ALGEBRAS OF HOLOMORPHIC FUNCTIONS IN RINGED SPACES. I

MAXWELL E. SHAUCK

1. Introduction. A pair $(X, \mathscr{A})$ is a ringed space if it is a subsheaf of rings with 1 of the sheaf of germs of continuous functions on X. If U is an open subset of X, we denote the set of sections over U relative to $\mathscr{A}$ by $\Gamma(U, \mathscr{A})$. If $\phi \in \Gamma(U, \mathscr{A})$, then $\phi(u) \in \mathscr{A}$ implies that there exists some open neighbourhood V of u, $V \subset U$, and some g continuous on V such that the germ of g at u, ${}_u\mathbf{g}$, is $\phi(u)$. Now we define $\phi(u)(u)$ to be $g(u)$ and in this way we obtain, in a unique fashion, a continuous complex-valued function on U. The collection of all such functions for a given set $\Gamma(U, \mathscr{A})$ is denoted by $\mathscr{A}(U)$ and is called the $\mathscr{A}$-holomorphic functions on U. The following theorem is a special case of a result of Quigley (**5**).

THEOREM. *Let X be a locally connected Hausdorff space and $(X, \mathscr{A})$ a ringed space. Then $\mathscr{A}$ is Hausdorff if and only if $\mathscr{A}(U)$ is quasi-analytic for all open connected subsets $U \subset X$.*

This result implies the existence of a general notion of continuation in the context of ringed spaces. In a natural generalization of the notions of continuation and domain of existence from the subject of analytic functions of several complex variables we have obtained necessary and sufficient conditions that a pair of $\mathscr{A}$-holomorphic functions are continuations and that an open subset of X be a domain of existence. The results obtained are similar to the Cartan-Thullen theorem.

We impose more of the conditions on $\mathscr{A}$ which are enjoyed by the sheaf of germs of analytic functions of several complex variables such as completeness in the compact open topology and the Montel and Baire properties. The notion of convexity is investigated and results concerning functions unbounded on the boundary of open convex subsets of X are obtained.

2. In the following, $(X, \mathscr{A})$ is a ringed space in which X is a locally connected Hausdorff space. If D is an open connected subset of X and f is in $\mathscr{A}(D)$, we let $\Omega(f)$ be the connected component of ${}_x\mathbf{f}$ in $\mathscr{A}$, where x is any element of D, and we let $\sigma(f)$ be the set of all ${}_x\mathbf{f}$, where x ranges over D.

Received February 7, 1968. The results in this paper are part of the author's doctoral thesis written while the author was an NSF Graduate Fellow at Tulane University under the direction of Professor F. D. Quigley. This research was supported, in part, by NSF grant GP-5493.

Definition 1. With the above notation, suppose that p is in $\mathrm{Cl}D - D$ and V is an open neighbourhood of p. If f is in $\mathscr{A}(D)$ and g is in $\mathscr{A}(V)$, then g is an *extension* of f *to* V if and only if $g = f$ on $V \cap D$. We say that p is a *local boundary singularity* for f if and only if f has no extension to any neighbourhood of p. The set of all p in $\mathrm{Cl}D - D$ such that p is a local boundary singularity for f is denoted by $G_3(f)$. If $\mathrm{Cl}D - D = G_3(f)$, we say that D is an *$\mathscr{A}$-domain of holomorphy for f.*

Definition 2. A function g in $\mathscr{A}(V)$ for V open in X is called a continuation of f in $\mathscr{A}(D)$ for D open in X if and only if there exist $W_1, \ldots, W_n$, open connected subsets of X, and f_i in $\mathscr{A}(W_i)$ such that $W_1 = D$, $W_n = V$, $W_i \cap W_{i+1} \neq \emptyset$ and f_{i+1} is an extension of f_i to W_{i+1}. An element p in $\mathrm{Cl}D - D$ is a *global boundary singularity* for f in $\mathscr{A}(D)$ if and only if f has no continuation to any open neighbourhood of p. $G_1(f)$ is, by definition, the set of all p in $\mathrm{Cl}D - D$ which are global boundary singularities for f. We say that D is an $\mathscr{A}$-domain of existence for f if and only if $G_1(f) = \mathrm{Cl}D - D$.

THEOREM 1. *Let D and V be open connected subsets of X, f in $\mathscr{A}(D)$, g in $\mathscr{A}(V)$. Then g is a continuation of f if and only if $\Omega(f) = \Omega(g)$.*

Proof. Suppose that g is a continuation of f. Let W_i, f_i in $\mathscr{A}(W_i)$, $1 \leqq i \leqq n$, be the relevant open sets and $\mathscr{A}$-holomorphic functions, as in Definition 2. Let $U_i = \sigma(f_i)$, $1 \leqq i \leqq n$. Then the U_is are open connected subsets of $\mathscr{A}$ and for s in $W_i \cap W_{i+1}$, ${}_s\mathbf{f}_i = {}_s\mathbf{f}_{i+1}$ which implies that $U_i \cap U_{i+1} \neq \emptyset$. Thus, $U = \bigcup[U_i: 1 \leqq i \leqq n]$ is an open connected subset of $\mathscr{A}$. Now $U_1 = \sigma(f) \subset U \subset \Omega(f)$ and $U_n = \sigma(g) \subset \Omega(g)$, and this implies that $\Omega(f) = \Omega(g)$.

Suppose that $\Omega(f) = \Omega(g)$, and s is in D, t is in V. Let C be the collection of all sets of the form $[{}_v\mathbf{h}: v \text{ in } W]$, where W is an open connected set in X and h is in $\mathscr{A}(W)$; then C is a base for the topology on $\mathscr{A}$ consisting of open connected sets. Since ${}_s\mathbf{f}$ and ${}_t\mathbf{g}$ are in the same component, there is a chain $U_1, \ldots, U_n$ of sets in C connecting ${}_s\mathbf{f}$ and ${}_t\mathbf{g}$. Let $\sigma(g_i) = U_i$ and $g_i \in \mathscr{A}(W_i)$, $1 \leqq i \leqq n$. Now s is in W_1, t is in W_n, and since $U_i \cap U_{i+1} \neq \emptyset$, there is some v_i in $W_i \cap W_{i+1}$ such that ${}_{v_i}\mathbf{g}_i = {}_{v_i}\mathbf{g}_{i+1}$, $1 \leqq i \leqq n - 1$. This means that there exists a V_i, open and connected, V_i contained in $W_i \cap W_{i+1}$, such that $g_i = g_{i+1}$ on V_i, $1 \leqq i \leqq n - 1$.

