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ON MULTIPLIERS INTO BERGMAN SPACES AND NEVANLINNA CLASS

BY

P. WOJTASZCZYK

ABSTRACT. We use the general factorisation theorems of Grothendieck, Nikishin and Maurey to characterise coefficient multipliers between Bergman spaces and into the Nevanlinna class.

Introduction. In this paper we discuss the multipliers between various classes of analytic functions on the unit disc. We are mainly interested in two scales of spaces; the Bergman spaces $B_p(D)$, $0 < p < \infty$, defined as

$$B_p(D) = \left\{ f(z) : f(z) \text{ is analytic for } |z| < 1 \text{ and } \|f\|_p = \left(\int_D |f(x)|^p \, d\nu(z) \right)^{1/p} < \infty \right\}$$

where ν is a normalised two-dimensional Lebesgue measure and spaces $X_\alpha(D)$, $0 \leqq \alpha < \infty$, where

$$X_\alpha(D) = \left\{ f(z) : f(z) \text{ is analytic for } |z| < 1 \text{ and } \sup_{|z|<1} |f(z)|(1 - |z|)^\alpha < \infty \right\}.$$

We characterise coefficient multipliers from X_α into $B_p(D)$ for $0 \leqq \alpha < \infty$ and $0 < p \leqq 2$. Using the known results this yields the description of multipliers from the Bloch space or from the Lipschitz spaces into B_p. Our methods also imply the description of multipliers from X_α into ℓ_p, $0 < p \leqq 2$, but those are known (cf. [BST] and [AS]). Actually we believe that our methods, which are based on general factorisation results, are of some interest.

We also present some results about multipliers in the Nevanlinna class which in particular generalise many results about the Bloch–Nevanlinna conjecture.

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Preliminaries. We say that a Banach space X is of cotype 2 if there exists a constant K such that

$$\left(\sum \|x_i\|^2\right)^{1/2} \leqq K \left(\int_0^1 \left\|\sum_i r_i(t)x_i\right\|^2 dt\right)^{1/2}$$

for every finite sequence $(x_i)_{i=1}^n \subset X$. (Recall that $r_i(t)$ denotes the sequence of Rademacher functions.) The following result of Maurey [M] will be our main tool.

THEOREM A. *Let X be a Banach space with the bounded approximation property such that X^* is of cotype 2. Then every operator $U : X \longrightarrow L_p(\Omega, \mu)$, $0 < p \leqq 2$, admits a factorisation*

$$\begin{tikzcd} X \arrow[rr, "U"] \arrow[dr, "U_1"'] & & L_p(\Omega, \mu) \\ & L_2(\Omega, \mu) \arrow[ur, "M"'] & \end{tikzcd}$$

where $M(f) = f \cdot g$ for some function g.

We will apply this theorem for X being X_α. For $\alpha > 0$ it is known that X_α is isomorphic to a complemented subspace of L_∞ (cf. [SW]) so it has the bounded approximation property and X^* has cotype 2 since L_1 has. In the case $\alpha = 0$ we have $X_0 = H_\infty$. It is clear that multipliers of H_∞ into $B_p(D)$ are the same as multipliers from the disc algebra $A(D)$ into $B_p(D)$. That $A(D)^*$ has cotype 2 is a deep result of Bourgain [B], while the bounded approximation follows from the Féjer theorem. We will use the similar passage to a smaller space also in the case $\alpha > 0$. Let

$$X_\alpha^0(D) = \left\{ f(z) \in X_\alpha(D) : \lim_{|z| \to 1} |f(z)|(1 - |z|)^\alpha = 0 \right\}.$$

It is known [SW] that $\left(X_\alpha^0(D)\right)^{**} = X_\alpha(D)$, so the coefficient multipliers into $B_p(D)$, $0 < p < \infty$, are the same for $X_\alpha^0(D)$ and for $X_\alpha(D)$.

We will also use the result of Grothendieck (cf. [LP1]) that every operator from L_∞ into the Hilbert space is 2-absolutely summing. The same holds with L_∞ replaced by $A(D)$, as was shown by Bourgain [B]. Using the Pietsch criterion we get the following fact, where by X_0^0 we mean $A(D)$.

