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THE CROSSED PRODUCT THEOREM FOR PROJECTIVE SCHUR ALGEBRAS

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Abstract. The projective Schur group of a commutative ring was introduced by Lorenz and Opolka. It was revived by Nelis and Van Oystaeyen, and later by Aljadeff and Sonn. In this paper we study the intriguing question that there seems to be no adequate version of the crossed product theorem for the projective Schur group. We present a radical group $R(k)$ (k a field) situated between the Schur group and the projective Schur group, and we prove the crossed product theorem for $R(k)$.

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1. Introduction. Let R be a commutative ring. A central separable R-algebra is called an *Azumaya algebra* and the set of similarity classes of Azumaya R-algebras forms the *Brauer group* $B(R)$ [**2**]. An Azumaya algebra which is a homomorphic image of a group ring RG for some finite group G is called a *Schur algebra*, and the set of similarity classes of Schur algebras is the *Schur subgroup* $S(R)$ of $B(R)$. An Azumaya algebra is called a *projective Schur algebra* if it is a homomorphic image of a twisted group ring $R^{\alpha}G$ for some finite group G and for a 2-cocycle α contained in the cohomology class $\bar{\alpha} \in H^2(G, u(R))$, where $u(R)$ is the set of units in R regarded as a G-module with respect to the trivial G-action. The set of classes of projective Schur algebras forms the projective Schur group $PS(R)$ [**7**]. Clearly, $PS(R)$ is a subgroup of $B(R)$ containing $S(R)$.

It is well known that there are interconnections between Azumaya algebras, crossed product algebras and second dimensional cohomology classes. The relation permits the use of computation of cohomology groups. If k is a field, $B(k)$ is identified with the inductive limit of the cohomology groups $H^2(L/k, L^*)$, where L ranges over all finite Galois extensions of k and L^* is the multiplicative group in the field L [**10**, p.28].

The purpose of the present article is to study relations between cohomology classes and some Azumaya algebras such as Schur and projective Schur algebras, and radical algebras. In Section 2 we study homomorphisms on projective Schur groups and radical groups. Also in Section 3 we prove that if $L = k(\Omega)$ is a finite Galois radical field extension over k, where $\Omega < L^*$, then a cohomology class $\bar{\alpha} \in H^2(L/k, L^*)$ that is an image of some elements in $H^2(L/k, \Omega)$ corresponds to a radical k-algebra split by L; (see Theorem 4). We then extend the result to cohomology groups and radical groups in Theorem 7 as well as to Schur groups.

The notations are standard. We denote the similarity of two Azumaya algebras A and A_1 by $A \sim A_1$. Let $[A] \in B(R)$ be the similarity classes of A. If L is a Galois extension field of k, we denote the Galois cohomology group by $H^2(L/k) = H^2(L/k, L^*) = H^2(\mathrm{Gal}(L/k), L^*)$ with respect to the natural Galois action.

2. Homomorphisms of Brauer subgroup. For a commutative ring homomorphism $f: R \to T$, there is an induced homomorphism $B(f): B(R) \to B(T)$ defined by $[A] \mapsto [T \otimes A]$ for $[A] \in B(R)$ that is called the Brauer homomorphism. Similarly, by replacing Brauer groups by Schur groups, we have a Schur homomorphism $S(f): S(R) \to S(T)$.

Let k be a field of characteristic 0. For a finite Galois extension L of k with Galois group G, let $A = \sum_{\sigma\in G} Lu_\sigma$ denote an algebra having basis $\{u_\sigma | \sigma \in G\}$ such that $u_\sigma x = \sigma(x)u_\sigma$ and $u_\sigma u_\tau = \alpha(\sigma, \tau)u_{\sigma\tau}$ for $x \in L, \sigma, \tau \in G$, where each $\alpha(\sigma, \tau) \in L^*$. Then $\alpha \in Z^2(L/k)$. Also A is called a *crossed product algebra* and is denoted by $(L/k, \alpha)$. We note that an Azumaya algebra over a field is a central simple algebra [**4**, (5.1.2)] and $(L/k, \alpha)$ is a central simple k-algebra [**8**, (29.6)], thus its similarity class $[(L/k, \alpha)]$ is contained in the Brauer group $B(k)$.

The following theorem, known as the crossed product theorem, gives the relation between the Brauer group and the Galois cohomology group.

THEOREM 1. [**8**, (29.12)]. *Let L be a finite Galois extension of a field k. Then $H^2(L/k)$ is isomorphic to the kernel of the Brauer homomorphism $B(k) \to B(L)$ under the relation $\bar{\alpha} \mapsto [(L/k, \alpha)]$.*

We denote the kernel of $B(k) \to B(L)$ by $B(L/k)$. Then $B(L/k)$ is a subgroup of $B(k)$ consisting of all similarity classes of Azumaya k-algebras split by L. If $H^2(*/k)$ is the direct limit of $H^2(L/k)$, where the limit runs over all finite Galois extensions L, then $H^2(*/k)$ is known as the *Brauer group* $B(k)$. We denote by $S(L/k)$ the kernel of the Schur homomorphism $S(k) \to S(L)$.

