triple homomorphism is always a JB*-triple ideal.

Every C^*-algebra $\mathfrak{A}$ becomes a JB*-triple system under the product

$$\{a, b, c\} := \tfrac{1}{2}(ab^*c + cb^*a). \tag{2.1}$$

More generally, every closed subspace of a C^*-algebra that is invariant under the product (2.1) is a JB*-triple, called a JC*-*triple system*. A JB*-triple system Z that is a dual Banach space is called a JBW*-*triple system*. Its predual is usually denoted by Z_*. The triple product of a JBW*-triple is separately $\sigma(Z, Z_*)$-continuous and its predual is unique.

An important example of JB*-triples is given by the *ternary rings of operators* (TROs). These are closed subspaces $T \subseteq B(H)$ such that

$$xy^*z \in T \tag{2.2}$$

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for all $x, y, z \in T$. *SubTROs* are closed subspaces $U \subseteq T$ closed under (2.2) and *TRO-ideals* are subTROs I of T such that $IT^*T + TI^*T + TT^*I \subseteq I$. TROs become JB*-triples under the product (2.1).

Let Z be a JB*-triple system. An element $e \in Z$ that satisfies $\{e, e, e\} = e$ is called a *tripotent*. The collection of all non-zero tripotents in Z is denoted by $\mathrm{Tri}(Z)$. A tripotent is called *minimal* if $\{e, Z, e\} = \mathbb{C}e$. If e is a non-zero tripotent, then e induces a decomposition of Z into the eigenspaces of $e \,\Box\, e$, the *Peirce decomposition*

$$Z = P_0^e(Z) \oplus P_1^e(Z) \oplus P_2^e(Z),$$

where $P_k^e(Z) := \{z \in Z \colon \{e, e, z\} = \frac{1}{2}kz\}$ is the $\frac{1}{2}k$-eigenspace, the *Peirce k-space*, of $e \,\Box\, e$, for $k = 0, 1, 2$. Each Peirce k-space, $k = 0, 1, 2$, is again a JB*-triple system. In the case of a TRO T, the Peirce 2-space $P_2^e(T)$ becomes a unital C^*-algebra under the product $a \bullet b := ae^*b$, denoted by $P_2^e(T)^{(e)}$.

Every finite-dimensional JB*-triple system Z is the direct sum of so-called Cartan factors $\mathcal{C}_1, \dots, \mathcal{C}_6$. The two exceptional Cartan factors $\mathcal{C}_5$ and $\mathcal{C}_6$ can be realized as subspaces of the 3×3 matrices over the complex Cayley algebra $\mathbb{O}$, and we call Z purely exceptional if it is composed of these two alone. Note that these JB*-triple systems admit no embedding into a space of bounded Hilbert space operators. The other four types are treated in detail in § 5.

3. Universal objects

We prove the existence of the universal enveloping TRO and the universal enveloping C^*-algebra of a JB*-triple system. As a corollary, we obtain a new proof of one of the main theorems of JB*-triple theory.

The following lemma and theorem are generalizations of classical results for real JB-algebras (cf. [**9**, Theorem 7.1.3] and [**1**, Theorem 4.36]).

Lemma 3.1. *Let Z be a JB*-triple system. Then there exists a Hilbert space H such that for every JB*-triple homomorphism $\varphi\colon Z \to B(K)$ the C^*-algebra $\mathfrak{A}_\varphi$ generated by $\varphi(Z)$ can be embedded $*$-isomorphically into $B(H)$.*

Proof. The cardinality of $\varphi(Z)$ is less than or equal to the cardinality of Z. One can now proceed with a proof similar to that of [**1**, Lemma 4.35]. □

Theorem 3.2. *Let Z be a JB*-triple system.*

(a) *There exist, up to $*$-isomorphism, a unique C^*-algebra $C^*(Z)$ and a JB*-triple homomorphism $\psi_Z \colon Z \to C^*(Z)$ such that*

   (i) *for every JB*-triple homomorphism $\varphi\colon Z \to \mathfrak{A}$, where $\mathfrak{A}$ is an arbitrary C^*-algebra, there exists a $*$-homomorphism $C^*(\varphi)\colon C^*(Z) \to \mathfrak{A}$ with $C^*(\varphi) \circ \psi_Z = \varphi$,*

   (ii) *$C^*(Z)$ is generated as a C^*-algebra by $\psi_Z(Z)$.*

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(b) *There exist, up to TRO-isomorphism, a unique TRO* $T^*(Z)$ *and a* JB**-triple homomorphism* $\rho_Z\colon Z \to T^*(Z)$ *such that*

(i) *for every* JB**-triple homomorphism* $\alpha\colon Z \to T$*, where* T *is an arbitrary TRO, there exists a TRO-homomorphism* $T^*(\alpha)\colon T^*(Z) \to T$ *with* $T^*(\alpha) \circ \rho_Z = \alpha$,

(ii) $T^*(Z)$ *is generated as a TRO by* $\rho_Z(Z)$.

Proof. Let H be the Hilbert space from Lemma 3.1 and let I be the family of JB*-triple homomorphisms from Z to $B(H)$. Let $\psi_Z := \rho_Z := \bigoplus_{\psi\in I} \psi$ and $\hat{H} := \bigoplus_{\psi\in I} H_\psi$ be l^2-direct sums with $H_\psi := H$. Then ψ_Z (and ρ_Z) are JB*-triple homomorphisms from Z to $B(\hat{H})$. Let $C^*(Z)$ be the C^*-algebra and $T^*(Z)$ be the TRO generated by $\rho_Z(Z)$ in $B(\hat{H})$. If $\mathfrak{A}$ is a C^*-algebra and $\varphi\colon Z \to \mathfrak{A}$ is a JB*-triple homomorphism, where $\varphi(Z)$ without loss of generality generates $\mathfrak{A}$ as a C^*-algebra, then we can suppose (by Lemma 3.1) that $\mathfrak{A}$ is a subalgebra of $B(H)$. Therefore, φ can be regarded as an element of I. Let $\pi_\varphi\colon \bigoplus_{\psi\in I} B(H_\psi) \to B(H_\varphi)$ be the projection onto the φ-component. Then $\pi_\varphi(\psi_Z(z)) = \pi_\varphi(\rho_Z(z)) = \varphi(z)$ for all $z \in Z$. We define $C^*(\varphi)$ and $T^*(\varphi)$ to be the restrictions of π_φ to $C^*(Z)$ and $T^*(Z)$, respectively. Uniqueness is proved in the usual way using the universal properties. □

We call $(T^*(Z), \rho_Z)$ the *universal enveloping TRO* and $(C^*(Z), \psi_Z)$ the *universal enveloping* C^**-algebra* of Z. Most of the time we only use the notation $T^*(Z)$ and $C^*(Z)$ for brevity.

Similar to the classical case [**1**, Proposition 4.40], there exists a TRO-antiautomorphism on $T^*(Z)$.

Proposition 3.3. *Let* Z *be a* JB**-triple system. There exists a TRO-antiautomorphism* θ *(i.e. a linear, bijective mapping from* $T^*(Z)$ *to* $T^*(Z)$ *such that* $\theta(xy^*z) = \theta(z)\theta(y)^*\theta(x)$ *for all* $x, y, z \in T^*(Z)$*) of* $T^*(Z)$ *of order* 2 *such that* $\theta \circ \rho_Z = \rho_Z$.

Proof. Denote by $T^*(Z)^{\mathrm{op}}$ the opposite TRO of $T^*(Z)$, i.e. the TRO that coincides with $T^*(Z)$ as a set and is equipped with the same norm. If $\gamma\colon T^*(Z) \to T^*(Z)^{\mathrm{op}}$, $\gamma(a) = a^{\mathrm{op}}$ denotes the (formal) identity mapping, then $(xy^*z)^{\mathrm{op}} = z^{\mathrm{op}}(y^{\mathrm{op}})^*x^{\mathrm{op}}$ for all $x, y, z \in T^*(Z)$.