THEOREM B. *Let $u : X_\alpha^0 \longrightarrow H$, $0 \leqq \alpha < \infty$, where H is a Hilbert space. Assume that $i : X_\alpha^0(D) \longrightarrow C(K)$ is any isometric embedding. Then there exists a probability measure μ on K such that the following factorisation holds*

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$$
\begin{tikzcd}
X_\alpha^0(D) \arrow[r, "u"] \arrow[d, "i"'] & H \\
C(K) \arrow[r, "id"] & L_2(K, \mu) \arrow[u, "\tilde{u}"']
\end{tikzcd}
$$

Multipliers into Bergman Spaces. As mentioned in the Introduction we are interested in multipliers from $X_\alpha(D)$ into $B_p(D)$, that is we want to describe sequences $\Lambda = (\lambda_n)_{n=0}^\infty$ such that $\sum_{n=0}^\infty \lambda_n a_n z^n \in B_p(D)$ for every function $\sum_{n=0}^\infty a_n z^n \in X_\alpha(D)$. The main idea of this note is to split this problem into two easier ones, as described in the following

PROPOSITION 1. *Suppose $\Lambda = (\lambda_n)_{n=0}^\infty$ is a multiplier from $X_\alpha(D)$ into $B_p(D)$, $0 \leqq \alpha < \infty$, $0 < p \leqq 2$. We can write $\lambda_n = \mu_n \cdot \nu_n$, $n = 0, 1, 2, \ldots$ in such a way that for every $\sum_{n=0}^\infty a_n z^n \in X_\alpha(D)$ the sequence $(\mu_n a_n)_{n=0}^\infty \in \ell_2$ and $\sum_{n=0}^\infty a_n \nu_n z^n \in B_p(D)$ for every $(a_n)_{n=0}^\infty \in \ell_2$.*

This proposition asserts that every multiplier from $X_\alpha(D)$ into $B_p(D)$, $0 \leqq \alpha < \infty$, $0 < p \leqq 2$, is a composition of a multiplier from $X_\alpha(D)$ into ℓ_2 and a multiplier from ℓ_2 into $B_p(D)$.

PROOF OF PROPOSITION 1. As explained above, Theorem A applies in our situation, since $B_p(D)$ is a subspace of $L_p(D, \nu)$. Thus there exists a function $g(z) \geqq 0$, $|z| < 1$ such that

$$\left\| \frac{\Lambda(f)}{g} \right\|_{L_2(D,\nu)} \leqq C \|f\| \quad \text{for} \quad f \in X_\alpha(D).$$

Since the operator Λ commutes with rotations of the disc and the norms involved are also rotation invariant, one can choose $g(z) = g(|z|)$ (average over rotations). But the sequence

$$\left(\frac{z^n}{g(|z|)} \right)_{n=0}^\infty$$

is orthogonal in $L_2(D, \nu)$ so we obtain

$$\left(\sum_{n=1}^\infty |\lambda_n a_n|^2 \beta_n^2 \right)^{1/2} \leqq C \|f\|$$

for every

$$f = \sum_{n=0}^\infty a_n z^n \in X_\alpha(D) \quad \text{and} \quad \beta_n = \left\| \frac{z^n}{g(|z|)} \right\|_{L_2(D,\nu)}.$$

This means that the sequence $\lambda_n \beta_n = \mu_n$, $n = 0, 1, 2, \ldots$ determines a multiplier from $X_\alpha(D)$ into ℓ_2. Since the sequence $\nu_n = \beta_n^{-1}$, $n = 0, 1, 2, \ldots$ is easily seen to define a multiplier from ℓ_2 into $B_p(D)$, the proposition is proved.

REMARK. Since the sequence $(z^n)_{n=0}^{\infty}$ is not unconditional both in $X_\alpha(D)$, $0 \leqq \alpha < \infty$, and in $B_p(D)$, $0 < p < \infty$, one would expect that conditions for the sequence $(\lambda_n)_{n=0}^{\infty}$ to be a multiplier from $X_\alpha(D)$ into $B_p(D)$ should take into account also arguments. The above Proposition 1 shows that this is not the case; only the size of λ_n's is important.