Now s is in $D \cap W_1$ and ${}_s\mathbf{f} = {}_s\mathbf{g}_1$; thus, there exists an open connected subset $V_0 \subset D \cap W_1$ such that $f = g_1$ on V_0. Let $h_0 = g_1$ on V_0. For t_1 in $W_1 \cap W_2$, let V_1 be an open connected subset of $W_1 \cap W_2$ such that $g_1 = g_2$ on V_1. Let $h_1 = g_1 = g_2$ on V_1. For t_2 in $W_2 \cap W_3$, let V_2 be an open connected subset of $W_2 \cap W_3$ such that $g_2 = g_3$ on V_2 and let $h_2 = g_2$ on V_2. Continue in this way to t in $W_n \cap V$ and let $V_n \subset W_n \cap V$ be such that $g_n = g$ on V_n and let $h_n = g_n$ on V_n. Now write the chain as follows:

$$\begin{array}{ll} W_1' = D; & g_1' = f, \\ W_2' = V_0; & g_2' = h_0, \\ W_3' = W_1; & g_3' = g_1, \\ W_4' = V_1; & g_4' = h_1, \\ W_5' = W_2; & g_5' = g_2, \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ W_{m-2}' = W_n; & g_{m-2}' = g_n, \\ W_{m-1}' = V_n; & g_{m-1}' = h_n, \\ W_m' = V; & g_m' = g. \end{array}$$

Then $W_i' \cap W_{i+1}' \neq \emptyset$, and $g_i' = g_{i+1}'$ on $W_i' \cap W_{i+1}'$, $1 \leqq i \leqq m - 1$, $f = g_1$, $g_m = g$, $D = W_1'$, and $W_m' = V$.

In the following, we let π be the local homeomorphism of $\mathscr{A}$ onto X.

LEMMA 1. *Let D be an open connected subset of X, p in $\mathrm{Cl}D - D, f \in \mathscr{A}(D)$. Then $p \in G_1(f)$ if and only if $p \notin \pi(\Omega(f))$.*

Proof. Suppose that $p \notin G_1(f)$, then there exists a V, an open neighbourhood of p, and an h in $\mathscr{A}(V)$ such that $\Omega(f) = \Omega(h)$. Since ${}_p\mathbf{h}$ is in $\Omega(f)$, we see that $\pi({}_p\mathbf{h}) = p \in \pi(\Omega(f))$.

Now suppose that $p \in \pi(\Omega(f))$. Then there exists $\mathbf{s} \in \Omega(f)$ such that $\pi(\mathbf{s}) = p$; thus there exists an open connected neighbourhood W of s such that $(\pi|W)^{-1} \in \Gamma(\mathscr{A}, \pi(W))$, and we let V be $\pi(W)$, $V \subset \pi(\Omega(f))$. If we let $\phi = (\pi|W)^{-1}$ and $h(u) = \phi(u)(u)$, then h is in $\mathscr{A}(V)$. Since p is in V, $\pi(p) = {}_p\mathbf{h} \in W \subset \Omega(f)$, and thus $\mathbf{s}$ is in $\sigma(h) \cap \Omega(f)$. This implies that $\Omega(f) = \Omega(h)$.

THEOREM 2. *Let D be an open connected subset of X. Then D is an $\mathscr{A}$-domain of existence for f in $\mathscr{A}(D)$ if and only if $\pi(\Omega(f)) = D$. Further, if $\mathscr{A}$ is Hausdorff and D is a domain of existence, then π is one-to-one on $\Omega(f)$ and, in fact, if D is not a domain of existence, there exists a p in $\mathrm{Cl}D - D$ such that $p \notin G_3(f)$.*

Proof. Suppose that $\pi(\Omega(f)) = D$; then if p is in $\mathrm{Cl}D - D$, $p \notin D$; thus, $p \notin \pi(\Omega(f))$ and by Lemma 1, p is in $G_1(f)$.

Now assume that $\mathrm{Cl}D - D = G_1(f)$ for f in $\mathscr{A}(D)$; then we show that $\pi(\Omega(f)) = D$. Since $\pi(\sigma(f)) = D \subset \pi(\Omega(f))$, it is sufficient to show that $\pi(\Omega(f)) \subset D$.

Let p be in $\pi(\Omega(f))$ and assume that $p \notin D$. Then there exists an $\mathbf{s}$ in $\Omega(f)$ such that $\pi(\mathbf{s}) = p$ and an open neighbourhood V of p and an h in $\mathscr{A}(V)$ such that $\Omega(h) = \Omega(f)$. Now let $V_1, \ldots, V_n$ be open connected sets and f_i in $\mathscr{A}(V_i)$, $1 \leqq i \leqq n$, where $V_n = V$, $f_1 = f$, $f_n = h$, the standard chain of Definition 2. Since p is not in D, there is some m such that $1 \leqq m \leqq n$, and $V_m \cap (\mathrm{Cl}D - D) \neq \emptyset$.

Thus, f_m is a continuation of f and $\Omega(f_m) = \Omega(f)$. This means that there exists an x in $V_m \cap (\mathrm{Cl}D - D)$, and hence ${}_x\mathbf{f}_m$ is in $\Omega(f)$. Therefore, x is in $\pi(\Omega(f)) \cap (\mathrm{Cl}D - D)$, which is a contradiction since $G_1(f) = \mathrm{Cl}D - D$. Thus p is in D.