The composed mapping $\gamma \circ \rho_Z\colon Z \to T^*(Z)^{\mathrm{op}}$ is a JB*-triple homomorphism and thus lifts to a TRO-homomorphism $T^*(\gamma \circ \rho_Z)\colon T^*(Z) \to T^*(Z)^{\mathrm{op}}$. We set

$$\theta := \gamma^{-1} \circ T^*(\gamma \circ \rho_Z)\colon T^*(Z) \to T^*(Z).$$

It can easily be seen (since, by construction, θ fixes $\rho_Z(Z)$, which generates $T^*(Z)$ as a TRO) using the universal properties of $T^*(Z)$ that θ is a TRO-antiautomorphism of order 2. □

We refer to θ as the *canonical TRO-antiautomorphism of order* 2 on $T^*(Z)$.

Corollary 3.4. *If the* JB**-triple system* Z *in Theorem 3.2 is a* JC**-triple, then the mappings* ψ_Z *and* ρ_Z *are injective.*

Obviously, ψ_Z and ρ_Z are the 0 mappings if Z is purely exceptional.

Lemma 3.5. *For every* JB*-*triple ideal* I *in a* JB*-*triple system* Z *and every* JB*-*triple homomorphism* $\varphi\colon I \to W$, *where* W *is a* JBW*-*triple system, there exists a* JB*-*triple homomorphism* $\Phi\colon Z \to W$ *that extends* φ.

Proof. We know from [**7**] that the second dual Z'' of Z is a JBW*-triple system and that the canonical embedding $\iota\colon Z \to Z''$ is an isometric JB*-triple isomorphism onto a norm closed w^*-dense subtriple of Z''. By [**4**, Remark 1.1], and since W is a JBW*-triple system, there exists a unique, w^*-continuous extension $\bar{\varphi}\colon I'' \to W$ of φ with $\bar{\varphi}(I'') = \overline{\varphi(I)}^{w^*}$. Let

$$I''^{\perp} := \{x \in Z''\colon y \mapsto \{x, i, y\} \text{ is the 0 mapping for all } i \in I''\}$$

be the w^*-closed orthogonal complement of I'' with $Z'' = I'' \oplus I''^{\perp}$ (cf. [**10**, Theorem 4.2 (4)]). If we denote the projection of Z'' onto I'' by π, we get the desired extension of φ by defining $\Phi := \bar{\varphi} \circ \pi \circ \iota$. □

We obtain a new proof of an important theorem of Friedman and Russo (see [**8**, Theorem 2]).

Corollary 3.6. *Any* JB*-*triple system* Z *contains a unique purely exceptional ideal* J *such that* Z/J *is* JB*-*triple isomorphic to a* JC*-*triple system.*

Proof. Let J be the kernel of the mapping $\rho_Z\colon Z \to T^*(Z)$, which is a JB*-triple ideal. We know that Z/J is a JB*-triple system that is JB*-triple isomorphic to the JB*-triple system $\rho_Z(Z) \subseteq T^*(Z)$ and hence to a JC*-triple system.

Let us assume that J is not purely exceptional, which means that there exists a non-zero JB*-triple homomorphism φ from J into some $B(H)$. By Lemma 3.5, this JB*-triple homomorphism extends to a JB*-triple homomorphism $\phi\colon Z \to B(H)$. Since $\phi = T^*(\phi) \circ \rho_Z$ holds, ϕ vanishes on J, which is a contradiction.

Now let I be another purely exceptional ideal such that Z/I is JB*-triple isomorphic to a JC*-triple system. On the one hand we have $I \subseteq \ker(\rho_Z) = J$. On the other hand, let $\varphi\colon Z \to B(H)$ be a JB*-triple homomorphism with kernel I. Then φ has to vanish on J and therefore $J \subseteq I$. □

4. Cartan factors

In this section we compute the universal enveloping TROs of the finite-dimensional Cartan factors. Since the universal enveloping TROs of the two exceptional factors are 0, we have to compute the factors of types I–IV. We do so by using the grids spanning these factors (cf. [**6**] and [**2**, Chapter 2]). We make much use of the elaborate work on grids in [**16**].

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4.1. Factors of type IV

A *spin system* is a subset $S = \{\mathrm{id}, s_1, \dots, s_n\}$, $n \geqslant 2$, of self-adjoint elements of $B(H)$ that satisfy the anti-commutator relation $s_i s_j + s_j s_i = 2\delta_{i,j}$ for all $i, j \in \{1, \dots, n\}$. The complex linear span of S is a JC*-algebra of dimension $n+1$ (cf. [**9**]). Every JC*-triple system that is JB*-isomorphic to such a JC*-algebra is called a *spin factor*. We now recall the definition of a spin grid: a *spin grid* is a collection $\{u_j, \tilde{u}_j \mid j \in J\}$ (or $\{u_j, \tilde{u}_j \mid j \in J\} \cup \{u_0\}$ in finite odd dimensions), where J is an index set with $0 \notin J$, for $j \in J$, u_j, $\tilde{u}_j$ are minimal tripotents and, if we let $i, j \in J$, $i \neq j$, then

(SPG1) $\{u_i, u_i, \tilde{u}_j\} = \frac{1}{2}\tilde{u}_j$, $\{\tilde{u}_j, \tilde{u}_j, u_i\} = \frac{1}{2}u_i$,

(SPG2) $\{u_i, u_i, u_j\} = \frac{1}{2}u_j$, $\{u_j, u_j, u_i\} = \frac{1}{2}u_i$,

(SPG3) $\{\tilde{u}_i, \tilde{u}_i, \tilde{u}_j\} = \frac{1}{2}\tilde{u}_j$, $\{\tilde{u}_j, \tilde{u}_j, \tilde{u}_i\} = \frac{1}{2}\tilde{u}_i$,

(SPG4) $\{u_i, u_j, \tilde{u}_i\} = -\frac{1}{2}\tilde{u}_j$,

(SPG5) $\{u_j, \tilde{u}_i, \tilde{u}_j\} = -\frac{1}{2}u_i$,

(SPG6) all other products of elements from the spin grid are 0.

In the case of finite odd dimensions (where u_0 is present) we have, for all $i \in J$, the additional conditions (as exceptions of (SPG6))

(SPG7) $\{u_0, u_0, u_i\} = u_i$, $\{u_i, u_i, u_0\} = \frac{1}{2}u_0$,

(SPG8) $\{u_0, u_0, \tilde{u}_i\} = \tilde{u}_i$, $\{\tilde{u}_i, \tilde{u}_i, u_0\} = \frac{1}{2}u_0$,

(SPG9) $\{u_0, u_i, u_0\} = -\tilde{u}_i, \{u_0, \tilde{u}_i, u_0\} = -u_i$.

It is known (see [**6**]) that every finite-dimensional spin factor is linearly spanned by a spin grid (but not necessarily by a spin system).

Let $\mathfrak{G} := \{u_i, \tilde{u}_i \colon i \in I\}$ (respectively, $\widetilde{\mathfrak{G}} := \mathfrak{G} \cup \{u_0\}$) be a spin grid that spans the JC*-triple Z and let $1 \in I$ be an arbitrary index. We define a tripotent $v := i(u_1 + \tilde{u}_1)$; Neal and Russo give in [**16**] a method of constructing from $\mathfrak{G}$ (respectively, $\widetilde{\mathfrak{G}}$) and v a JC*-triple system that is JB*-triple isomorphic to Z and contains a spin system. First they showed for the Peirce 2-space $P_2^v(Z)$ of v that $P_2^v(Z) = Z$ and that if $\mathfrak{A}$ is any von Neumann algebra containing Z, then $P_2^v(\mathfrak{A})^{(v)}$ is a C^*-algebra TRO-isomorphic to $P_2^v(\mathfrak{A})$ (the isomorphism is the identity mapping). Moreover, they proved the following.