PROPOSITION 2. *The sequence $\Lambda = (\lambda_n)_{n=0}^{\infty}$ is a multiplier from $X_\alpha(D)$, $0 < \alpha < \infty$, into ℓ_2 if and only if*

$$\sum_{n=0}^{\infty} \left(\sup_{2^k \leqq n < 2^{k+1}} n^\alpha |\lambda_n| \right)^2 = K < \infty. \tag{1}$$

PROOF. As it was explained in the introduction, it is enough to consider multipliers from $X_\alpha^0(D)$ into ℓ_2. Let

$$C_0(D) = \left\{ f(z) : f \text{ continuous for } |z| < 1 \text{ and } |f(z)| \underset{|z| \to 1}{\longrightarrow} 0 \right\}.$$

Then the operator $i : X_\alpha^0(D) \longrightarrow C_0(D)$ defined as $i(f)(z) = f(z)(1 - |z|)^\alpha$ is an isometric embedding. Given a multiplier $\Lambda = (\lambda_n) : X_\alpha^0 \longrightarrow \ell_2$, we get a factorisation as in Theorem B

$$\begin{tikzcd}
X_\alpha^0(D) \arrow[r, "\Lambda"] \arrow[d, "i"'] & l_2 \\
C_0(D) \arrow[r, "id"'] & L_2(D, \mu) \arrow[u, "\tilde{\Lambda}"']
\end{tikzcd}$$

Since all operators and norms involved are invariant under rotation of the disc, we can assume that

$$\int_D \varphi(z)\, d\mu(z) = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} \varphi(re^{i\theta})\, d\theta\, d\mu_1(r)$$

for all $\varphi \in C_0(D)$ and some probability measure μ_1 on $[0, 1)$.

With this assumption we see that $\tilde{\Lambda}$ is bounded if and only if

$$|\lambda_n| \leqq C \|i(z^n)\|_{L_2(D,\mu)} = C \left(\int_0^1 r^{2n}(1 - r)^{2\alpha}\, d\mu_1(r) \right)^{1/2}. \tag{2}$$

This reasoning is clearly reversible, so we get that $\Lambda = (\lambda_n)_{n=0}^{\infty}$ is a multiplier from $X_\alpha(D)$ into ℓ_2 if and only if (2) holds for some probability measure μ_1 on $[0, 1)$.

Assume now that (1) holds. We can additionally assume $K = 1$. Let us fix s_k, $2^k \leqq s_k < 2^{k+1}$ such that

$$\sup_{2^k \leqq n < 2^{k+1}} n^\alpha |\lambda_n| = s_k^\alpha |\lambda_{s_k}|, \quad k = 0, 1, 2, \ldots .$$

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Let $a_k = (1 - s_k^{-1})$ and define

$$v = \sum_{k=0}^{\infty} s_k^{2\alpha} |\lambda_{s_k}|^2 \delta_{\alpha_k}$$

where δ_{α_k} is the Dirac measure concentrated at the point α_k. This is clearly a probability measure on $[0, 1)$.

Take arbitrary n and let k_0 be such that $2^{k_0} \leqq n \leqq 2^{k_0+1}$. Then we have

$$\begin{aligned}\int_0^1 r^{2n}(1-r)^{2\alpha}\, dv(r) &= \sum_{k=0}^{\infty} \alpha_k^{2n}(1-\alpha_k)^{2\alpha} s_k^{2\alpha} |\lambda_{s_k}|^2 \\ &= \sum_{k=0}^{\infty} (1 - s_k^{-1})^{2n} |\lambda_{s_k}|^2 \geqq (1 - s_{k_0}^{-1})^{2n} |\lambda_{s_{k_0}}|^2 \geqq C|\lambda_n|^2,\end{aligned}$$

so (2) holds. Conversely, assuming (2) we have

$$\begin{aligned}(3) \qquad &\sum_k \sup_{2^k \leqq n < 2^{k+1}} n^{2\alpha} \int_0^1 r^{2n}(1-r)^{2\alpha}\, d\mu_1(r) \\ &\leqq \sum_k 2^{2k\alpha} \int_0^1 r^{2\cdot 2^k}(1-r)^{2\alpha}\, d\mu_1(r) \\ &= \int_0^1 (1-r)^{2\alpha} \left(\sum_k 2^{2k\alpha} r^{2\cdot 2^k} \right) d\mu_1(r).\end{aligned}$$