Now assume that $\mathscr{A}$ is Hausdorff. We will show that π is one-to-one on $\Omega(f)$. It is sufficient to prove that if ϕ in $\Gamma(\mathscr{A}, D)$ is such that $f(u) = \phi(u)(u)$, then $\phi(D) = \sigma(f) = \Omega(f)$, for π is one-to-one on the range of any section. Since $\phi(D)$ is contained in $\Omega(f)$, we take $\mathbf{s}$ in $\Omega(f)$ and show that $\mathbf{s}$ is in $\sigma(f)$. To do this it is sufficient to prove that if $\mathbf{s}$ is the germ at x of some $\mathscr{A}$-holomorphic function g over a sufficiently small connected open set V, then ${}_x\mathbf{g} = {}_x\mathbf{f}$. Since ${}_x\mathbf{g} \in \Omega(f)$, then $\Omega(f) \cap \Omega(g) \neq \emptyset$ and this means that $\Omega(f) = \Omega(g)$. Thus by Theorem 1, g is a continuation of f and we have a connecting chain $V_1, \ldots, V_n$, $g_i \in A(V_i)$, where $V_1 = D$, $g_1 = f$, $V_n = V$, $g_n = g$, $V_i \cap V_{i+1} \neq \emptyset$ and $g_i = g_{i+1}$ on $V_i \cap V_{i+1}$. Further, by the first part of this theorem and Theorem 1 we have $\pi(\Omega(g_i)) = \pi(\Omega(f)) = D$, $1 \leqq i \leqq n$. Thus, $\sigma(g_i)$, contained in $\Omega(g_i)$, implies that

$$\pi(\Omega(g_i)) = V_i \subset \pi(\Omega(f)) = D.$$

Now using the quasi-analyticity of $\mathscr{A}(D)$, we have the following: $f|V_2 = g_2$, thus $g_3 = f$ on $V_2 \cap V_3$ which implies that $f|V_3 = g_3, \ldots, f|V = g$. Hence, ${}_x\mathbf{f} = {}_x\mathbf{g}$, which implies that π is one-to-one on $\Omega(f)$, as noted.

The last statement is clearly a consequence of the Hausdorff property on $\mathscr{A}$.

Definition 3. Let D be an open subset of X. Then D is said to be *locally connected* at p *in* $\mathrm{Cl}D - D$ if and only if p has a base $B(p)$ of open connected neighbourhoods whose intersections with D are connected.

THEOREM 3. *If D is locally connected at every x in* $\mathrm{Cl}D - D$, *$\mathscr{A}$ is Hausdorff, and there is some f in $\mathscr{A}(D)$ such that $G_3(f)$ is equal to* $\mathrm{Cl}D - D$, *then D is an $\mathscr{A}$-domain of existence for f.*

Proof. Suppose that D is not an $\mathscr{A}$-domain of existence for f. We show that in this case there is some p in $\mathrm{Cl}D - D$ which is not a local boundary singularity for f.

Since $\Omega(f)$ is a connected component in $\mathscr{A}$ and $\mathrm{Cl}\sigma(f)$ is connected, then $\mathrm{Cl}\sigma(f) \subset \Omega(f)$. If $\mathrm{Cl}\sigma(f) = \sigma(f)$, then $\sigma(f)$ is equal to $\Omega(f)$ and $\pi(\Omega(f)) = D$ which implies that D is an $\mathscr{A}$-domain of existence for f. Thus

$$\mathrm{Cl}\sigma(f) - \sigma(f) \neq \emptyset.$$

Let $\mathbf{s}$ be in $\mathrm{Cl}\sigma(f) - \sigma(f)$ and $\pi(\mathbf{s}) = p$; then p is in $\mathrm{Cl}D$. Suppose that p is in D; then there exists an open connected W in the filter base of open neighbourhoods of $\mathbf{s}$ such that $\pi|W$ is one-to-one and $\pi(W) = V \subset D$, where V is open. Let g, in $\mathscr{A}(V)$, be such that $g(u) = (\pi|W)^{-1}(u)(u)$; then $\sigma(g) = W$ and ${}_p\mathbf{g} = \mathbf{s}$. Since $\mathbf{s}$ is in $\mathrm{Cl}\sigma(f)$, then $\sigma(f) \cap \sigma(g)$ is not empty; hence, there exists a v in V such that ${}_v\mathbf{g} = {}_v\mathbf{f}$. This means that there is some $U \subset V$ for which $g = f$ on U, U an open neighbourhood of v. Since $\mathscr{A}$ is Hausdorff, this means that $f|V = g$; thus, ${}_p\mathbf{g} = {}_p\mathbf{f} = \mathbf{s}$, which means that $\mathbf{s}$ is in $\sigma(f)$, a contradiction; thus p is not in D.

Now we have V as an open neighbourhood of p, p in $\mathrm{Cl}D - D$, and thus there is some V' in $B(p)$ such that $V' \subset V$ and $V' \cap D$ is connected. Since $g|V'$ is in $\mathscr{A}(V')$, then $\sigma(g|V')$ being a neighbourhood of $\mathbf{s}$ implies that $(g|V') \cap \sigma(f) \neq \emptyset$. Thus, there exists v in $V' \cap D$ such that ${}_v\mathbf{g} = {}_v\mathbf{f}$, and hence $f = g$ on $V' \cap D$. Thus, p is not a local boundary singularity for f.

Definition 4. Let D be an open subset of X, and let p be in $\mathrm{Cl}D - D$. Then p is a *semi-local boundary singularity for f in* $\mathscr{A}(D)$ if and only if for all open neighbourhoods V of p and all connected components U of $V \cap D$ such that p is in $\mathrm{Cl}\,U$, $f|U$ has no extension to V, and $G_2(f)$ is defined to be the set of all p in $\mathrm{Cl}D - D$ such that p is a semi-local boundary singularity for f.