Theorem 4.1 (Neal and Russo [16, Theorem 3.1]). *The space $P_2^v(Z)^{(v)}$ is the linear span of a spin grid. More precisely, let $s_j = u_j + \tilde{u}_j$, $j \in I \setminus \{1\}$; $t_j := i(u_j - \tilde{u}_j)$, $j \in I$. Then a spin system that linearly spans $P_2^v(Z)^{(v)}$ is given by*

$$\{s_j, t_k, v \colon j \in I \setminus \{1\},\ k \in I\}$$

or, if the spin factor is of odd finite dimension,

$$\{s_j, t_k, v, u_0 \colon j \in I \setminus \{1\},\ k \in I\}.$$

Lemma 4.2. *Let T be a TRO and let $v \in \mathrm{Tri}(T)$.*

(a) *We have $P_2^v(T) = \{z \in T\colon v(vz^*v)^*v = z\}$.*

(b) *Let $Z \subseteq B(H)$ be a* JC**-triple system and let T be the TRO generated by Z. If $Z = P_2^v(Z)$, then $T = P_2^v(T)$.*

(c) *If v is a tripotent in the TRO T, then the Peirce 2-space $P_2^v(T)$ is a subTRO of T.*

Proof. (a) Let $z \in T$ with $vv^*z + zv^*v = 2z$. Then vv^* and v^*v are projections with $vv^*zv^*v + zv^*v = 2zv^*v$ and $vv^*zv^*v + vv^*z = 2vv^*z$. Thus, we have $vv^*zv^*v = zv^*v = vv^*z$ and therefore $vv^*zv^*v = \frac{1}{2}(vv^*z + zv^*v) = z$.

If $z \in Z$ with $vv^*zv^*v = z$, then $vv^*zv^*v = zv^*v$ and $vv^*zv^*v = vv^*z$. We get $\frac{1}{2}(vv^*z + zv^*v) = vv^*zv^*v = z$.

(b) Let $x = z_1z_2^*z_3 \cdots z_{2n}z_{2n+1} \in T$, with $z_j \in Z = P_2^v(Z)$. By (a) we get $vv^*z_jv^*v = z_j$ and $z_j = vv^*z_j = z_jvv^*$. Thus,

$$vv^*xv^*v = (vv^*z_1)z_2^*z_3 \cdots z_{2n}^*(z_{2n+1}v^*v) = z_1z_2^*z_3 \cdots z_{2n}^*z_{2n+1} = x,$$

and it follows that $x \in P_2^v(T)$.

(c) Let $a, b, c \in P_2^v(T)$, then

$$vv^*ab^*cv^*v = vv^*a(vv^*bv^*v)^*cv^*v = (vv^*av^*v)b^*(vv^*cv^*v) = ab^*c.$$

□

As a first result we get an upper bound for the dimension of the universal enveloping TRO of a spin system.

Proposition 4.3. *Let Z be a spin factor of dimension $k+1 < \infty$. Then*

$$\dim T^*(Z) \leqslant 2^k.$$

Proof. For $k = 2n$ let

$$\mathfrak{G} = \{u_1, \tilde{u}_1, \ldots, u_n, \tilde{u}_n\}$$

(or, $\mathfrak{G} = \{u_1, \tilde{u}_1, \ldots, u_n, \tilde{u}_n\} \cup \{u_0\}$ for $k = 2n+1$, respectively) be a spin grid generating Z. Then $\rho_Z(\mathfrak{G})$ is a spin grid in $\rho_Z(Z) \subseteq T^*(Z)$. By Lemma 4.2 we have for $v := i(u_1 + \tilde{u}_1)$ that $P_2^v(T^*(Z)) = T^*(Z)$, which is TRO-isomorphic to $P_2^v(T^*(Z))^{(v)}$. By Theorem 4.1, the unital C^*-algebra $P_2^v(T^*(Z))^{(v)}$ contains a spin system $\{\mathrm{id}, s_1, \ldots, s_k\}$, which generates it as a C^*-algebra. It is easy to observe (see [**9**, Remark 7.1.12]) that $P_2^v(T^*(Z))^{(v)}$ is linearly spanned by the 2^k elements $s_{i_1} \cdots s_{i_j}$, where $1 \leqslant i_1 < i_2 < \cdots < i_j$ and $0 \leqslant j \leqslant k$. □

From the proof of Proposition 4.3 we can deduce that the universal enveloping TRO of a spin factor is TRO-isomorphic to its universal enveloping C^*-algebra, once we have shown that $\dim T^*(Z) = 2^k$.

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In Jordan C^*-theory, the following famous spin system appears (cf. [**9**, 6.2.1]).

Let

$$\sigma_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

be the Pauli spin matrices.

For matrices $a = (\alpha_{i,j}) \in \mathbb{M}_k$ and $b \in \mathbb{M}_l$ we define $a \otimes b := (\alpha_{i,j} b) \in M_k(\mathbb{M}_l) = \mathbb{M}_{kl}$. If $a \in \mathbb{M}_k$ and $l \in \mathbb{N}$, we denote by $a^{\otimes l}$ the l-fold tensor product of a with itself.

The so-called standard spin system, which linearly generates a $(k+1)$-dimensional spin factor in $\mathbb{M}_{2^n}$ when $k \leqslant 2n$, is given via $\{\mathrm{id}, s_1, \dots, s_k\}$ with

$$\begin{aligned} s_1 &:= \sigma_1 \otimes \mathrm{id}^{\otimes(n-1)}, & s_2 &:= \sigma_2 \otimes \mathrm{id}^{\otimes(n-1)}, \\ s_3 &:= \sigma_3 \otimes \sigma_1 \otimes \mathrm{id}^{\otimes(n-2)}, & s_4 &:= \sigma_3 \otimes \sigma_2 \otimes \mathrm{id}^{\otimes(n-2)}, \\ s_{2l+1} &:= \sigma_3^{\otimes l} \otimes \sigma_1 \otimes \mathrm{id}^{\otimes(n-l-1)}, & s_{2l+2} &:= \sigma_3^{\otimes l} \otimes \sigma_1 \otimes \mathrm{id}^{\otimes(n-l-1)} \end{aligned}$$

for $1 \leqslant l \leqslant n-1$.

Lemma 4.4. *Let $S = \{\mathrm{id}, s_1, \dots, s_k\}$ be the standard spin system. If $k = 2n$, then the TRO generated by S in $\mathbb{M}_{2^n}$ is $\mathbb{M}_{2^n}$. If $k = 2n-1$, then the generated TRO is TRO-isomorphic to $\mathbb{M}_{2^{n-1}} \oplus \mathbb{M}_{2^{n-1}}$.*

Proof. Let T be the TRO generated by S.

Let $k = 2n$. It suffices to show that the $3k$ elements

$$\begin{aligned} a_j &:= \mathrm{id}^{\otimes(j-1)} \otimes \sigma_1 \otimes \mathrm{id}^{\otimes(n-j)}, \\ b_j &:= \mathrm{id}^{\otimes(j-1)} \otimes \sigma_2 \otimes \mathrm{id}^{\otimes(n-j)}, \\ c_j &:= \mathrm{id}^{\otimes(j-1)} \otimes \sigma_1 \otimes \mathrm{id}^{\otimes(n-j)}, \end{aligned}$$

for every $j = 1, \dots, k$, are elements of T, since a_j, b_j, c_j and $\mathrm{id} \otimes \cdots \otimes \mathrm{id}$ span $\mathbb{C} \otimes \cdots \otimes \mathbb{C} \otimes \mathbb{M}_2 \otimes \mathbb{C} \otimes \cdots \otimes \mathbb{C}$.

Obviously, $a_1 = s_1 \in T$. Suppose we show $a_j \in T$ for a fixed $j \geqslant 1$. Then

$$s_{2j} s_{2j+1}^* a_j = \mathrm{id}^{\otimes(j-1)} \otimes \sigma_2 \sigma_3 \sigma_1 \otimes \sigma_1 \mathrm{id}^{\otimes(n-j-1)} = \mathrm{i} a_{j+1}.$$

Similarly, we have $b_1 = s_2 \in T$. If we show for a fixed $j \geqslant 1$ that $b_j \in T$, then

$$s_{2j} s_{2j+2}^* a_j = \mathrm{i} b_{j+1}.$$

Another easy induction shows that $c_j \in T$ for all $j = 1, \dots, n$.

