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A COUNTABLE NOWHERE FIRST COUNTABLE HAUSDORFF SPACE

BY

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ABSTRACT. Here an example of a countable, nowhere first countable Hausdorff space is given.

1. **Introduction.** Steen and Seebach [1, nos. 26, 35, 98, 114] give four examples of countable spaces which are not first countable and cite no. 26 as being interesting for that reason. However each of these spaces fails to be first countable at precisely one point. Here we construct an example of a countable Hausdorff (actually, completely regular) space which is first countable at no point. Applications of some properties of this example will be given in [2].

2. **Description and proofs.** Let $\mathscr{K}$ denote the collection of all finite simplicial complexes triangulating $I=[0, 1]$ and having rational vertices. Of this collection, let L denote the complex with vertices 0 and 1. Then X will be the collection of all simplicial maps $K\to L$, where $K \in \mathscr{K}$, with the induced (from I^I) product topology. Since for fixed K the set of all simplicial maps $K\to L$ is finite, and $\mathscr{K}$ is countable, X has only countably many points. Let $I(x)$ denote the ordinary Riemann integral of $x \in X$, and for $\varepsilon>0$, $X_\varepsilon=\{x \in X : I(x)\leq\varepsilon\}$. Then by a routine geometric argument we can show that given points $t_0, t_1, \ldots, t_n \in I$, open subsets $U_0, U_1, \ldots, U_n$ of I, and a point $x \in X$ such that $x(t_i) \in U_i$, $i=0, 1, \ldots, n$, there exists a point $x_\varepsilon \in X_\varepsilon$ such that $x_\varepsilon(t_i) \in U_i$, $i=0, 1, \ldots, n$. Consequently, we have the following lemma.

LEMMA 2.1. *For each* $\varepsilon>0$, X_ε *is dense in* X.

The main theorem now follows.

THEOREM 2.1. *X is nowhere first-countable.*

Proof. Let 0 and 1 denote respectively the constant functions $0(t)=0$ and $1(t)=1$. If $x\neq 0$, choose $\varepsilon=0$ such that $\varepsilon<I(x)$. Then x is a limit point of X_ε but no sequence in X_ε converges to x, since by Lebesgue's dominated convergence theorem [3, pp. 209–210], X_ε is sequentially closed. Hence, X admits no countable base at x. If $x=0$, we conclude by observing that the mapping $h(x)(t)=1-x(t)$ is a homeomorphism between the pointed spaces $(X, 0)$ and $(X, 1)$.

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