For $1 - 2^{-N} \leqq r \leqq 1 - 2^{-N-1}$ we have for some q, $0 < q < 1$:

$$\begin{aligned}\sum_{k=0}^{\infty} 2^{2k\alpha} r^{2\cdot 2^k} &\leqq \sum_{k=0}^{N} 2^{2\alpha k}(1 - 2^{-N-1})^{2\cdot 2^k} + \sum_{k=N+1}^{\infty} 2^{2\alpha k}(1-2^{-N-1})^{2\cdot 2^k} \\ &\leqq \sum_{k=0}^{N} 2^{2\alpha k} + \sum_{k=N+1}^{\infty} 2^{2\alpha k} q^{2^{k-(N+1)}} \\ &\leqq C2^{2\alpha N} + C2^{2\alpha N} \sum_{k=1}^{\infty} 2^{2\alpha k} q^{2k} \leqq C2^{2\alpha N} \leqq C(1-r)^{-2\alpha}.\end{aligned}$$

This shows that the integrand in (3) is bounded so (1) follows.

PROPOSITION 3. *The sequence $\Lambda = (\lambda_n)_{n=0}^{\infty}$ is a multiplier from $A(D)$ into ℓ_2 if and only if it is bounded.*

This is easy and well known. One possible proof follows immediately from Theorem B.

PROPOSITION 4. *A sequence $\Lambda = (\lambda_n)_{n=0}^{\infty}$ is a multiplier from ℓ_2 into $B_p(D)$, $0 < p \leqq 2$, if and only if*

$$(4) \qquad \sum_k \left(\sup_{2^k \leqq n < 2^k} n^{-1/p} |\lambda_n| \right)^s < \infty \quad \textit{with } s^{-1} + 2^{-1} = p^{-1}.$$

PROOF. For every sequence $(n_k)_{n=0}^{\infty}$ such that $2^k \leqq n_k < 2^{k+1}$ we have

$$\left\| \sum_k \alpha_k 2^{n_k} \right\|_p \geqq C \left(\sum_k |\alpha_k|^p n_k^{-1} \right)^{1/p}.$$

Thus for any multiplier $\Lambda = (\lambda_n)_{n=0}^{\infty}$ from ℓ_2 into $B_p(D)$ we have

$$\left(\sum_{k=1}^{\infty} |\alpha_{n_k}|^p |\lambda_{n_k}|^p n_k^{-1} \right)^{1/p} \leqq C \left\| \sum_k \alpha_{n_k} \lambda_{n_k} z^{n_k} \right\|_p \leqq c \left(\sum_k |\alpha_{n_k}|^2 \right)^{1/2}. \tag{5}$$

Since (5) holds with the same constant c for every square summable sequence (α_{n_k}) and every (n_k) as above we get (4).

In order to prove the other implication we will use the following inequality

$$\left\| \sum_{n=0}^{\infty} \alpha_n z^n \right\|_p \leqq C_p \left(\sum_k \left\| \sum_{2^k}^{2^{k+1}-1} a_n z^n \right\|_p^p \right)^{1/p}. \tag{6}$$

For $p \leqq 1$ this inequality follows from the p-convexity of the space $B_p(D)$ (cf. [R]). It is obvious for $p = 2$ and the rest follows by interpolation.

For a sequence $(a_n)_{n=0}^{\infty}$ such that $\Sigma |a_n|^2 = 1$ and $(\lambda_n)_{n=0}^{\infty}$ satisfying (4) we obtain from (6) and Hölder's inequality

$$\begin{aligned}
\left\| \sum_{n=0}^{\infty} \alpha_n \lambda_n z^n \right\|_p &\leqq C_p \left(\sum_k \left\| \sum_{2^k}^{2^{k+1}-1} a_n \lambda_n z^n \right\|_p^p \right)^{1/p} \\
&= C_p \left(\sum_k \int_0^1 \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{2^k}^{2^{k+1}-1} a_n \lambda_n r^n e^{in\theta} \right|^p d\theta \, r\, dr \right)^{1/p} \\
&\leqq C_p \left(\sum_k \int_0^1 \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{2^k}^{2^{k+1}-1} a_n \lambda_n r^n e^{in\theta} \right|^2 d\theta \right)^{p/2} r\, dr \right)^{1/p} \\
&\leqq C_p \left(\sum_k \int_0^1 \left(\sum_{2^k}^{2^{k+1}-1} |a_n \lambda_n|^2 r^{2n} \right)^{p/2} r\, dr \right)^{1/p}
\end{aligned}$$