If $k = 2n-1$, we have $a_n \in T$, $b_n, c_n \notin T$. Since σ_1 and $\mathrm{id} \otimes \cdots \otimes \mathrm{id}$ generate the diagonal matrices, the statement is clear.

Alternatively, we could argue that T contains the identity, so T has to be a C^*-algebra. Then the statement follows from [**9**, Theorem 6.2.2]. $\square$

Theorem 4.5. *For the universal enveloping TRO of a spin factor Z with* $\dim Z = k+1$ *we have*

$$T^*(Z) = \begin{cases} \mathbb{M}_{2^{n-1}} \oplus \mathbb{M}_{2^{n-1}} & \textit{if } k = 2n-1, \\ \mathbb{M}_{2^n} & \textit{if } k = 2n. \end{cases}$$

Proof. The JC*-triple system Z is JB*-isomorphic to the JC*-algebra J linearly generated by the standard spin system $\{1, s_1, \dots, s_k\}$. Since J generates $\mathbb{M}_{2^{n-1}} \oplus \mathbb{M}_{2^{n-1}}$ if $k = 2n-1$ (respectively, $\mathbb{M}_{2^n}$ if $k = 2n$) as a TRO, by the universal property of $T^*(Z)$ we obtain a surjective TRO-homomorphism from $T^*(Z)$ onto $\mathbb{M}_{2^{n-1}} \oplus \mathbb{M}_{2^{n-1}}$ if $k = 2n-1$ (respectively, $\mathbb{M}_{2^n}$ if $k = 2n$). By Proposition 4.3, this has to be an isomorphism. □

4.2. Factors of type III

A Hermitian grid is a family $\{u_{ij} \colon i, j \in I\}$ of tripotents in Z such that, for all $i, j, k, l \in I$,

(HG1) $u_{ij} = u_{ji}$ for all $i, j \in I$,

(HG2) $\{u_{kl}, u_{kl}, u_{ij}\} = 0$ if $\{i, j\} \cap \{k, l\} = \emptyset$,

(HG3) $\{u_{ii}, u_{ii}, u_{ij}\} = \frac{1}{2} u_{ij}$, $\{u_{ij}, u_{ij}, u_{ii}\} = u_{ii}$ if $i \neq j$,

(HG4) $\{u_{ij}, u_{ij}, u_{jk}\} = \frac{1}{2} u_{jk}$, $\{u_{jk}, u_{jk}, u_{ij}\} = \frac{1}{2} u_{ij}$ if i, j, k are pairwise distinct,

(HG5) $\{u_{ij}, u_{jk}, u_{kl}\} = \frac{1}{2} u_{il}$ if $i \neq l$,

(HG6) $\{u_{ij}, u_{jk}, u_{ki}\} = u_{ii}$ if at least two of these tripotents are distinct.

(HG7) all other products of elements from the Hermitian grid are 0.

Let Z be a finite-dimensional TRO. Then the direct sum

$$T = \bigoplus_{\alpha=1}^{r} \mathbb{M}_{n_\alpha, m_\alpha}$$

can be described by so-called *rectangular matrix units*: let $E(\alpha, i, j) := E_{i,j} \in \mathbb{M}_{n_\alpha, m_\alpha}$ be the matrix in $\mathbb{M}_{n_\alpha, m_\alpha}$ that is 0 everywhere except in the (i, j)-component for all $1 \leqslant i \leqslant n_\alpha$, $1 \leqslant j \leqslant m_\alpha$ and $\alpha \in \{1, \dots, r\}$, where it takes the value 1. Set

$$e_{i,j}^{(\alpha)} := (0, \dots, 0, E(\alpha, i, j), 0, \dots, 0) \in T,$$

where $E(\alpha, i, j)$ is in the αth summand. The rectangular matrix units satisfy the following:

(i) $e_{i,j}^{(\alpha)} (e_{l,j}^{(\alpha)})^* e_{l,k}^{(\alpha)} = e_{i,k}^{(\alpha)}$;

(ii) $e_{i,j}^{(\alpha)} (e_{n,m}^{(\beta)})^* e_{p,q}^{(\gamma)} = 0$ for $j \neq m$, $n \neq p$, $\alpha \neq \beta$ or $\beta \neq \gamma$;

(iii) $T = \operatorname{lin}\{e_{i,j}^{(\alpha)} \colon 1 \leqslant \alpha \leqslant r,\ 1 \leqslant i \leqslant n_\alpha,\ 1 \leqslant j \leqslant m_\alpha\}$.

If U is another TRO which contains elements $f_{i,j}^{(\beta)}$ satisfying the analogues of (i)–(iii) for $1 \leqslant i \leqslant n_\alpha$, $1 \leqslant j \leqslant m_\alpha$ and $\alpha, \beta \in \{1, \dots, r\}$, then it is easy to see that the mapping sending $e_{i,j}^{(\alpha)}$ to $f_{i,j}^{(\alpha)}$ for $1 \leqslant i \leqslant n_\alpha$, $1 \leqslant j \leqslant m_\alpha$ and $\alpha \in \{1, \dots, r\}$ is a TRO-isomorphism.

Let Z be a finite-dimensional JC*-triple system spanned by a Hermitian grid $\{u_{ij} : 1 \leqslant i, j \leqslant n\}$ and T the TRO generated by this grid. Define

$$e_{ij} := u_{ii}\biggl(\sum_{k=1}^{n} u_{kk}\biggr)^{*} u_{ji} \in T$$

for $1 \leqslant i, j \leqslant n$. From [**16**, Lemma 3.2 (a)] we can conclude that $\{e_{ij}\}$ forms a system of rectangular matrix units in T. We get that

$$T^*(Z) \simeq \mathbb{M}_n.$$

4.3. Factors of type II

A symplectic grid is a family $\{u_{ij} : i, j \in I,\ i \neq j\}$ of minimal tripotents such that, for all $i, j, k, l \in I$,

(SYG1) $u_{ij} = -u_{ji}$ for $i \neq j$,

(SYG2) $\{u_{ij}, u_{ij}, u_{kl}\} = \frac{1}{2}u_{kl}$, $\{u_{kl}, u_{kl}, u_{ij}\} = \frac{1}{2}u_{ij}$ for $\{i, j\} \cap \{k, l\} \neq \emptyset$,

(SYG3) $\{u_{kl}, u_{kl}, u_{ij}\} = 0$ if $\{i, j\} \cap \{k, l\} = \emptyset$,

(SYG4) $\{u_{ij}, u_{il}, u_{kl}\} = \frac{1}{2}u_{kj}$ for i, j, k, l pairwise distinct,

(SYG5) all other triple products in the symplectic grid are 0.

The standard example of a finite-dimensional symplectic grid is the collection $\{U_{i,j} : 1 \leqslant i, j \leqslant n,\ i \neq j\} \subseteq \mathbb{M}_n$, where $U_{i,j}$, for $i < j$, is a complex $n \times n$ matrix, which is 0 everywhere except for the (i, j)-entry, which is 1, and the (j, i)-entry, which is -1. This grid spans linearly the JC*-triple system $\{A \in \mathbb{M}_n : A^{\mathrm{t}} = -A\}$ of skew-symmetric $n \times n$ matrices; its TRO span is $\mathbb{M}_n$.

Let $\mathfrak{G} := \{u_{ij} : i, j \in I,\ i \neq j\}$ be a symplectic grid, let Z be the JC*-triple system spanned by $\mathfrak{G}$ and let T be the TRO generated by it. Since for $\dim Z = 3$ Z is JB*-triple isomorphic to a type I Cartan factor and for $\dim Z = 6$ it is JB*-triple isomorphic to a type IV Cartan factor, both covered in other sections, let $\dim Z \geqslant 10$.