THEOREM 2. *Let $f: R \to T$ be a homomorphism of commutative rings with $f(1_R) = 1_T$. Then there is an induced group homomorphism $PS(f): PS(R) \to PS(T)$ such that $[A] \mapsto [T \otimes_R A]$, for $[A] \in PS(R)$.*

Proof. For $[A] \in PS(R)$, let $\psi: R^\alpha G \to A$ be the surjection with finite group G and $\alpha \in Z^2(G, u(R))$. Let $\{u_g | g \in G\}$ be a basis of $R^\alpha G$ with $u_g u_x = \alpha(g, x)u_{gx}$ for $g, x \in G$. Clearly $\alpha' = f\alpha$ defined by $f\alpha(g, x) = f(\alpha(g, x))$ is a 2-cocycle in $Z^2(G, u(T))$. Consider the twisted group ring $T^{\alpha'}G$ with basis $\{v_g | g \in G\}$ such that $v_g v_x = \alpha'(g, x)v_{gx}$. Then it can be regarded as an R-module by defining an action $r \cdot \sum_{g\in G} t_g v_g = \sum_{g\in G} f(r)t_g v_g$ for $r \in R$, $t_g \in T$.

We claim that $T \otimes R^\alpha G \cong T^{\alpha'}G$ as R-modules. Indeed, there is a map

$$\theta: T \otimes R^\alpha G \to T^{\alpha'}G, \quad t \otimes \sum_{g\in G} r_g u_g \mapsto \sum_{g\in G} tf(r_g)v_g \quad (t \in T, r_g \in R)$$

which is induced from the bilinear map $T \times R^\alpha G \to T^{\alpha'}G$ defined by $(t, \sum r_g u_g) \mapsto \sum tf(r_g)v_g$. On the other hand, we have a map

$$\phi: T^{\alpha'}G \to T \otimes R^\alpha G, \quad \sum_{g\in G} t_g v_g \mapsto \sum_{g\in G} t_g \otimes u_g \quad (t_g \in T)$$

which is T-linearly extended from the map $\phi(v_g) = 1 \otimes u_g$. Since $t \otimes r_g u_g = tf(r_g) \otimes u_g \in T \otimes R^\alpha G$, we have that $\phi(v_g)\phi(v_x) = 1 \otimes u_g u_x = f(\alpha(g,x)) \otimes u_{gx} = \alpha'(g,x) \otimes u_{gx} = \phi(\alpha'(g,x)v_{gx})$ which is equal to $\phi(v_g v_x)$. Also $\phi\theta(t \otimes \sum r_g u_g) = \phi(\sum tf(r_g)v_g) = \sum tf(r_g) \otimes u_g = t \otimes \sum r_g u_g$, and $\theta\phi(\sum t_g v_g) = \theta(\sum t_g \otimes u_g) = \sum t_g v_g$, which yields $T \otimes R^\alpha G \cong T^{\alpha'} G$.

This shows that there is a surjection $T^{\alpha'} G \to T \otimes_R A$ and hence $[T \otimes A]$ belongs to $PS(T)$. Moreover $PS(f)$ is a homomorphism, since $T \otimes_R (A \otimes_R B) = (T \otimes_R A) \otimes_T (T \otimes_R B)$ for $[A], [B] \in PS(R)$. This completes the proof.

In particular if $f : R \to T$ is an inclusion then $\alpha' = \alpha$ and $T \otimes R^\alpha G \cong T^\alpha G$ as T-algebras. Thus if an Azumaya R-algebra A is a homomorphic image of $R^\alpha G$ then $T \otimes A$ is an Azumaya T-algebra which is an image of $T^\alpha G$ hence $[T \otimes A] \in PS(T)$. If L is an extension of a field k, we denote by $PS(L/k)$ the kernel of $PS(k) \to PS(L)$.

According to Theorem 1, the Brauer group $B(L/k)$ is the cohomology group $H^2(L/k)$. In order to find a cohomology group that corresponds to $PS(L/k)$, we study classes of field extensions L of k that split projective Schur k-algebras.