It is easy to show that if D is open in X and f is in $\mathscr{A}(D)$, then

$$G_1(f) \subset G_2(f) \subset G_3(f).$$

LEMMA 2. *Let $\mathscr{A}$ be Hausdorff, D an open connected subset of X, and f in $\mathscr{A}(D)$; then if $\mathrm{Cl}D - D = G_2(f)$, D is an $\mathscr{A}$-domain of existence.*

Proof. Suppose that there exists some p in $G_2(f)$ such that p is not in $G_1(f)$. Then there exist $V_1, \ldots, V_n$ open connected subsets of X, g_i in $\mathscr{A}(V_i)$, $1 \leqq i \leqq n$, such that $g_1 = f$, $V_1 = D$, p in V_n, and $g_i = g_{i+1}$ on $V_i \cap V_{i+1}$. Let j be the least integer such that $V_j \cap (\mathrm{Cl}D - D) \neq \emptyset$. Then $V_{j-1} \subset D$ since the V_is are connected. Further, $g_i = f$ on V_i for $1 \leqq i \leqq j - 1$ since $\mathscr{A}$ is Hausdorff. Now $V_j \cap V_{j-1}$ meets some component of $V_j \cap D$. Call this component U. Since D and V_j are connected, $V_j \cap D \neq \emptyset$, $V_j \not\subset D$, and U is a component of $D \cap V_j$, there exists a q in $\mathrm{Cl}D - D$ such that q is in $\mathrm{Cl}\,U \cap V_j$.

We also know that $g_j = g_{j-1} = f$ on $V_j \cap V_{j-1} \cap U$, and hence $g_j = f$ on U. Therefore g is in $G_2(f)$. Thus, we have shown that if $G_1(f)$ is contained in but not equal to $G_2(f)$, then $G_2(f)$ is contained in but not equal to $\mathrm{Cl}D - D$.

Definition 5. Let D be an open subset of X. Then D is $\mathscr{A}$*-convex* if and only if whenever K is a compact subset of D, the set $[x \in D\colon |f(x)| \leqq ||f||_K$, for all $f \in \mathscr{A}(D)]$, which is denoted by $\mathrm{hull}_{\mathscr{A}(D)}K$, is a compact subset of D.

If X is a locally compact space, we say that $\mathscr{A}$ is a c.o. (compact open) complete sheaf if and only if for all U open in X, $\mathscr{A}(U)$ is closed in the c.o. topology on $C(U)$.

LEMMA 3. *Let X be locally compact, locally connected, D an open subset of X, and $[K_n]$ a sequence of relatively compact open subsets of D such that $\bigcup K_n = D$, $K_n \subset K_{n+1}$, and $\mathrm{hull}_{\mathscr{A}(D)}K_n$ is a compact subset of D. Then if $[q_n] \subset D$ is such that q_n is in $\mathrm{hull}_{\mathscr{A}(D)}K_{n+1} - \mathrm{hull}_{\mathscr{A}(D)}K_n$, then given any sequence $[r_n]$ of positive real numbers, there is some f in $\mathscr{A}(D)$ such that $|f(q_n)| \geqq r_n$.*

Proof. This is essentially a lemma due to Quigley (**4**, pp. 85–86).

Definition 6. Let D be an open subset of X and let B be the topological boundary of D. A set $S \subset B$ is said to be *countably connected in* B if and only if for all s in S there exists a basis of neighbourhoods $[V_{si}: i \in Z_+]$, where Z_+ denotes the positive integers, such that for each i in Z_+ there are at most countably many connected components $[V_{sij}: j \in Z_+]$ such that s is in $\operatorname{Cl} V_{sij}$.

If S is countably connected in B, then S is *spread* in B if and only if for every p in B and every open neighbourhood V of p, every component of $V \cap B$ meets S, and if W is a component of $V \cap D$ such that p is in $\operatorname{Cl} W$ and s is some element of S in the component of $V \cap D$ containing p, then for some i in Z_+ such that $V_{si} \subset V$ there exists a j in Z_+ such that $V_{sij} \cap W \neq \emptyset$.

LEMMA 4. *Let X be a locally compact, locally connected Hausdorff space, $\mathscr{A}$ a Hausdorff sheaf which is c.o. complete, D an open $\mathscr{A}$-convex subset of X countable at ∞ such that there exists a countable set $S \subset B$, the topological boundary of D, which is countably connected. Then there exists an f in $\mathscr{A}(D)$ such that S is contained in $G_2(f)$.*

Proof. Let $[w_u: u \in Z_+]$ be the set S, and for each u in Z_+ let $[V_{uk}': k \in Z_+]$ be a countable nested basis of neighbourhoods of w_u and $[V_{iuk}': i \in Z_+]$ the set of components of $V_{uk}' \cap D$ such that w_u is in $\operatorname{Cl} V_{iuk}'$ for each i in Z_+. Now let $[p_j: j \in Z_+]$ be the sequence $w_1, w_1, w_2, w_1, w_2, w_3, w_1, \ldots$, so that each w_u occurs infinitely often in $[p_j]$. Whenever $p_j = w_u$, let $V_{jk} = V_{uk}'$ and $V_{ijk} = V_{iuk}'$.

Now let $D = \bigcup K_n'$, where K_n' is a relatively compact subset of D, $\operatorname{Cl} K_n' \subset K_{n+1}'$, and define K_n to be $\operatorname{hull}_{\mathscr{A}(D)} K_n'$. Since $V_{111} \cap (\operatorname{Cl} D - D) \neq \emptyset$, there exists a $q_{11u(1)}'$ in V_{111} such that $q_{11u(1)}'$ is not in K_1. Let $K_1 = K_{111}$ and $q_{11u(1)}' = q_{111}$ by definition. Now there exists a $v(1)$ in Z_+ such that q_{111} is in $K_{v(1)}$ and a $q_{11u(2)}'$ in V_{112} such that $q_{11u(2)}'$ is not in $K_{v(1)}$. Define $K_{v(1)}$ to be K_{112} and $q_{11u(2)}'$ to be q_{112}.