If we define

$$e_{ii} := u_{ik} u_{kl}^* u_{il}$$

and

$$e_{ij} := e_{ii} e_{ii}^* u_{ij} e_{jj}^* e_{jj}$$

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for $1 \leqslant i, j, k, l \leqslant n$ pairwise distinct, using [**16**, Lemmas 4.1 and 4.3] yields that the elements e_{ii} and e_{ij} are well defined and that, for $v := \sum e_{kk}$,

$$ve_{ij}^* v = e_{ji} \quad \text{and} \quad e_{ij} v^* e_{kl} = \delta_{jk} e_{il}.$$

Using this result we see that

$$\begin{aligned} e_{ij} e_{kl}^* e_{mn} &= e_{ij} v^* e_{lk} v^* e_{mn} \\ &= \delta_{jl} \delta_{km} e_{in}, \end{aligned}$$

which shows that $\{e_{ij}\}$ is a set of rectangular matrix units.

Theorem 4.6. *If Z is a* JC*-*triple system spanned by a symplectic grid with* $\dim Z \geqslant 10$, *then*

$$T^*(Z) = \mathbb{M}_n.$$

4.4. Factors of type I

Let Δ and Σ be two index sets. A rectangular grid is a family $\{u_{ij} \colon i \in \Delta,\ j \in \Sigma\}$ of minimal tripotents such that

(RG1) $\{u_{il}, u_{il}, u_{jk}\} = 0$ if $i \neq j, k \neq l$,

(RG2) $\{u_{il}, u_{il}, u_{jk}\} = \frac{1}{2} u_{jk}$, $\{u_{jk}, u_{jk}, u_{il}\} = \frac{1}{2} u_{il}$ if either $j = i, k \neq l$ or $j \neq i, k = l$,

(RG3) $\{u_{jk}, u_{jl}, u_{il}\} = \frac{1}{2} u_{ik}$ if $j \neq i$ and $k \neq l$,

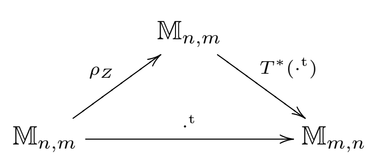
(RG4) all other triple products in the rectangular grid equal 0.

Let Z be the JC*-triple system generated by a finite rectangular grid. We assume that Z is finite dimensional and hence JB*-triple isomorphic to $\mathbb{M}_{n,m}$ with $m = |\Delta|$ and $n = |\Sigma|$.

We first exclude some candidates for $T^*(Z)$.

Lemma 4.7. *For the* JC*-*triple system* $Z = \mathbb{M}_{n,m}$, *its universal enveloping TRO* $T^*(Z)$ *is not TRO-isomorphic to* $\mathbb{M}_{n,m}$ *or to* $\mathbb{M}_{m,n}$.

Proof. Assume that $T^*(Z)$ is TRO-isomorphic to $\mathbb{M}_{n,m}$. Let $\cdot^{\mathrm{t}} \colon \mathbb{M}_{n,m} \to \mathbb{M}_{m,n}$ be the transposition mapping. According to the universal property of $T^*(Z)$ there is a mapping $T^*(\cdot^{\mathrm{t}})$ such that



commutes. Since ρ_Z is bijective, there is a TRO-isomorphism $T^*(\rho_Z) \colon \mathbb{M}_{n,m} \to \mathbb{M}_{n,m}$ with $T^*(\rho_Z) \circ \rho_Z = \mathrm{id}$. This means $T^*(\rho_Z) = \rho_Z^{-1}$; in particular, ρ_Z is a complete isometry.

Since ρ_Z and $\cdot^{\mathrm{t}}$ are bijective, the same holds for $T^*(\cdot^{\mathrm{t}})$ and it follows that $\cdot^{\mathrm{t}}$ is a complete isometry. We get a contradiction because $\cdot^{\mathrm{t}}$ is not even completely bounded. The other statement can be proved analogously. □

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Lemma 4.8 (Neal and Russo [16, Lemmas 5.1 (b) and 5.2 (b)]). *Let $\{u_{ij}\}$ be a rectangular grid spanning Z.*

(a) *If, for $i \in \Delta$, $k, l \in \Sigma$, where $k \neq l$, we have $u_{il}u_{ik}^* = 0$ or for $i, j \in \Delta$, $k \in \Sigma$, where $i \neq j$, we have $u_{ik}^* u_{jk} = 0$, then Z is TRO-isomorphic to $\mathbb{M}_{n,m}$.*

(b) *If, for $i \in \Delta$, $k, l \in \Sigma$, where $k \neq l$, we have $u_{il}^* u_{ik} = 0$ or for $i, j \in \Delta$, $k \in \Sigma$, where $i \neq j$, we have $u_{ik}u_{jk}^* = 0$, then Z is TRO-isomorphic to $\mathbb{M}_{m,n}$.*

By this lemma we obtain the following.

Lemma 4.9. *Let $\{e_{ij}\}$ be a rectangular grid spanning $\rho_Z(Z) \subseteq T^*(Z)$, then we have*

$$e_{ik}e_{il}^* \neq 0 \quad \text{and} \quad e_{ik}^* e_{il} \neq 0 \quad \text{for all } i \in \Delta,\ k, l \in \Sigma, \tag{4.1}$$

as well as

$$e_{ik}e_{jk}^* \neq 0 \quad \text{and} \quad e_{ik}^* e_{jk} \neq 0 \quad \text{for all } i, j \in \Delta,\ k \in \Sigma. \tag{4.2}$$

Proof. If one of these conditions is not satisfied, by Lemma 4.8 and since $\rho_Z(Z)$ generates $T^*(Z)$ as a TRO, we obtain that $\rho_Z(Z) = T^*(Z)$ and hence is isomorphic to $\mathbb{M}_{n,m}$ (respectively, $\mathbb{M}_{m,n}$). But this is a contradiction to Lemma 4.7. □

Lemma 4.10. *Let $\operatorname{rank} Z \geqslant 2$ and let $\{e_{ij}\}$ be a rectangular grid spanning $\rho_Z(Z)$. Then*

$$p := \sum_{i \in \Delta} \prod_{j \in \Sigma} e_{ij}e_{ij}^* \in C^*(Z)$$

is a sum of non-zero orthogonal projections. We have

$$pT^*(Z) \subseteq T^*(Z), \qquad (1-p)T^*(Z) \subseteq T^*(Z).$$

Proof. Since (4.1) and (4.2) hold, we can use [**16**, Lemma 5.5] to obtain that

$$\prod_{j \in \Sigma} e_{ij}e_{ij}^* \neq 0$$

are orthogonal projections for all $i \in \Delta$.

The fact that p leaves $T^*(Z)$ invariant is obvious. □

Lemma 4.11. *For all $i, k, a \in \Delta$, $j, l, b \in \Sigma$ we have*

$$pe_{ij}(pe_{kl})^* pe_{ab} = pe_{ij}e_{kl}^* pe_{ab} \in \operatorname{lin}\{pe_{ij}\},$$

and for $q := (1-p)$

$$qe_{ij}(qe_{kl})^* qe_{ab} = qe_{ij}e_{kl}^* qe_{ab} \in \operatorname{lin}\{qe_{ij}\}.$$

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Proof. Since $\{e_{ij}\}$ is a rectangular grid, we know for $i \neq k$ and $j \neq l$ that

$$e_{ij}e_{kl}^* = 0 \quad \text{and} \quad e_{ij}^* e_{kl} = 0,$$

and therefore, for $i \neq k$ and $j \neq l$,

$$pe_{il}(pe_{kl})^* = pe_{il}e_{kl}^* p = 0 \tag{4.3}$$

as well as

$$\begin{aligned}(pe_{il})^* pe_{kl} &= e_{il}^* pe_{kl} \\ &= e_{il}^* \Big(\sum_{\alpha\in\Delta} \prod_{\beta\in\Sigma} e_{\alpha\beta}e_{\alpha\beta}^* \Big) e_{kl} \\ &= e_{il}^* e_{i1} e_{i1}^* \cdots e_{in} \underbrace{e_{in}^* e_{kl}}_{=0 \text{ if } n\neq l} \\ &= 0, \end{aligned} \tag{4.4}$$

since the range projections of collinear tripotents commute by [**16**, Lemma 5.4].