An extension L of k is called a *radical extension* if there is a multiplicative subgroup Ω of L^* such that $L = k(\Omega)$ and $\Omega k^*/k^*$ is torsion. A crossed product algebra $A = (L/k, \alpha)$ is an (abelian) radical k-algebra if for a multiplicative subgroup $\Omega < L^*$, $L = k(\Omega)$ is an (abelian) finite radical $\mathrm{Gal}(L/k)$-Galois extension over k (i.e., Ω is $\mathrm{Gal}(L/k)$-invariant) and $\bar{\alpha} \in H^2(L/k)$ is the image of some $\bar{\alpha}' \in H^2(L/k, \Omega)$. A radical algebra is a projective Schur algebra, and conversely every projective Schur division algebra is itself a radical abelian algebra [**1**, p.797, Theorem 1]. The set of similarity classes of radical k-algebras forms a subgroup $R(k)$ called a *radical group* [**3**]. Clearly $S(k) < R(k) < PS(k) < B(k)$.

THEOREM 3. *For a finite Galois extension L of k, there is a homomorphism $\psi : R(k) \to R(L)$ such that $[A] \mapsto [L \otimes A]$, for any $[A] \in R(k)$.*

Proof. For $[A] \in R(k)$, we write $A = (E/k, \alpha)$, where $E = k(\Omega)$ is a Galois radical extension for some multiplicative subgroup Ω of E^*, and $\bar{\alpha} \in H^2(E/k)$ is an image of some $\bar{\alpha}' \in H^2(E/k, \Omega)$ under the canonical homomorphism $\iota : H^2(E/k, \Omega) \to H^2(E/k)$.

Let $F = L \cap E$. By Galois theory, $\mathrm{Gal}(LE/L) \cong \mathrm{Gal}(E/F) \subset \mathrm{Gal}(E/k)$ and $\bar{\alpha} \in H^2(E/k, E^*)$ restricts to $\bar{\alpha}_1 \in H^2(LE/L, E^*)$. Thus it follows from [**8**, (29.13)] that $L \otimes (E/k, \alpha) \sim (LE/L, \alpha_1)$. Clearly LE is a radical extension $L(\Omega)$ of L because $\Omega < (LE)^*$ and $\Omega L^*/L^*$ is torsion.

Now for the $\bar{\alpha}' \in H^2(E/k, \Omega)$, let $\bar{\alpha}'_1 \in H^2(LE/L, \Omega)$ be an image of $\bar{\alpha}'$ under the restriction $\mathrm{res} : H^2(E/k, \Omega) \to H^2(LE/L, \Omega)$. Consider the following commutative diagram.

$$\begin{array}{ccccc} H^2(E/k,\Omega) & \overset{\text{res}}{\to} & H^2(LE/L,\Omega) & \qquad & \bar{\alpha}' \mapsto \bar{\alpha}'_1 \\ \iota\downarrow & & \downarrow\iota & & \downarrow \quad \downarrow \\ H^2(E/k,E^*) & \overset{\text{res}}{\to} & H^2(LE/L,E^*) & & \bar{\alpha} \mapsto \bar{\alpha}_1 \end{array}$$

Then $\bar{\alpha}_1$ is the image of $\bar{\alpha}'_1 \in H^2(LE/L, \Omega)$ under ι, and thus $[L \otimes A] = [(LE/L, \alpha_1)]$ belongs to $R(L)$. This proves the theorem.

We denote the kernel of $R(k) \to R(L)$ by $R(L/k)$. Then $R(L/k) < R(k)$ consists of similarity classes of the radical k-algebra A split by L, and clearly $S(L/k) < R(L/k) < PS(L/k) < B(L/k)$.

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3. Cohomology groups of radical extension. If L is a finite Galois radical extension with $L = k(\Omega)$, for some multiplicative subgroup $\Omega < L^*$, then $\iota : \Omega \hookrightarrow L^*$ induces a natural homomorphism $H^2(L/k, \Omega) \to H^2(L/k)$. We shall use the same notation ι for the induced homomorphism. In this section we find the relation between radical algebras and certain cohomology classes, and then extend to the radical group and the cohomology group.

THEOREM 4. *Let L be a finite Galois radical extension of k. Then a radical k-algebra split by L corresponds to a cohomology class in $H^2(L/k)$ that is an image of some elements in $H^2(L/k, \Omega)$, for some multiplicative subgroup, $\Omega < L^*$ such that $L = k(\Omega)$ and $\Omega k^*/k^*$ is torsion. The converse is also true.*

Proof. Let A be a radical k-algebra split by L. Then $A = (E/k, \alpha)$ for some Galois radical extension E of k with multiplicative group $\Omega_E < E^*$, $\Omega_E k^*/k^*$ torsion, and $\bar{\alpha} \in H^2(E/k)$ is an image of some $\bar{\alpha}' \in H^2(E/k, \Omega_E)$. Moreover $1 = [L \otimes A] = [(LE/L, \alpha_1)] \in B(L)$ and $1 = \bar{\alpha}_1 = \mathrm{res}_{E/k\to LE/L}\bar{\alpha}$ in $H^2(LE/L)$.