Now we continue in this fashion and exhaust the set of indices

$$Q = [(i, j, k): i, j, k \text{ in } Z_+, i \leqq j \leqq k]$$

using the following ordering; if (i, j, k) and (l, m, n) are in Q, then $(i, j, k) \leqq (l, m, n)$ if and only if $k \leqq n$, and if $k = n$, then $j \leqq m$ and if $j = m$, then $i \leqq l$. Now define $\sigma: Q \to Q$ by the rule

$$\sigma(i, j, k) = \begin{cases} (1, 1, k + 1) & \text{if } i = j = k, \\ (i, j + 1, k) & \text{if } i \leqq j < k, \\ (i + 1, j, k) & \text{if } i < j = k. \end{cases}$$

Then it follows that for the set $[q_{ijk}, K_{ijk}: (i, j, k) \text{ in } Q]$, q_{ijk} is not in K_{lmn} if $(l, m, n) \leqq (i, j, k)$ and q_{ijk} is in $K_{\sigma(i,j,k)}$. Since $\bigcup K_{v(i)} = D$, there exists f in $\mathscr{A}(D)$ such that $f(q_{ijk}) \to \infty$ by Lemma 3. Now suppose that there is some u in Z_+ such that w_u is not in $G_2(f)$. Then there is an open neighbourhood V of w_u and some component W of $V \cap D$ such that $f|W$ has an extension to V.

https://doi.org/10.4153/CJM-1969-141-8 Published online by Cambridge University Press

Let K in Z_+ be such that $V_{uk}' \subset V$ for $k \geqq K$ and let the set $[j(k): k \geqq K]$ be a subsequence of Z_+ such that $j(k) \leqq k$ and $p_{j(k)} = w_u$ for all k. Since W is a component, we know that for each $k \geqq K$ there is some $i(k)$ in Z_+ such that $i(k) \leqq j(k)$ and $V_{i(k),j(k),k} \subset W$. Thus, $[q_{i(k),j(k),k}: k \geqq K]$ is a subsequence of $[q_{ijk}: (i, j, k) \text{ in } Q]$ contained in W with the property that $q_{i(k),j(k),k} \to w_u$. Since $f(q_{i(k),j(k),k}) \to \infty$, we have a contradiction. Thus $S \subset G_2(f)$.

THEOREM 4. *Let X, $\mathscr{A}$, and D be as in Lemma 4. If there exists a countable set S spread in B, then D is an $\mathscr{A}$-domain of existence.*

Proof. Let $S = [p_k: k \text{ in } Z_+]$. Suppose that there exists a p in B such that p is not in $G_2(f)$. Then there exists an open neighbourhood V of p and a component W of $V \cap D$ such that p is in $\mathrm{Cl}\,W$ and $f|W$ has some extension f' to V. Now let p_k be an element of S which is in the same component of $V \cap B$ as p and V_{ikm} a component of $V_{km} \cap D$ such that p_k is in $\mathrm{Cl}\,V_{ikm}$ and $V_{ikm} \cap W \neq \emptyset$. Then $f = f'$ on $V_{ikm} \cap W$; thus $f = f'$ on V_{ikm} since $\mathscr{A}$ is Hausdorff, and thus p_k is not in $G_2(f)$. This is a contradiction to Lemma 4, and thus $B = G_2(f)$ and by Lemma 3, D is an $\mathscr{A}$-domain of existence.

THEOREM 5. *Let X and $\mathscr{A}$ be as in Theorem 4, D open, $\mathscr{A}$-convex, countable at ∞, separable and such that $\mathrm{Cl}D - D$ is second countable. Then D is an $\mathscr{A}$-domain of existence.*

Proof. Let $[V_j \cap (\mathrm{Cl}D - D): j \in Z_+]$ be a countable basis for the topology of $\mathrm{Cl}D - D$. Since D is separable, each $V_j \cap D$ has at most countably many components V_{jk}. For each j and k in Z_+ there exists a q_{jk} in

$$\mathrm{Cl}\,V_{jk} \cap (\mathrm{Cl}D - D) \cap V_j.$$

Clearly, the set $[q_{jk}: j, k \text{ in } Z_+]$ is a countable, countably connected subset of $\mathrm{Cl}D - D$, and thus Lemma 4 implies that there is some f in $\mathscr{A}(D)$ such that $[q_{jk}: j, k \text{ in } Z_+] \subset G_2(f)$.

Assume that there is some p in $\mathrm{Cl}D - D$ such that p is in $G_2(f)$. Let V be an open neighbourhood of p and let W be a component of $V \cap D$ such that p is in $\mathrm{Cl}\,W$ and $f|W$ has some extension to V. Then there is some $V_j \subset V$ and some component V_{jk} of $V_j \cap D$ such that $V_{jk} \subset W$. Thus, q_{jk} is in

$$\mathrm{Cl}\,V_{jk} \cap (\mathrm{Cl}D - D) \subset \mathrm{Cl}\,W \cap (\mathrm{Cl}D - D),$$

which means that f cannot be extended across $\mathrm{Cl}\,W$; hence, p is in $G_2(f)$. As in Theorem 4, this implies that D is a domain of existence.

Definition 7. Let D be open in X and let p be an element of $\mathrm{Cl}D - D$. Then p is said to have property g_3 if and only if for all V (an open neighbourhood of p) there is some f in $\mathscr{A}(D)$ which has no extension to $D \cup V$.

Definition 8. $\mathscr{A}$ is said to be *Montel* if and only if whenever U is an open set in X and H is a subset of $\mathscr{A}(U)$ with the property that for every compact subset K of U there is some $M > 0$ such that $||f||_K < M$ for all f in H, then H

is relatively compact in $\mathscr{A}(U)$. $\mathscr{A}$ is *Baire* if no $\mathscr{A}(U)$ is a countable union of closed nowhere dense subsets for any U open in X.

LEMMA 5. *Let X be a locally compact, locally connected Hausdorff space, $\mathscr{A}$ Hausdorff, c.o. complete, Montel, and Baire, D open in X, and p an element of* $\mathrm{Cl}D - D$ *such that X is first countable at p. Then if p has property* g_3, *there exists an f in $\mathscr{A}(D)$ such that p is in $G_3(f)$.*

Proof. Let $[V_n]$ be a basis of nested, open, relatively compact neighbourhoods of p. For n in Z_+, define $\mathscr{A}(V_m, n)$ to be the set of all f in $\mathscr{A}(D)$ such that there exists some f' in $\mathscr{A}(D \cup V_m)$ extending f to $D \cup V_m$ and such that $\|f'\|_{V_m} \leqq n$. Let $[f_k]$ be any net in $\mathscr{A}(V_m, n)$ converging c. o. to some f. Each f_k is uniformly bounded on compact subsets of D and since $[f_k]$ is c. o. convergent, $[f_k]$ is uniformly bounded on compact subsets of D. For f_k in $\mathscr{A}(V_m, n)$ let g_k be an element in $\mathscr{A}(V_m \cup D)$ such that $\|g_k\|_{V_m} \leqq n$ and $g_k = f_k$ on D. Then $[g_k]$ is uniformly bounded on compact subsets of $D \cup V_m$, and since $\mathscr{A}$ is Montel, $[g_k]$ is relatively compact in $\mathscr{A}(D \cup V_m)$.