Equations (4.3) and (4.4) lead us to the fact that we only have to prove (for arbitrary $a \in \Delta$, $b \in \Sigma$) that

$$\begin{array}{llll} pe_{ik}(pe_{il})^* pe_{ab}, & k \neq l, & pe_{jk}(pe_{ik})^* pe_{ab}, & i \neq j, \\ pe_{il}(pe_{il})^* pe_{ab}, & & pe_{ab}(pe_{il})^* pe_{ik}, & k \neq l, \\ pe_{ab}(pe_{il})^* pe_{jl}, & i \neq j, & pe_{ab}(pe_{il})^* pe_{il} & \end{array}$$

are elements of $\text{lin}\{pe_{ij}\}$.

Using (4.3) and (4.4) again, we have to prove this in the following cases:

$$\begin{array}{llll} pe_{ik}(pe_{il})^* pe_{ib}, & k \neq l,\ k \neq b \neq l, & pe_{ik}(pe_{il})^* pe_{ik}, & k \neq l, \\ pe_{ik}(pe_{il})^* pe_{il}, & k \neq l, & pe_{ik}(pe_{il})^* pe_{al}, & k \neq l,\ a \neq i, \\ pe_{jk}(pe_{ik})^* pe_{ib}, & b \neq k,\ i \neq j & pe_{jk}(pe_{ik})^* pe_{ik}, & i \neq j, \\ pe_{jk}(pe_{ik})^* pe_{jk}, & i \neq j, & pe_{jk}(pe_{ik})^* pe_{ak}, & i \neq j,\ a \neq i, \\ pe_{il}(pe_{il})^* pe_{ib}, & b \neq l, & pe_{il}(pe_{il})^* pe_{al}, & a \neq i, \\ pe_{il}(pe_{il})^* pe_{il}. & & & \end{array}$$

We obtain a similar list for q. Luckily, Neal and Russo calculated all these products to show that $\{pe_{ij}\}$ is a rectangular grid (see the proof of [**16**, Lemma 5.6]) and it is true that all of them are elements of $\{pe_{ij}\}$. One can show by similar methods that all products in the list for q are elements of the rectangular grid $\{(1-p)e_{ij}\}$. □

Proposition 4.12. *If* rank $Z \geqslant 2$, *for the universal enveloping TRO of Z we have*

$$T^*(Z) = \text{lin}\{pe_{ij}, (1-p)e_{ij} \colon 1 \leqslant i \leqslant n,\ 1 \leqslant i \leqslant m\};$$

in particular,

$$\dim T^*(Z) \leqslant 2nm.$$

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Proof. The rectangular grid $\{e_{ij}\}$ spans $\rho_Z(Z)$, which generates $T^*(Z)$ as a TRO, so an element $x \in T^*(Z)$ has to be of the form

$$x = \sum_{\alpha=1}^{n} \lambda_\alpha e_1^\alpha (e_2^\alpha)^* e_3^\alpha \cdots (e_{2n}^\alpha)^* e_{2k_\alpha+1}^\alpha,$$

with $e_1^\alpha, \dots, e_{2k_\alpha+1}^\alpha \in \{e_{ij}\}$, $\lambda_\alpha \in \mathbb{C}$ and $k_\alpha \in \mathbb{N}$ for all $1 \leqslant \alpha \leqslant n$, $n \in \mathbb{N}$. Let $e_1, \dots, e_{2n+1}$ and $e := e_1 e_2^* e_3 \cdots e_{2n} e_{2n+1}^* \in T^*(Z)$. Then

$$\begin{aligned} e &= (pe_1 + (1-p)e_1)(pe_2 + (1-p)e_2)^* \cdots (pe_{2n+1} + (1-p)e_{2n+1}) \\ &= pe_1(pe_2)^* \cdots pe_{2n+1} + (1-p)e_1((1-p)e_2)^* \cdots (1-p)e_{2n+1} \\ &\qquad\qquad\qquad\qquad\qquad\qquad + \text{mixed terms in } p \text{ and } (1-p) \\ &= pe_1(pe_2)^* \cdots pe_{2n+1} + (1-p)e_1((1-p)e_2)^* \cdots (1-p)e_{2n+1}, \end{aligned}$$

since $\{pe_{ij}\} \perp \{(1-p)e_{ij}\}$ by [**16**, Lemma 5.6]. An inductive use of Lemma 4.11 gives us $e \in \{pe_{ij}, (1-p)e_{ij} : 1 \leqslant i \leqslant n,\ 1 \leqslant i \leqslant m\}$. □

Theorem 4.13. *Let Z be a* JC**-triple system of* rank $\geqslant 2$ *and isomorphic to a finite-dimensional Cartan factor of type I. Let $\{u_{ij};\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m\}$ be a grid spanning Z. Then*

$$T^*(Z) = \mathbb{M}_{n,m} \oplus \mathbb{M}_{m,n}.$$

Proof. We identify Z with $\mathbb{M}_{n,m}$. The mapping $\Phi\colon \mathbb{M}_{n,m} \to \mathbb{M}_{n,m} \oplus \mathbb{M}_{m,n}$, $A \mapsto (A, A^{\mathrm{t}})$, is a JB*-triple isomorphism onto a JB*-subtriple of $\mathbb{M}_{n,m} \oplus \mathbb{M}_{m,n}$ that generates $\mathbb{M}_{n,m} \oplus \mathbb{M}_{m,n}$ as a TRO. Since by Proposition 4.12 $\dim T^*(Z) \leqslant 2nm$, the induced mapping $T^*(\Phi)\colon T^*(Z) \to \mathbb{M}_{n,m} \oplus \mathbb{M}_{m,n}$ has to be a TRO isomorphism. □

For the rest of this section we assume that $\operatorname{rank} Z = 1$ and Z is of finite dimension. This implies that if $\{u_{ij};\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m\}$ is a rectangular grid spanning Z, then n or m have to be equal to 1. In this special case the definition of a rectangular grid becomes simpler.

A finite rectangular grid of rank 1 is a set $\{u_1, \dots, u_n\}$ of tripotents where

(RG′1) $\{u_i, u_j, u_i\} = 0$ for $i \neq j$,

(RG′2) $\{u_i, u_i, u_k\} = \frac{1}{2} u_k$ for $i \neq k$,

(RG′3) all other products are 0.

Let Z be an n-dimensional type I Cartan factor of rank 1. We fix a finite rectangular grid $\{e_1, \dots, e_n\}$ of rank 1 spanning $\rho_Z(Z) \subseteq T^*(Z)$.

Lemma 4.14. *Let Z be as above. Then*

$$\dim T^*(Z) \leqslant \sum_{k=1}^{n} \binom{n}{k-1} \binom{n}{k}.$$

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Proof. Using the grid properties (RG′1)–(RG′3) we show that

$$T^*(Z) = \mathrm{lin}\{e_{i_1}e^*_{i_2}e_{i_3}\cdots e^*_{i_{2k}}e_{i_{2k+1}} : i_j < i_{j+2},\ 1 \leqslant j \leqslant 2k-1,\ 0 \leqslant k \leqslant \tfrac{1}{2}(n-1)\}.$$

For a fixed k we have $\binom{n}{k-1}\binom{n}{k}$ choices for $e_{i_1}e^*_{i_2}e_{i_3}\cdots e^*_{i_{2k}}e_{i_{2k+1}}$. This is true because $i_j < i_{j+2}$. We have $\binom{n}{k}$ choices for $i_1 < i_3 < \cdots < i_{2k+1}$ and $\binom{n}{k-1}$ choices for $i_2 < i_4 < \cdots < i_{2k}$.

To prove that $T^*(Z)$ is the above-mentioned linear span, we give an induction that takes $x = e_{i_1}e^*_{i_2}e_{i_3}\cdots e^*_{i_{2k}}e_{i_{2k+1}} \in T^*(Z)$ and rearranges the grid elements such that x is a sum of elements of the form $e_{j_1}e^*_{j_2}e_{j_3}\cdots e^*_{j_{2k}}e_{j_{2k+1}}$ with $j_1 \leqslant j_3 \leqslant \cdots \leqslant j_{2l+1}$ and $j_2 \leqslant j_4 \leqslant \cdots \leqslant j_{2l}$. Since the grid elements are tripotents, we can assume that we do not have three equal indices in a row. If we have the case $e_\alpha e^*_\beta e_\alpha$, where $\alpha \neq \beta$, this equals 0 by the minimality of the tripotents (according to (RG′1)). Therefore, $j_a < j_{a+2}$ for all $1 \leqslant a \leqslant 2l-1$. In particular, $l \leqslant \frac{1}{2}(n-1)$.