Since $LE = L(\Omega_E)$ is radical over L, $\bar{\alpha}_1$ is an image of some element $\bar{\alpha}_1'$ in $H^2(LE/L, \Omega_E)$, as shown in the proof of Theorem 3. Also by Galois theory, $\bar{\alpha}_1$ and $\bar{\alpha}_1'$ are considered as elements in $H^2(E/F)$ and $H^2(E/F, \Omega_E)$, respectively, where $F = L \cap E$. Clearly F is a Galois extension over k and $\mathrm{res}_{E/k\to E/F}\, \alpha' = \alpha_1'$. We consider the inflation-restriction sequences

$$\begin{array}{ccccc} H^2(F/k, \Omega_E^{\mathrm{Gal}(E/F)}) & \overset{\inf_{F\to E}}{\to} & H^2(E/k, \Omega_E) & \overset{\mathrm{res}_{E/k\to E/F}}{\to} & H^2(E/F, \Omega_E) \\ & & \iota\downarrow & & \iota\downarrow \\ H^2(F/k) & \overset{\inf_{F\to E}}{\to} & H^2(E/k) & \overset{\mathrm{res}_{E/k\to E/F}}{\to} & H^2(E/F). \end{array}$$

Since $H^1(E/F) = 1$ ([**9**, (1.5.4)]), the lower sequence is exact [**9**, (3.4.3)]. Hence from $1 = \bar{\alpha}_1 = \mathrm{res}\bar{\alpha}$, we have $\bar{\alpha} \in \ker(\mathrm{res}) = \mathrm{Im}(\inf_{F\to E})$, and so

$$\text{there is } \bar{\beta} \in H^2(F/k) \text{ such that } \inf_{F\to E}\bar{\beta} = \bar{\alpha}. \qquad (1)$$

Consider another inflation map $\inf_{F\to L} : H^2(F/k) \to H^2(L/k)$ and let $\inf_{F\to L}\bar{\beta} = \bar{\gamma} \in H^2(L/k)$. Then for any $x_i \in \mathrm{Gal}(L/k)$ with $\bar{x}_i = x_i\mathrm{Gal}(L/F)$ $(i = 1, 2)$, we have from [**8**, (29.16)]

$$\gamma(x_1, x_2) = \inf_{F\to L}\beta(x_1, x_2) = \beta(\bar{x}_1, \bar{x}_2). \qquad (2)$$

Since $\bar{\alpha}$ is an image of $\bar{\alpha}'$ and since $\alpha'(g_1, g_2) \in \Omega_E$ for all $g_i \in \mathrm{Gal}(E/k)$, the order of $\alpha(g_1, g_2)$ mod k^* is finite. Due to (1) and (2), the values of both β mod k^* and γ mod k^* are all of finite order. Because L is radical over k, we may write $L = k(\Omega_L)$ for some multiplicative subgroup Ω_L of L^* such that $\Omega_L k^*/k^*$ is torsion. Now we define a set Ω by

$$\Omega = \langle \Omega_L \cup \{\gamma(x_1, x_2) \mid x_i \in \mathrm{Gal}(L/k)\}\rangle. \qquad (3)$$

Then $\Omega < L^*$, $\Omega k^*/k^*$ is torsion and $L = k(\Omega_L) < k(\Omega) < L$ implies $L = k(\Omega)$. This shows that $\gamma(x_1, x_2) \in \Omega$, for all $x_i \in \mathrm{Gal}(L/k)$, so that $\bar{\gamma} \in H^2(L/k, \Omega)$.

Therefore we can assume A corresponds to $\bar{\gamma} = \inf_{F\to L}\bar{\beta}$.

Conversely, choose any $\bar{f}$ in $H^2(L/k)$ that is an image of some $\bar{f}' \in H^2(L/k, \Omega)$ for some multiplicative subgroup $\Omega < L^*$ with $L = k(\Omega)$ and $\Omega k^*/k^*$ torsion, and consider the crossed product algebra $(L/k, f)$. Then it is a radical k-algebra split by L, because $L \otimes (L/k, f) \sim (LL/L, f_1)$ with $f_1 = \mathrm{res}_{L/k \to LL/L} f \sim 1$. Hence $[(L/k, f)]$ is contained in $R(L/k)$. Thus $\bar{f}$ corresponds to $(L/k, f)$.

Using the correspondence between classes of radical algebras and cohomologies, we show that a radical group is isomorphic to a cohomology group.