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$$\leqq C_p\left(\sum_k \int_0^1 r^{p\cdot 2^k}\left(\sum_{2^k}^{2^{k+1}-1}|a_n|^2|\lambda_n|^2\right)^{p/2} r\,dr\right)^{1/p}$$

$$\leqq C_p\left(\sum_k \frac{1}{2^k}\left(\sum_{2^k}^{2^{k+1}-1}|a_n|^2|\lambda_n|^2\right)^{p/2}\right)^{1/p}$$

$$\leqq C_p\left(\max_{2^k\leqq n<2^{k+1}} n^{-1}|\lambda_n|^p\left(\sum_{2^k}^{2^{k+1}-1}|a_n|^2\right)^{p/2}\right)^{1/p}$$

$$\leqq C_p\left(\sum_k\left(\max_{2^k\leqq n<2^{k+1}} n^{-1}|\lambda_n|^p\right)^{\frac{2}{2-p}}\right)^{\frac{2-p}{2p}} \leqq \text{const.}$$

This completes the proof.

We will also need the following elementary

LEMMA 5. *The sequence* $(\lambda_n)_{n=0}^\infty$ *can be written as* $\lambda_n = \mu_n\cdot\zeta_n$, $n = 0, 1, 2, \dots$ *with*

$$\left(\sup_{2^k\leqq n<2^k} n^a|\mu_n|\right)\in \ell_r \quad \textit{with } a \textit{ real and } 0<r\leqq\infty$$

and

$$\left(\sup_{2^k\leqq n<2^k} n^b|\zeta_n|\right)\in \ell_p \quad \textit{with } b \textit{ real and } 0<p\leqq\infty$$

if and only if

$$\left(\sup_{2^k\leqq n<2^{k+1}} n^c|\lambda_n|\right)\in \ell_q$$

where $q^{-1}=p^{-1}+r^{-1}$ *and* $c=a+b$.

PROOF. The direct implication is an immediate consequence of the Hölder's inequality while for the converse implication we write

$$\mu_n = n^\alpha|\lambda_n|^{q/r} \quad\text{and}\quad \zeta_n = n^{-\alpha}|\lambda_n|^{q/p}\cdot\operatorname{sgn}\lambda_n$$

where $\alpha = b - c(q/p)$.

Now we are ready to state our main results.

THEOREM 6. *The sequence* $\Lambda = (\lambda_n)_{n=0}^\infty$ *is a multiplier from the disc algebra* $A(D)$ *into* $B_p(D)$, $0<p\leqq 2$, *if and only if*

$$\left(\sup_{2^k\leqq n<2^{k+1}} n^{-1/p}|\lambda_n|\right)\in\ell_s \quad\text{for } s^{-1}+2^{-1}=p^{-1}.$$

PROOF. This is an immediate consequence of Propositions 1, 3 and 4.

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Theorem 7. *The sequence* $\Lambda = (\lambda_n)_{n=0}^{\infty}$ *is a multiplier from* $X_\alpha(D)$, $\alpha > 0$, *into* $B_p(D)$, $0 < p \leqq 2$, *if and only if*

$$\left(\sup_{2^k \leqq n < 2^{k+1}} n^{\alpha - 1/p}|\lambda_n|\right) \in \ell_p. \tag{7}$$

Proof. This follows immediately from Propositions 1, 2, and 4 and Lemma 5.

Our Theorem 6 gives in particular the known result (cf. [J] or [K]) that there exists $f \in A(D)$ with $f' \notin B_1(D)$. A little bit sharper result can be obtained using a different argument.

Proposition 8. *There exists a function* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in A(D)$ *such that the series* $\sum_{n=0}^{\infty} a_n z^n$ *converges uniformly in D and such that* $f' \notin B_1(D)$.

Proof. Let $\mathcal{U}$ denote the space of functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in A(D)$$

such that

$$|||f||| = \sup_N \left\| \sum_{n=0}^{N} a_n z^n \right\|_{A(D)} < \infty.$$

If our Proposition is false then we have a continuous linear map $D : \mathcal{U} \longrightarrow B_1(D)$ defined as $Df = f'$. By the results of [B2] the space $\mathcal{U}$ does not contain a complemented subspace isomorphic to ℓ_1. On the other hand $B_1(D)$ is isomorphic to ℓ_1 (cf. [LP2]), so D has to be a compact operator. But $\|D(z^{2^k}) - D(z^{2^s})\| \geqq c$ if $k \neq s$, so D is not compact. This shows that D is not continuous so the Proposition follows.