Therefore, let $x = e_{i_1}e^*_{i_2}e_{i_3}\cdots e^*_{i_{2k}}e_{i_{2k+1}} \in T^*(Z)$. Since the e_{i_a} are all minimal tripotents, we can assume $e_{i_a} \neq e_{i_{a+2}}$.

For $k=0$ there is nothing to prove. Additionally, we prove the case when $k=1$. Let $x = e_{i_1}e^*_{i_2}e_{i_3}$.

- If $i_1 < i_3$, we are done.
- If $i_1 = i_2 > i_3$, we can use (RG′2) to get $x = e_{i_3} - e_{i_3}e^*_{i_1}e_{i_1}$.
- If $i_1 > i_2 = i_3$, we can also use (RG′2) to get $x = e_{i_1} - e_{i_2}e^*_{i_2}e_{i_1}$.
- If $i_1 \neq i_2 \neq i_3$, then if $i_1 > i_3$ we can use (RG′3) and we deduce $x = -e_{i_3}e^*_{i_2}e_{i_1}$.

Now we assume that we have shown the statement for $2k+1 \in \mathbb{N}$, $2k+3 \leqslant n$ and for all lesser indices. If we apply our induction statement to the first $2k+1$ grid elements in the product and then apply the beginning of the induction to all of the last three elements of the products in the resulting sum, then we can easily convince ourselves that in at most three repetitions of this procedure we get the desired form for x. □

Again we have to give a faithful representation T of $T^*(Z)$. This happens to be more complicated than in the other cases. Again we can use the work of Neal and Russo. In [**16**] they showed that a JC*-triple system that is linearly spanned by a finite rectangular grid of rank 1 with n elements has to be completely isometric (in particular, JB*-triple isomorphic) to one of the spaces H^k_n, $k = 1, \dots, n$, that are generalizations of the row and column Hilbert space.

We recall the construction of the spaces H^k_n (see [**16**, §§ 6 and 7] or [**17**, § 1]). Let $1 \leqslant k \leqslant n$ and let I and J be subsets of $\{1, \dots, n\}$ such that I has $k-1$ elements and J has $n-k$ elements. There are $q_k := \binom{n}{k-1}$ choices for I and

$$p_k := \binom{n}{n-k} = \binom{n}{k}$$

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choices for J. We assume that the collections $\mathcal{I} := \{I_1, \ldots, I_{q_k}\}$ and $\mathcal{J} := \{J_1, \ldots, J_{p_k}\}$ of such sets are ordered lexicographically. Let $e_{I_1}, \ldots, e_{I_{q_k}}$ and $e_{J_1}, \ldots, e_{J_{p_k}}$ be the canonical bases of $\mathbb{C}^{p_k}$ and $\mathbb{C}^{q_k}$. We can define an element in $\mathbb{M}_{p_k,q_k}$ by $E_{I,J} := E_{i,j}$, when $I = I_i \in \mathcal{I}$ and $J = J_j \in \mathcal{J}$. The space H_n^k is the linear span of matrices $b_i^{n,k}$, $1 \leqslant i \leqslant n$, given by

$$b_i^{n,k} := \sum_{\substack{I \cap J = \emptyset, \\ (I \cup J)^c = \{i\}}} \operatorname{sgn}(I, i, J) E_{J,I}, \tag{4.5}$$

where $\operatorname{sgn}(I, i, J)$ is the signature of the permutation taking $(i_1, \ldots, i_{k-1}, i, j_1, \ldots, j_{n-k})$ to $(1, \ldots, n)$, when $I = \{i_1, \ldots, i_{k-1}\}$, $i_1 < i_2 < \cdots < i_{k-1}$, and $J = \{j_1, \ldots, j_{n-k}\}$, where $j_1 < j_2 < \cdots < j_{n-k}$.

One can show that the TRO spanned by $b_1^{n,k}, \ldots, b_n^{n,k}$ equals $\mathbb{M}_{p_k,q_k}$, so if we represent our JC*-triple system Z as $\bigoplus_{k=1}^n H_n^k$, with Lemma 4.14 we get the following.

Theorem 4.15. *If Z is a JC*-triple system spanned by a finite rectangular grid of rank 1, then*

$$T^*(Z) = \bigoplus_{k=1}^n \mathbb{M}_{p_k,q_k},$$

where

$$p_k = \binom{n}{k} \quad \text{and} \quad q_k = \binom{n}{k-1} \quad \text{for all } k = 1, \ldots, n.$$

With this result the list of universal enveloping TROs of the finite-dimensional Cartan factors is complete.

5. The radical

We use the theory of reversibility developed in [**5**] to prove some facts for the universal enveloping TRO of a universally reversible TRO T. We consider the case in which a universally reversible TRO T contains an ideal of codimension 1 that is not covered in [**5**]. We show that there exists an ideal $\boldsymbol{R}(T)$ in T that is universally reversible and that does not contain an ideal of codimension 1 itself, such that $T/\boldsymbol{R}(T)$ is an abelian JB*-triple system. We obtain an exact sequence

$$0 \to \boldsymbol{R}(T) \oplus \theta(\boldsymbol{R}(T)) \to T^*(T) \to C_0^{\mathbb{T}}(\operatorname{Epi}(T/\boldsymbol{R}(T), \mathbb{C})) \to 0,$$

where the notation is given below.

We adopt the following definition from [**5**]. It is the generalization of reversibility of JC*-algebras.

Definition 5.1. A JC*-triple system $Z \subseteq B(H)$ is said to be *reversible* if

$$\tfrac{1}{2}(x_1 x_2^* x_3 \cdots x_{2n}^* x_{2n+1} + x_{2n+1} x_{2n}^* \cdots x_3 x_2^* x_1) \in Z$$

for all $x_1, \ldots, x_n \in Z$ and $n \in \mathbb{N}$. We call a JC*-triple system *universally reversible* if it is reversible in every representation.

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Obviously, every TRO, and therefore every C^*-algebra, is reversible (but not necessarily universally reversible, since we have to cope with JB*-triple homomorphisms). A JC*-triple system is universally reversible if and only if it is reversible when embedded in its universal enveloping TRO, as follows from [**5**, Lemma 4.2].

Lemma 5.2 (Bunce *et al.* [5, Theorem 4.4]). *Let Z be a universally reversible* JC*-*triple system and let $\varphi\colon Z \to B(H)$ be an injective triple homomorphism. Suppose there exists a TRO antiautomorphism Ψ of the TRO-span* $\mathrm{TRO}(\varphi(Z))$ *such that $\Psi \circ \varphi = \varphi$. Then $T^*(\varphi)\colon T^*(Z) \to \mathrm{TRO}(\varphi(Z))$ is a TRO-isomorphism.*

Lemma 5.3 (Bunce *et al.* [5, Corollary 4.5]). *Let T be a universally reversible TRO in a C^*-algebra $\mathfrak{A}$. Suppose T has no TRO-ideals of codimension* 1 *and there is a TRO antiautomorphism $\theta\colon \mathfrak{A} \to \mathfrak{A}$ of order* 2*. Then $T^*(T) \simeq T \oplus \theta(T)$ with universal embedding $a \mapsto (a, \theta(a))$.*

In order to establish the announced generalization of Lemma 5.3 we define an ideal such that the quotient of T by this ideal is abelian. We first recall some facts about abelian JB*-triple systems that allow us to compute the universal enveloping TRO of a general abelian triple, before showing that every ideal of a universal reversible JC*-triple system is universally reversible.