Lemma 5. [**9**, (2.3.7)] *Let G_i be normal subgroups of G such that $G_2 < G_1$ and M be a G-module. Then the following inf-res-diagram commutes:*

$$\begin{array}{ccc} H^2(G/G_2, M^{G_2}) & \overset{inf}{\to} & H^2(G, M) \\ \downarrow res & & \downarrow res \\ H^2(G_1/G_2, M^{G_2}) & \overset{inf}{\to} & H^2(G_1, M) \end{array}$$

Lemma 6. *Let L be a radical extension of k. If $[A] \in R(L/k)$ such that $A = (E/k, \alpha)$ with some Galois radical extension E of k, then for any Galois radical extension V of k containing E, $[A] = [(V/k, \inf_{E \to V} \alpha)] \in R(L/k)$.*

Proof. Due to [**8**, (29.16)], $(E/k, \alpha) \sim (V/k, \inf_{E \to V} \alpha)$. Hence it is enough to show that $\mathrm{res}_{V/k \to VL/L} \inf_{E \to V} \bar{\alpha} = 1 \in H^2(VL/L)$. Consider the following diagram.

$$\begin{array}{ccc} H^2(\frac{E}{k}) \cong H^2(\frac{V}{k} / \frac{V}{E}, E^*) & \overset{\inf_{E \to V}}{\to} & H^2(\frac{V}{k}) \\ \downarrow \mathrm{res}_{\frac{E}{k} \to \frac{EL}{L}} & & \downarrow \mathrm{res}_{\frac{V}{k} \to \frac{VL}{L}} \\ H^2(\frac{E}{E \cap L}) \cong H^2(\frac{EL}{L}) \cong H^2(\frac{VL}{L} / \frac{VL}{EL}, E^*) & \overset{\inf_{EL \to VL}}{\to} & H^2(\frac{V}{V \cap L}) \cong H^2(\frac{VL}{L}, V^*) \end{array}$$

Regarding $\mathrm{Gal}(V/k)$, $\mathrm{Gal}(V/V \cap L)$ and $\mathrm{Gal}(VL/EL) \cong \mathrm{Gal}(V/V \cap EL)$ as G, G_1, G_2 in Lemma 5 respectively, the diagram is commutative. Thus we have $\mathrm{res}_{V/k \to VL/L} \inf_{E \to V} \bar{\alpha} = \inf_{EL \to VL} \mathrm{res}_{E/k \to EL/L} \bar{\alpha} = 1$, as needed.

Theorem 7. *Let L be a finite Galois radical extension and let $H_0^2(L/k)$ be the image of the homomorphism $\iota : H^2(L/k, \Omega) \to H^2(L/k)$, for some multiplicative subgroup Ω of L^* with $L = k(\Omega)$ and $\Omega k^*/k^*$ torsion. Then the radical subgroup $R(L/k)$ of $B(k)$ is isomorphic to the subgroup $H_0^2(L/k)$ of $H^2(L/k)$.*

Proof. Keeping all the notations of Theorem 4, we let

$$\begin{aligned} \phi &: H_0^2(L/k) \to R(L/k), \ \bar{f} \mapsto [(L/k, f)], \\ \psi &: R(L/k) \to H_0^2(L/k), \ [(E/k, \alpha)] \mapsto \inf_{E \cap L \to L} \bar{\beta}, \end{aligned}$$

where $\bar{\beta} \in H^2(E \cap L/k)$ such that $\inf_{E \cap L \to E} \bar{\beta} = \bar{\alpha}$, due to (1). Moreover by (3), we may take the multiplicative subgroup Ω containing all values of $\inf_{E \cap L \to L} \beta$.

It is easy to see that ϕ is a well-defined homomorphism. Now for ψ, let $[A_1], [A_2]$ be any elements in $R(L/k)$. For $i = 1, 2$ we write $A_i = (E_i/k, \alpha_i)$, where $E_i = k(\Omega_i)$ is a finite Galois radical extension of k, $\bar{\alpha}_i \in H^2(E_i/k)$ is an image of some $\bar{\alpha}_i' \in H^2(E_i/k, \Omega_i)$, and moreover

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$$\mathrm{res}_{E_i/k \to E_iL/L}\bar{\alpha}_i = 1 \in H^2(E_iL/L). \tag{4}$$

In order to prove that ψ is well-defined, we first assume that $[A_1] = [A_2]$. By the construction of ψ, we have that

$$\psi[A_i] = \mathrm{inf}_{E_i \cap L \to L}\bar{\beta}_i \quad \text{with} \quad \mathrm{inf}_{E_i \cap L \to E_i}\bar{\beta}_i = \bar{\alpha}_i \quad (i = 1, 2). \tag{5}$$

Let V be a Galois radical extension of k containing both E_1 and E_2. Then it follows from Lemma 6 that $A_i \sim (V/k, \mathrm{inf}_{E_i \to V}\alpha_i)$ in $R(L/k)$. Since $A_1 \sim A_2$ we have

$$\mathrm{inf}_{E_1 \to V}\bar{\alpha}_1 = \mathrm{inf}_{E_2 \to V}\bar{\alpha}_2 \in H^2(V/k). \tag{6}$$