The space of Bloch functions is defined as

$$B = \left\{ f(z) : f \text{ is analytic for } |z| < 1 \text{ and } \sup_{|z|<1} |f'(z)|(1 - |z|) < \infty \right\}.$$

From Theorem 7 we immediately obtain

Corollary 9. *The sequence* $\Lambda = (\lambda_n)_{n=0}^{\infty}$ *is a multiplier from B into* $B_p(D)$ *if and only if*

$$\left(\sup_{2^k \leqq n < 2^{k+1}} n^{-1/p}|\lambda_n|\right) \in \ell_p.$$

From classical results of Hardy and Littlewood (cf. [D] chapter 5) we get that $f \in \Lambda_\alpha(D)$, $0 < \alpha \leqq 1$ if and only if $f' \in X_{1-\alpha}(D)$. As is usual,

$$\Lambda_\alpha(D) = \{f(z) : f \in A(D) \quad \text{and} \quad |f(e^{i\theta}) - f(e^{i(\theta+h)})| \leqq C|h|^\alpha\}.$$

Exactly as before we get

COROLLARY 10. *The sequence* $\Lambda = (\lambda_n)_{n=0}^{\infty}$ *is a multiplier from* $\Lambda_\alpha(D)$, $0 < \alpha < 1$, *into* $B_p(D)$, $0 < p \leqq 2$ *if and only if*

$$\left(\sup_{2^k \leqq n < 2^k} n^{-\alpha - 1/p} |\lambda_n|\right) \in \ell_p.$$

We can also describe multipliers from $B_q(D)$ into $B_p(D)$ for $0 < p \leqq 2 \leqq q < \infty$. This follows from the fact that $B_s(D)^* = B_{s'}(D)$, $s^{-1} + (s')^{-1} = 1$ (cf. [SW]), thus $B_q(D)^*$ has cotype 2. Using Theorem A like in Proposition 1, we get that every multiplier Λ from $B_q(D)$ into $B_p(D)$ admits a factorisation

$$B_q(D) \xrightarrow{\Lambda^1} \ell_2 \xrightarrow{\Lambda^2} B_p(D)$$

where Λ^1 and Λ^2 are multipliers. The above mentioned duality also implies that $(\Lambda^1)^*$ is a multiplier from ℓ_2 into $B_{q'}(D)$, $1/q + 1/q' = 1$, given by the sequence

$$\mu_n = \lambda_n^1 \left(\int_D |z^{2n}|\, d\nu(z)\right)^{-1}$$

where $\Lambda^1 = (\lambda_n^1)_{n=0}^{\infty}$.

Thus Proposition 4 and Lemma 5 give

THEOREM 11. *The sequence* $\Lambda = (\lambda_n)_{n=0}^{\infty}$ *is a multiplier from* $B_q(D)$ *into* $B_p(D)$, $0 < p \leqq 2 \leqq q < \infty$, *if and only if*

$$\left(\sup_{2^k \leqq n < 2^k} n^{-1/r} |\lambda_n|\right) \in \ell_r \quad \text{with } r^{-1} = p^{-1} - q^{-1}.$$

REMARK. Clearly our methods also allow to compute multipliers from $X_\alpha(D)$ into ℓ_p, $0 < p \leqq 2$, or from $B_q(D)$ into ℓ_p, $0 < p \leqq 2 \leqq q \leqq \infty$, simply because Theorem A works in those cases. The results however are known in those cases (cf. [AS], [BST]), so we do not go into details. Also we can replace $B_p(D)$ by spaces $B_p^\varphi(D)$ with the norm

$$\|f\| = \left(\int_D |f(z)|^p \varphi(|z|)\, d\nu(z)\right)^{1/p}$$

for appropriate weight function φ. Also the spaces $X_\alpha(D)$ can be replaced by spaces $A_\infty(\varphi)$ considered in [SW], Theorem 1. The main feature of our proofs, i.e., the use of Theorems A and B, would remain the same and proofs will go in the same manner, only the presentation would be more complicated. We decided to avoid those complications, since one of our objectives is to present the power and usefulness of the general, abstract methods summarised in Theorems A and B.