Recall that a JB*-triple system Z is called abelian if

$$\{\{a,b,c\},d,e\} = \{a,\{b,c,d\},e\} = \{a,b,\{c,d,e\}\}$$

for all $a,b,c,d,e \in Z$. The importance of abelian JB*-triple systems derives from the fact that every JB*-triple system is locally abelian, which means that every element in a JB*-triple system generates an abelian subtriple. Every commutative C^*-algebra is an abelian JB*-triple system with the product $\{a,b,c\} = ab^*c$. We call the elements of

$$\mathrm{Epi}(Z,\mathbb{C}) := \{\varphi\colon Z \to \mathbb{C}\colon \varphi \neq 0 \text{ is a triple homomorphism}\}$$

the *characters* of Z. Following [**12**, § 1], we consider $\mathrm{Epi}(Z,\mathbb{C})$ as a subspace of $Z' = B(Z,\mathbb{C})$ and endow it with the $\sigma(Z^*,Z)$ topology. Then $\mathrm{Epi}(Z,\mathbb{C})$ becomes a locally compact space and a principal $\mathbb{T}$-bundle for the group $\mathbb{T} = \{t \in \mathbb{C}\colon |t| = 1\}$. The base space $\mathrm{Epi}(Z,\mathbb{C})/\mathbb{T}$ can be identified with the set of all JB*-triple ideals $I \subseteq Z$ such that Z/I is isometric to $\mathbb{C}$. The space

$$C_0^{\mathbb{T}}(\mathrm{Epi}(Z,\mathbb{C})) := \{f \in C_0(\mathrm{Epi}(Z,\mathbb{C})) \mid \forall t \in \mathbb{T},\ \forall \lambda \in \mathrm{Epi}(Z,\mathbb{C})\colon f(t\lambda) = tf(\lambda)\}$$

is a subtriple of the abelian C^*-algebra $C_0(\mathrm{Epi}(Z,\mathbb{C}))$, the continuous functions on $\mathrm{Epi}(Z,\mathbb{C})$ vanishing at infinity. The mapping

$$\widehat{\cdot}\colon Z \to C_0^{\mathbb{T}}(\mathrm{Epi}(Z,\mathbb{C})) \tag{5.1}$$

defined by $\hat{x}(\lambda) = \lambda(x)$ for all $x \in Z$ and $\lambda \in \mathrm{Epi}(Z,\mathbb{C})$ is called the *Gelfand transform* of Z.

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Theorem 5.4 (Kaup [13, Theorem 6.2]). *For every* JB**-triple system* Z *the following assertions are equivalent:*

(a) Z *is abelian;*

(b) Z *is a subtriple of a commutative* C^**-algebra;*

(c) *the Gelfand transform of* Z *is a surjective isometry onto* $C_0^{\mathbb{T}}(\mathrm{Epi}(Z,\mathbb{C}))$.

In particular, every abelian JB*-triple system is a TRO.

Lemma 5.5. *Let* Z *be an abelian* JC**-triple. Then* Z *is a universally reversible TRO.*

Proof. We only have to show that every abelian JC*-triple system is already a TRO, since every TRO is already reversible, but by Theorem 5.4 we know that Z is a subtriple of an abelian C^*-algebra and therefore a TRO. □

Proposition 5.6. *Let* Z *be an abelian* JC**-triple system. Then*

$$T^*(Z) \simeq C_0^{\mathbb{T}}(\mathrm{Epi}(Z,\mathbb{C}))$$

and the universal embedding $\rho_Z\colon Z \to C_0^{\mathbb{T}}(\mathrm{Epi}(Z,\mathbb{C}))$ *is given by the Gelfand transform of* Z.

Proof. The abelian JC*-triple system Z is, by Lemma 5.5, a universally reversible TRO. Let $\widehat{\cdot}\colon Z \to C_0^{\mathbb{T}}(\mathrm{Epi}(Z,\mathbb{C}))$ be the Gelfand transform, which is, by Theorem 5.4, a JB*-triple isomorphism. Since we are in the abelian world, the identity mapping id on $C_0^{\mathbb{T}}(\mathrm{Epi}(Z,\mathbb{C}))$ is also an antiautomorphism, satisfying $\mathrm{id} \circ \widehat{\cdot} = \widehat{\cdot}$. Since $\hat{Z}$ generates $C_0^{\mathbb{T}}(\mathrm{Epi}(Z,\mathbb{C}))$ as a TRO, we obtain the statement from Lemma 5.2. □

Definition 5.7. Let Z be a universally reversible JC*-triple system. Define the *radical* of Z to be the set

$$\boldsymbol{R}(Z) := \bigcap_{\varphi \in \mathrm{Epi}(Z,\mathbb{C}) \cup \{0\}} \ker(\varphi).$$

In the case that $\mathrm{Epi}(Z,\mathbb{C}) = \emptyset$ we have $\boldsymbol{R}(Z) = Z$. It should be mentioned that radicals have been defined for Jordan triple systems in, for example, [**14, 15, 18**]. The above definition is tailored to our purposes here and modelled after the definition for commutative Banach algebras.

The next proposition helps us to show that the radical of a universal reversible JC*-triple system is universally reversible.

Proposition 5.8. *Let* Z *be a universally reversible* JC**-triple system and* $I \subseteq Z$ *a* JB**-triple ideal. Then* I *is also universally reversible.*

Proof. We assume that $T^*(I) \subseteq T^*(Z)$. It suffices to show that $\rho_Z(I) \subseteq T^*(Z)$ is reversible. Since $T^*(I)$ is a TRO-ideal and $\rho_Z(Z)$ is reversible by definition, we know that $\rho_Z(I)$ is reversible if

$$\rho_Z(I) = T^*(I) \cap \rho_Z(Z).$$

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Let $x \in T^*(I) \cap \rho_Z(Z)$ and let $\pi\colon \rho_Z(Z) \to \rho_Z(Z)/\rho_Z(I)$ be the JB*-quotient homomorphism. It follows from [**2**, Theorem 4.2.4] that

$$T^*(Z)/T^*(I) \simeq T^*(\rho_Z(Z)/\rho_Z(I)),$$

and therefore $\pi(x) = \tau(\pi)(x) = 0$, which yields $x \in \rho_Z(I)$. □

Since the radical is always a JB*-triple ideal, the next corollary follows immediately.

Corollary 5.9. *Let Z be a universally reversible* JC*-*triple system. Then $\boldsymbol{R}(Z)$ is universally reversible.*

Theorem 5.10. *Let T be a universally reversible TRO embedded in a C^*-algebra $\mathfrak{A}$ such that there exists a TRO antiautomorphism $\theta\colon \mathfrak{A} \to \mathfrak{A}$ of order 2. Then we have an exact sequence of TROs*

$$0 \to \boldsymbol{R}(T) \oplus \theta(\boldsymbol{R}(T)) \to T^*(T) \to C_0^{\mathbb{T}}(\mathrm{Epi}(T/\boldsymbol{R}(T), \mathbb{C})) \to 0. \tag{5.2}$$

Proof. By Corollary 5.9 we know that the radical $\boldsymbol{R}(T)$ is universally reversible and does not contain a TRO-ideal of codimension 1 by Lemma 3.5. Using Lemma 5.3, we get

$$T^*(\boldsymbol{R}(T)) = \boldsymbol{R}(T) \oplus \theta(\boldsymbol{R}(T)).$$

The quotient $T/\boldsymbol{R}(T)$ is an abelian JB*-triple system and, with Proposition 5.6 and by Theorem 5.4, we get that

$$T^*(T/(\boldsymbol{R}(T))) = C_0^{\mathbb{T}}(\mathrm{Epi}(T/\boldsymbol{R}(T), \mathbb{C})).$$

The exactness of (5.2) now follows from the exactness of

$$0 \to \boldsymbol{R}(T) \to T \to T/\boldsymbol{R}(T) \to 0$$

and [**2**, Theorem 4.2.4]. □

Theorem 5.10 is a generalization of Lemma 5.3. If we add the additional assumption that T does not contain a one-codimensional TRO-ideal, then $\boldsymbol{R}(T) = T$, and thus (5.2) becomes

$$0 \to T \oplus \theta(T) \to T^*(T) \to 0 \to 0.$$

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