Thus, there is a subnet $[g_{k(i)}]$ of $[g_k]$ which converges c. o. to some g in $\mathscr{A}(D \cup V_m)$. Now $g_{k(i)}|D = f_{k(i)}$; thus $\lim g_{k(i)}|D = \lim f_{k(i)} = g|D = f|D$, and $\|g_{k(i)}\|_{V_m} \leqq n$ implies that $\|g\|_{V_m}$ is at most n. Hence, f is in $\mathscr{A}(V_m, n)$ and $\mathscr{A}(V_m, n)$ is closed in the c. o. topology of $\mathscr{A}(D)$.

Now by hypothesis there is some h in $\mathscr{A}(D)$ which has no extension to $\mathscr{A}(D \cup V_m)$. Define $f_j = f - (1/j)(h)$, where f is any element of $\mathscr{A}(V_m, n)$. Then f_j does not extend to $D \cup V_m$, for if it does then we let g_1 and g_2 be elements of $\mathscr{A}(D \cup V_m)$ such that $g_1|D = f$ and $g_2|D = f_j$. Since

$$h = j(f - f_j) = j(g_1|D) - j(g_2|D)$$

and $jg_1 - jg_2$ is in $\mathscr{A}(D \cup V_m)$, we have a contradiction. Now

$$\lim f_j = \lim f - \lim(1/j)(h) = f,$$

however f_j is not in $\mathscr{A}(V_m, n)$; thus f is not in the interior of $\mathscr{A}(V_m, n)$, and we have shown that the interior of $\mathscr{A}(V_m, n)$ is empty.

If $B = \bigcup[\mathscr{A}(V_m, n)\colon m, n \text{ in } Z_+]$, then B is a countable union of closed nowhere dense subsets of $\mathscr{A}(D)$, and hence B is contained in but not equal to $\mathscr{A}(D)$. Now define $B' = [f \text{ in } \mathscr{A}(D)\colon p \text{ is not in } G_3(f)]$, then for f in B' there exists an open neighbourhood V of p and f' extending f to $D \cup V$. Let V_m be an open neighbourhood of p such that $\mathrm{Cl}\, V_m \subset V$, and let n in Z_+ be such that $\|f'\|_{V_m} \leqq n$. Since $f'|V_m \cap D$ is in $\mathscr{A}(V_m \cap D)$, f is in B, and thus $B' \subset B$. Now let h in $\mathscr{A}(D)$ be such that h is not in B; then h is not in B', and thus p is in $G_3(h)$.

COROLLARY 1. *Let X, $\mathscr{A}$, D, and p be as in Lemma 5. Then if f in $\mathscr{A}(D)$ implies that p is not in $G_3(f)$, there is some open neighbourhood V of p such that every g in $\mathscr{A}(D)$ has an extension to $D \cup V$.*

Definition 9. Let D be an open subset of X and let p be an element of $\mathrm{Cl}D - D$. Then p has property g_2 if and only if for all open neighbourhoods V of p and every component U of $V \cap D$ such that p is in $\mathrm{Cl}\,U$ there exists an f in $\mathscr{A}(D)$ such that $f|U$ has no extension to V.

LEMMA 6. *Let X and $\mathscr{A}$ be as in Lemma* 5, *D an open, separable subset of X and p in* $\mathrm{Cl}D - D$ *such that X is first countable at p. Then if p has property* g_2, *there exists an f in $\mathscr{A}(D)$ such that p is in $G_2(f)$.*

Proof. Let $[V_n]$ be a countable nested basis of open relatively compact neighbourhoods of p. There is at most a countable number of components of $V_n \cap D$ for each n. Let $[U_n: n \text{ in } Z_+]$ be the subset of these for which p is in $\mathrm{Cl}\,U$. If V is an open neighbourhood of p and U is a component of $V \cap D$ such that p is in $\mathrm{Cl}\,U$, define $\mathscr{A}(V, U, m)$ to be the set of all g in $\mathscr{A}(D)$ such that there is some g' in $\mathscr{A}(V)$ which extends $g|U$ and $||g'||_V \leqq m$. Now the method used in Lemma 5 may be applied here to show that $\mathscr{A}(V, U, m)$ is closed and nowhere dense in $\mathscr{A}(D)$.

If B is defined to be $\bigcup[\mathscr{A}(V_m, U_j, n): m, j, n \text{ in } Z_+]$, then B is contained in but not equal to $\mathscr{A}(D)$. Let B' be the set of all f in $\mathscr{A}(D)$ such that p is not in $G_2(f)$. Then $B' \subset B$, and as before we let h in $\mathscr{A}(D)$ be such that h is not in B. Thus h is not in B' and p is in $G_2(h)$.

COROLLARY 2. *Let X, $\mathscr{A}$, D, and p be as in Lemma* 6. *If f in $\mathscr{A}(D)$ implies that p is not in $G_2(f)$, then there is some open neighbourhood V of p and some component U of $V \cap D$ with p in* $\mathrm{Cl}\,U$ *such that for all g in $\mathscr{A}(D)$, $g|U$ has some extension to V.*

THEOREM 6. *Let X be a locally compact, locally connected, Hausdorff space, D an open subset of X such that* $\mathrm{Cl}D - D$ *is separable and X is first countable on* $\mathrm{Cl}D - D$, *$\mathscr{A}$ Hausdorff, c. o. complete, Montel, and Baire. Then if for each p in* $\mathrm{Cl}D - D$ *there exists an f_p in $\mathscr{A}(D)$ such that p is in $G_3(f)$, there exists an f in $\mathscr{A}(D)$ such that* $\mathrm{Cl}D - D = G_3(f)$.