Multipliers into the Nevanlinna Class. This section is based on a version of Nikishin Theorem (cf. [N]). Actually an earlier Theorem of Stein [S] is enough. We are interested in the following situation. Let $L_0(T)$ denote the space of all finite, measurable functions on T, with the topology of convergence in measure. Let X be a Banach space of functions on D such that $\|\cdot\|_X$ is invariant under rotations. The following is a special case of the above mentioned Theorems.

PROPOSITION N–S. *Let $T : X \longrightarrow L_0(T)$ be a linear operator which commutes with rotations. Then actually $T : X \longrightarrow L_p(T)$ for every $p < 1$.*

The clear presentation of those facts can be found in [GR].

Let X be a space of analytic functions on D with the norm $\|\cdot\|$ invariant under rotations and such that convergence in norm implies almost uniform convergence in D. We say that the sequence $\Lambda = (\lambda_n)_{n=0}^{\infty}$ is a multiplier from X into $L_0(T)$ if for every $f = \sum_{n=0}^{\infty} a_n z^n \in X$ the series $\sum_{n=0}^{\infty} \lambda_n a_n z^n$ defines an analytic function in D and this function has radial limits almost everywhere. The operator $\Lambda : X \longrightarrow L_0(T)$ defined as

$$\Lambda(f)(e^{i\theta}) = \lim_{r\longrightarrow 1} \sum_{n=0}^{\infty} \lambda_n a_n r^n e^{in\theta}$$

is then a continuous linear operator from X into $L_0(T)$. This follows from the Banach–Steinhaus Theorem (see [R]), since

$$\Lambda f = \lim_{k\longrightarrow\infty} T_k(f) \quad \text{where} \quad T_k(f)(e^{i\theta}) = \sum_{n=0}^{\infty} a_n \lambda_n \left(1 - \frac{1}{k}\right)^n e^{in\theta}.$$

Let $W(D)$ denote the space of absolutely convergent Taylor series, i.e. the space of functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\|f\| = \sum_{n=0}^{\infty} |a_n| < \infty$.

THEOREM 12. *The sequence $\Lambda = (\lambda_n)_{n=0}^{\infty}$ is a multiplier from $W(D)$ into $L_0(T)$ if and only if $\Lambda \in \ell_\infty$.*

PROOF. From Proposition N–S and the above comments we see that if Λ is such a multiplier then

$$\sup_n \|\lambda_n z^n\|_{L_{1/2}(T)} = \sup_n |\lambda_n| < \infty.$$

The converse is obvious.

Note that this shows in particular that for every $(\lambda_n)_{n=0}^{\infty} \notin \ell_\infty$ there exists $(a_n)_{n=0}^{\infty}$ with $\sum_{n=0}^{\infty} |a_n| = 1$ such that $\sum_{n=0}^{\infty} a_n \lambda_n z^n$ is not of bounded characteristic. For a survey of this type of results see [CW].

Putting together Theorem A, Proposition N–S and the results of the previous section we get

THEOREM 13.(a) *The sequence $\Lambda = (\lambda_n)_{n=0}^{\infty}$ is a multiplier from $X_\alpha(D)$, $0 < \alpha < \infty$, into $L_0(T)$ if and only if Λ is a multiplier from $X_\alpha(D)$ into $H_2(T)$ if and only if* (1) *holds.*

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(b) *The sequence* $\Lambda = (\lambda_n)_{n=0}^{\infty}$ *is a multiplier from* $B_p(D)$, $p \geqq 2$, *into* $L_0(T)$ *if and only if* Λ *is a multiplier from* $B_p(D)$ *into* $H_2(T)$ *if and only if*

$$\left(\sup_{2^k \leqq n < 2^{k+1}} n^{1/p}|\lambda_n|\right) \in \ell_r \quad \textit{where} \quad r^{-1} = 2^{-1} - p^{-1}.$$

Note that analogously to Corollary 9 we get that Λ is a multiplier from B into $L_0(T)$ if and only if

$$\left(\sup_{2^k \leqq n < 2^{k+1}} |\lambda_n|\right) \in \ell_2.$$

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Institute of Mathematics
Polish Academy of Sciences
00–950 Warszawa, POLAND