Now using the transitivity of inflation maps together with (5) and (6), we have

$$\begin{aligned}
\mathrm{inf}_{L \to VL}\mathrm{inf}_{E_1 \cap L \to L}\bar{\beta}_1 &= \mathrm{inf}_{E_1 \cap L \to VL}\bar{\beta}_1 = \mathrm{inf}_{E_1 \to VL}\mathrm{inf}_{E_1 \cap L \to E_1}\bar{\beta}_1 \\
&= \mathrm{inf}_{E_1 \to VL}\bar{\alpha}_1 = \mathrm{inf}_{V \to VL}\mathrm{inf}_{E_1 \to V}\bar{\alpha}_1 \\
&= \mathrm{inf}_{V \to VL}\mathrm{inf}_{E_2 \to V}\bar{\alpha}_2 = \mathrm{inf}_{E_2 \to VL}\bar{\alpha}_2 \\
&= \mathrm{inf}_{E_2 \to VL}\mathrm{inf}_{E_2 \cap L \to E_2}\bar{\beta}_2 = \mathrm{inf}_{L \to VL}\mathrm{inf}_{E_2 \cap L \to L}\bar{\beta}_2.
\end{aligned}$$

Since $0 \to H^2(L/k) \overset{\mathrm{inf}_{L \to VL}}{\to} H^2(VL/k) \overset{\mathrm{res}}{\to} H^2(VL/L)$ is an exact sequence ([**9**, (3.4.3)]), the $\mathrm{inf}_{L \to VL}$ is a monomorphism. Hence the relation above yields the equality $\mathrm{inf}_{E_1 \cap L \to L}\bar{\beta}_1 = \mathrm{inf}_{E_2 \cap L \to L}\bar{\beta}_2$, so that (5) yields

$$\psi[A_1] = \mathrm{inf}_{E_1 \cap L \to L}\bar{\beta}_1 = \mathrm{inf}_{E_2 \cap L \to L}\bar{\beta}_2 = \psi[A_2].$$

Now for $[A_1], [A_2]$ described in (4), we consider the composite fields $E = E_1E_2$ and EL. Since E_i $(i = 1, 2)$ and L are finite Galois radical extensions of k, so are E and EL. By [**8**, (29.16)] again, we have $(E_i/k, \alpha_i) \sim (EL/k, \mathrm{inf}_{E_i \to EL}\alpha_i)$; hence it follows that

$$[A_1][A_2] = [(EL/k, \mathrm{inf}_{E_1 \to EL}\alpha_1\mathrm{inf}_{E_2 \to EL}\alpha_2)].$$

Moreover the commutativity of inf-res-diagram in Lemma 5 gives rise to

$$\begin{aligned}
&\mathrm{res}_{EL/k \to EL/L}(\mathrm{inf}_{E_1 \to EL}\bar{\alpha}_1\mathrm{inf}_{E_2 \to EL}\bar{\alpha}_2) \\
&= (\mathrm{res}_{EL/k \to EL/L}\mathrm{inf}_{E_1 \to EL}\bar{\alpha}_1)(\mathrm{res}_{EL/k \to EL/L}\mathrm{inf}_{E_2 \to EL}\bar{\alpha}_2) \\
&= (\mathrm{inf}_{E_1L \to EL}\mathrm{res}_{E_1/k \to E_1L/L}\bar{\alpha}_1)(\mathrm{inf}_{E_2L \to EL}\mathrm{res}_{E_2/k \to E_2L/L}\bar{\alpha}_2),
\end{aligned}$$

which is equal to 1 in $H^2(EL/L)$ by (4). Thus $[(EL/k, \mathrm{inf}_{E_1 \to EL}\alpha_1\mathrm{inf}_{E_2 \to EL}\alpha_2)]$ belongs to $R(L/k)$.

Therefore we have $\psi([A_1][A_2]) = \mathrm{inf}_{EL \cap L \to L}\bar{\beta}$ with $\bar{\beta} \in H^2(EL \cap L/k)$ such that $\mathrm{inf}_{EL \cap L \to EL}\bar{\beta} = \mathrm{inf}_{E_1 \to EL}\bar{\alpha}_1\mathrm{inf}_{E_2 \to EL}\bar{\alpha}_2$. Since $EL \cap L = L$ it shows that

$$\psi([A_1][A_2]) = \bar{\beta} \quad \text{with } \mathrm{inf}_{L \to EL}\bar{\beta} = \mathrm{inf}_{E_1 \to EL}\bar{\alpha}_1\mathrm{inf}_{E_2 \to EL}\bar{\alpha}_2. \tag{7}$$