Proof. As in Lemma 5, for p in $\mathrm{Cl}D - D$ we let $[V_n]$ be a countable basis of nested, relatively compact, open neighbourhoods of p. Let $\mathscr{A}(p, V_n, m)$ be the set of all f in $\mathscr{A}(D)$ such that there exists some extension g of f to $D \cup V_n$ such that $||g||_{V_n} \leqq m$, where m is in Z_+. Then the same proof as in Lemma 5 shows that $\mathscr{A}(p, V_n, m)$ is closed in $\mathscr{A}(D)$ and that if there is some h such that p is in $G_3(h)$, $\mathscr{A}(p, V_n, m)$ is nowhere dense in $\mathscr{A}(D)$.

Now let $[p_i]$ be a countable dense subset of $\mathrm{Cl}D - D$ and let h_j be an element of $\mathscr{A}(D)$ such that p_j is in $G_3(h_j)$ for j in Z_+. Then for each p_i define $\mathscr{A}(p_i, V_n^i, m)$ as above and define

$$B = \bigcup[\mathscr{A}(p_i, V_n^i, m): i, n, m \text{ in } Z_+].$$

Then B is contained in but not equal to $\mathscr{A}(D)$. Now let B' be the set of all f in $\mathscr{A}(D)$ which have an extension to a neighbourhood of D, then $B \subset B'$ by

https://doi.org/10.4153/CJM-1969-141-8 Published online by Cambridge University Press

definition. Now if f is in B', there are p in $\mathrm{Cl}D - D$, an open neighbourhood V of p, and a g in $\mathscr{A}(D \cup V)$ which extends f to $D \cup V$. Let p_i be in $V \cap (\mathrm{Cl}D - D)$ and $V_n{}^i$ an open neighbourhood of p_i such that $\mathrm{Cl}V_n{}^i \subset V$ and let m be the least integer such that $||g||_{V_n{}^i} \leqq m$. Thus, f is in $\mathscr{A}(p_i, V_n{}^i, m)$ and we have shown that $B = B'$. Thus there is some f in $\mathscr{A}(D) - B'$ and we have $\mathrm{Cl}D - D$ equal to $G_3(f)$.

THEOREM 7. *Let X be a locally compact, locally connected, Hausdorff space, D open in X, $\mathscr{A}$-convex, countable at ∞, $\mathrm{Cl}D - D$ separable and first-countable, and $\mathscr{A}$ c.o. complete. Then there exists an f in $\mathscr{A}(D)$ such that* $\mathrm{Cl}D - D = G_3(f)$.

Proof. Let $[w_k]$ be a countable dense subset of $\mathrm{Cl}D - D$ and $[p_i]$ a sequence such that each w_k occurs infinitely often in $[p_i]$. Let $[U_j{}^i]$ be a countable nested basis of open relatively compact neighbourhoods of p_i and when $p_i = p_m$, let $[U_l{}^i: l \text{ in } Z_+]$ and $[U_j{}^m: j \text{ in } Z_+]$ be the same system. Now let $D = \cup K_j$, where $K_j = \mathrm{hull}_{\mathscr{A}(D)} K_j$, which is compact in D, $K_j \subset K_{j+1}$; and let $[q_j]$ be a sequence in D such that $q_j \in K_{j+1} - K_j$, $q_j \in U_j{}^j$. We also let f_j be an element of $\mathscr{A}(D)$ such that $||f_j||_{K_j} \leqq 1$, $|f_j(q_j)| > 1$. Then there is some f in $\mathscr{A}(D)$ such that $|f(q_n)| \to \infty$.

Now each w_n occurs infinitely often in $[p_i]$, so we let $p_{n(j)} = w_n$ and we have $q_{n(j)}$ in $U_{n(j)}{}^{n(j)} = U_n{}^{n(j)}$. Thus $q_{n(j)} \to w_n$ and $|f(q_{n(j)})| \to \infty$. Consequently, f is unbounded on a dense subset of $\mathrm{Cl}D - D$, and thus on all of $\mathrm{Cl}D - D$; thus $\mathrm{Cl}D - D = G_3(f)$.

Definition 10. For U open in X and f in $C(U)$, we say that f *belongs $\mathscr{A}$-locally to $\mathscr{A}(U)$* if and only if for all x in U there is some open neighbourhood V of x and g in $\mathscr{A}(V)$ such $V \subset U$ and $f|V = g$.

Now it is immediate from the definition that f belongs $\mathscr{A}$-locally to $\mathscr{A}(U)$ if and only if f belongs to $\mathscr{A}(U)$.

LEMMA 7. *If X is locally compact, locally connected, and $\mathscr{A}$ is c. o. complete, then for each U open in X, $\mathscr{A}(U)$ is inverse closed.*

Proof. Let U be an open set in X and f an element of $\mathscr{A}(U)$ such that the zero set of f, $Z(f)$, is empty. It is sufficient to show that f^{-1} belongs $\mathscr{A}$-locally to $\mathscr{A}(U)$.

Let x be in X, $c = 2^{-1}(|f(x)|)$ and $D(c) = [s \text{ in } C: |s - f(x)| < c]$. Now z^{-1} belongs to the boundary-value algebra on $D(c)$; thus for any $e > 0$ there is some polynomial $P(z)$ such that

$$||P(z) - z^{-1}||_{\mathrm{Cl}D(c)} < e.$$

Since f is continuous on U, there is some open neighbourhood V of x such that $f(V) \subset D(c)$. Let K be a compact subset of V; then $f(K)$ is contained in $f(V)$ which is contained in $D(c)$, and thus

$$||P(f) - f^{-1}||_K < e.$$

Since $P(f)|V$ is in $\mathscr{A}(V)$ and $\mathscr{A}(V)$ is c. o. complete, we have shown that $f^{-1}|V$ is in $\mathscr{A}(V)$.

PROPOSITION 1. *Let X be a locally compact, locally connected Hausdorff space, D an open subset of X which is $\mathscr{A}$-convex and countable at ∞ and such that* $\mathrm{Cl}D - D$ *is separable, and X is first-countable at* $\mathrm{Cl}D - D$ *and let $\mathscr{A}$ be c. o. complete, Montel, and Baire. Then if there is some f in $\mathscr{A}(D)$ such that $D = X - Z(f)$, there is some h in $\mathscr{A}(D)$ such that* $\mathrm{Cl}D - D = G_3(h)$.

Proof. Since f^{-1} is in $\mathscr{A}(D)$ for every p in $\mathrm{Cl}D - D$, define $f_p = (f - f_p)^{-1}$. Then f_p is in $\mathscr{A}(D)$ and p is in $G_3(f_p)$. Thus by Theorem 7 we have the desired conclusion.

PROPOSITION 2. *Let X be a locally compact, locally connected, separable and second-countable space which is countable at ∞ and whose topology is given by the Gelfand topology relative to $\mathscr{A}(X)$, where $\mathscr{A}$ is c. o. complete and Hausdorff. Then for every x in X there is some open neighbourhood V of x such that V is an $\mathscr{A}$-domain of existence.*

Proof. It is sufficient by Theorem 5 to demonstrate, for each x in X, the existence of some open neighbourhood V of x which is $\mathscr{A}$-convex.

Since the topology on X is the Gelfand topology, for x in X there exist $f_1, \ldots, f_n$ in $\mathscr{A}(X)$ such that if we let

$$V = [y \text{ in } X\colon |f_i(y)| < 1, 1 \leqq i \leqq n],$$

then V is a relatively compact, connected neighbourhood of x.

Let K be a compact subset of V, and let

$$K' = [y \text{ in } X\colon |f_i(y)| \leqq m, 1 \leqq i \leqq n],$$

where $m = \max[||f_i||_K\colon 1 \leqq i \leqq n]$. Clearly, $m < 1$. Now y in $\mathrm{hull}_{\mathscr{A}(V)}K$ implies that $|f_i(y)| \leqq ||f_i||_K \leqq m$, hence y is in K'. Since $m < 1$, $K' \subset V$, and thus $\mathrm{hull}_{\mathscr{A}(V)}K$ is a compact subset of V, and hence V is $\mathscr{A}$-convex.

Many of the following results have the same proof as the corresponding results in the case of analytic functions. When this is the case, the proofs are omitted.

Definition 11. Let D be an open subset of X, $S \subset \mathscr{A}(D)$, and $D_0 \subset D$. Then if B is defined to be the following set:

$$[x \text{ in } D_0\colon |f(x)| < 1, \text{ for all } f \text{ in } S],$$

we say that B is the $\mathscr{A}$-holomorphic polyhedron in D defined by D_0 and S. We call B finite if S is finite.

PROPOSITION 3. *Let D be an open subset of X, B an $\mathscr{A}$-holomorphic polyhedron in D defined by D and $S \subset \mathscr{A}(D)$ such that B is open and* $\mathrm{Cl}B \subset D$, *and let $\mathscr{A}$ be Hausdorff and c. o. complete. Then for every p in* $\mathrm{Cl}B - B$ *there exists some g in $\mathscr{A}(B)$ such that if $[q_n]$ is any net converging to p, then $|g(q_n)| \to \infty$.*

Proof. $B = [s \text{ in } D: |f(s)| < 1 \text{ for all } f \text{ in } S]$ by hypothesis. Thus for p in $\mathrm{Cl}B - B$ there exists some f in S such that $|f(p)| \geqq 1$ and therefore the zero set of the function $g = (f|B) - f(p)$ is empty, and hence g^{-1} is in $A(B)$. Clearly, g^{-1} is unbounded on any net $[q_n]$ converging to p in the sense that $|g(q_n)| \to \infty$.

PROPOSITION 4. *Let D be open in X with the property that whenever $[q_n]$ is a net with no convergent subnet then there is some f in $\mathscr{A}(D)$ such that $|f(q_n)| \to \infty$. Then D is $\mathscr{A}$-convex.*

Proof. Let K be compact in D such that $\mathrm{hull}_{\mathscr{A}(D)}K$, which we denote by K^*, is not compact. Then there is some net $[q_n]$ in K^* which has no convergent subnet. Since $[q_n] \subset K^*$ and for any f in $\mathscr{A}(D)$, $||f||_K = ||f||_{K^*}$, we have that no f in $\mathscr{A}(D)$ is unbounded on $[q_n]$.

COROLLARY 3. *Let D be an open subset of X. Assume also that B is an $\mathscr{A}$-holomorphic polyhedron in D defined by D and some $S \subset \mathscr{A}(D)$. Then B is $\mathscr{A}$-convex.*

PROPOSITION 5. *Let D be an open subset of X, D_0 an open subset of D which is $\mathscr{A}$-convex and B a finite $\mathscr{A}$-holomorphic polyhedron in D defined by D_0 and $f_1, \ldots, f_n$ in $\mathscr{A}(D)$. Then B is $\mathscr{A}$-convex.*

LEMMA 8. *Let D be an open subset of X which is $\mathscr{A}$-convex, and K a compact subset of D. If D_0 is an open subset of D such that*

$$\mathrm{hull}_{\mathscr{A}(D)}K \ll D_0 \ll D,$$

then there exists an analytic polyhedron B such that B is $\mathscr{A}$-convex and

$$\mathrm{hull}_{\mathscr{A}(D)}K \ll B \ll D.$$

THEOREM 8. *Let X be a locally compact, locally connected Hausdorff space, and D an open subset of X which is $\mathscr{A}$-convex and countable at ∞. Then there exists a sequence $[B_j]$ of $\mathscr{A}$-convex subsets of D such that $B_j \ll B_{j+1} \ll D$ for j in Z_+ and $D = \bigcup B_j$.*

REFERENCES

1. S. Bochner and W. Martin, *Several complex variables*, Princeton Mathematical Series, Vol. 10 (Princeton Univ. Press, Princeton, N.J., 1948).
2. R. Godement, *Théorie des faisceaux* (Hermann, Paris, 1958).
3. K. Hoffman, *Domains of holomorphy*, Mimeographed notes, Massachusetts Institute of Technology, Cambridge, Mass., 1958.
4. F. Quigley, *Approximation by algebras of functions*, Math. Ann. *135* (1958), 81–92.
5. ——— *Lectures on several complex variables*, Tulane University, New Orleans, Louisiana, 1964–65, 1965–66.

Tulane University,
New Orleans, Louisiana;
Yale University,
New Haven, Connecticut