On the other hand from $[A_i] = [(E_i/k, \alpha_i)] = [(EL/k, \mathrm{inf}_{E_i \to EL}\alpha_i)]$, we have $\psi[A_1]\psi[A_2] = \mathrm{inf}_{EL \cap L \to L}\bar{\beta}_1\mathrm{inf}_{EL \cap L \to L}\bar{\beta}_2$, where $\bar{\beta}_i \in H^2(EL \cap L/k)$, such that $\mathrm{inf}_{EL \cap L \to EL}\bar{\beta}_i = \mathrm{inf}_{E_i \to EL}\bar{\alpha}_i$ for $i = 1, 2$. Again since $EL \cap L = L$, we have

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$$\psi[A_1]\psi[A_2] = \overline{\beta_1\beta_2} \quad \text{with } \inf_{L\to EL}\bar{\beta}_i = \inf_{E_i\to EL}\bar{\alpha}_i. \tag{8}$$

From (7) and (8) we have

$$\inf_{L\to EL}\bar{\beta} = \inf_{L\to EL}\bar{\beta}_1 \inf_{L\to EL}\bar{\beta}_2 = \inf_{L\to EL}\overline{\beta_1\beta_2}.$$

Due to the fact that $\inf_{L\to EL}$-$\text{res}_{EL/k\to EL/L}$-sequence is exact, $\inf_{L\to EL}$ is injective so that $\bar{\beta} = \overline{\beta_1\beta_2}$ and ψ is a homomorphism.

In order to finish the proof we must show that $\psi\phi = 1$ and $\phi\psi = 1$. If $A = (L/k, \alpha)$ (i.e., if $L = E$), then $\inf_{E\cap L\to L}\bar{\beta} = \inf_{E\cap L\to E}\bar{\beta} = \bar{\alpha}$; hence $\psi([A]) = \bar{\alpha}$. This shows that $\psi\phi(\bar{f}) = \psi[(L/k, f)] = \bar{f}$, for any $\bar{f} \in H_0^2(L/k)$.

Conversely for any $[A] \in R(L/k)$ with $A = (E/k, \alpha)$, we have

$$\phi\psi([A]) = \phi(\inf_{E\cap L\to L}\bar{\beta}) = [(L/k, \inf_{E\cap L\to L}\beta)] \quad \text{for } \bar{\beta} \in H^2(E\cap L/k).$$

This algebra is similar to $(E\cap L/k, \beta)$ and to $(E/k, \inf_{E\cap L\to E}\beta) = (E/k, \alpha)$. Hence $\phi\psi([A]) = [A]$. This completes the proof.

We conclude this section with an application of Theorem 7 to the Schur group $S(k)$. We may assume char $k = 0$, because $S(k) = 0$ otherwise.

Let $L = k(\varepsilon_m)$ (ε_m: a primitive m-th root of unity, $m > 0$) be a cyclotomic extension field in the algebraic closure of k and let $\mu(L)$ be the group of roots of unity in L. Then $\mu(L)$ is a (multiplicative) $\text{Gal}(L/k)$-submodule of L^*. The crossed product algebra $(L/k, \alpha)$, where values of $\alpha \in Z^2(L/k)$ are in $\mu(L)$, is called a *cyclotomic k-algebra*. If we denote the set of all Azumaya algebra classes of $B(k)$ that are represented by a cyclotomic k-algebra by $C(k)$, then $C(k)$, is a subgroup of $B(k)$, and indeed $C(k) = S(k)$, due to the Brauer-Witt theorem [**10**].

Let $H_0^2(L/k)$ be the image of the homomorphism $\iota : H^2(L/k, \mu(L)) \to H^2(L/k)$ induced by $\mu(L) \hookrightarrow L^*$. Then ι is injective, since $\mu(L)$ is a subgroup of the torsion group of L^* [**6**, p.91], and thus we may identify $H^2(L/k, \mu(L)) = H_0^2(L/k) < H^2(L/k)$.

Corollary 8. *For $L = k(\varepsilon_m)$ ($m > 0$), $S(L/k)$ is isomorphic to $H_0^2(L/k)$.*

Proof. If A is a Schur k-algebra split by L, then A is a cyclotomic algebra $(E/k, \alpha)$ for some $E = k(\varepsilon_t)$ ($t > 0$) and the values of α are in $\mu(E)$. Moreover $\text{res}_{E/k\to LE/L}\bar{\alpha} = 1$. Thus the proof follows immediately from Theorem 7.

If $k \subset Q(\varepsilon_l)$ (Q: rational number field), Janusz proved the next lemma.

Lemma 9. [**5**], [**10**, (7.12)] *Let $l, m > 0$ be either odd or divisible by 4 and $k \subset Q(\varepsilon_l)$. Let $L = k(\varepsilon_m)$ and $K = k(\varepsilon_{m'})$, where $m' = 4^\delta p_1 \cdots p_s$ with distinct prime divisors p_i of m and $\delta = 1$ if $4 \nmid l$, $4|m$ and $\delta = 0$ otherwise. Then the inflation map $\inf_{K\to L}$ yields an isomorphism $H_0^2(K/k) \cong H_0^2(L/k)$.*

This result enables us to reduce the cyclotomic algebra $(k(\varepsilon_m)/k, \alpha)$ to a smaller algebra $(k(\varepsilon_{m'})/k, \alpha')$ ([**10**, (7.9)]). Together with Lemma 9 and Corollary 8, we have the next theorem.

Theorem 10. *Let $k \subset Q(\varepsilon_l)$. If $k(\varepsilon_m)$ and $k(\varepsilon_{m'})$ are the same fields in Lemma 9, then we have the following commutative diagram:*

$$\begin{array}{ccc} H_0^2(k(\varepsilon_{m'})/k) & \overset{inf_{k(\varepsilon_{m'})\to k(\varepsilon_m)}}{\rightarrow} & H_0^2(k(\varepsilon_m)/k) \\ \phi_1\downarrow & & \downarrow\phi_2 \\ S(k(\varepsilon_{m'})/k) & \rightarrow & S(k(\varepsilon_m)/k) \end{array}$$

where ϕ_i $(i = 1, 2)$ *are the isomorphisms defined in Corollary* 8.

Proof. Let $\chi = \inf_{k(\varepsilon_{m'})\to k(\varepsilon_m)}$ be the isomorphism in Lemma 9. Now ϕ_1 maps $\bar{\alpha}$ to $[(k(\varepsilon_{m'})/k, \alpha)]$, for $\bar{\alpha} \in H_0^2(k(\varepsilon_{m'})/k)$, and ϕ_2 is defined similarly.

If $[B] \in S(k(\varepsilon_{m'})/k)$, then $B = (k(\varepsilon_t)/k, \beta)$ with $\bar{\beta} \in H^2(k(\varepsilon_t)/k)$ having values in $\mu(k(\varepsilon_t))$ $(t > 0)$. Also $\mathrm{res}_{k(\varepsilon_t)/k\to k(\varepsilon_{m'},\varepsilon_t)/k(\varepsilon_{m'})}\bar{\beta} = 1$. Then we have $\phi_2\chi\phi_1^{-1}[B] = \phi_2\chi(\inf_{F\to k(\varepsilon_{m'})}\bar{f})$, for $F = k(\varepsilon_t) \cap k(\varepsilon_{m'})$, where $\bar{f} \in H^2(F/k)$ and $\inf_{F\to k(\varepsilon_t)}\bar{f} = \bar{\beta}$, due to Theorem 7. Thus we have

$$\phi_2\chi\phi_1^{-1}[B] = \phi_2(\inf_{k(\varepsilon_{m'})\to k(\varepsilon_m)}\inf_{F\to k(\varepsilon_{m'})}\bar{f}) = \phi_2(\inf_{F\to k(\varepsilon_m)}\bar{f})$$

and it follows that we may define $\xi : S(k(\varepsilon_{m'})/k) \to S(k(\varepsilon_m)/k)$ by

$$\xi[B] = \xi[(k(\varepsilon_t)/k, \beta)] = [(k(\varepsilon_m)/k, \inf_{F\to k(\varepsilon_m)}f)].$$

Moreover for any $\bar{\alpha} \in H_0^2(k(\varepsilon_{m'})/k)$, we have $\phi_2\chi(\bar{\alpha}) = \phi_2(\inf_{k(\varepsilon_{m'})\to k(\varepsilon_m)}\bar{\alpha}) = [(k(\varepsilon_m)/k, \inf_{k(\varepsilon_{m'})\to k(\varepsilon_m)}\alpha)] = [(k(\varepsilon_m)/k, \inf_{F\to k(\varepsilon_m)}\alpha)] = \xi[(k(\varepsilon_{m'})/k, \alpha)] = \xi\phi_1(\bar{\alpha})$ since $F = k(\varepsilon_m) \cap k(\varepsilon_{m'})$. This completes the proof.

The diagram in Theorem 10 yields stronger relations between Schur and cohomology groups than that of Brauer and cohomology groups [**8**, p.252]: for a Galois extension E/k containing L/k, the diagram is commutative.

$$\begin{array}{ccc} H^2(L/k) & \overset{\inf}{\longrightarrow} & H^2(E/k) \\ \downarrow & & \downarrow \\ B(L/k) & \longrightarrow & B(E/k) \end{array}$$

In the diagram, vertical arrows are isomorphisms from Theorem 1, while $H^2(L/k) \to H^2(E/k)$ and $B(L/k) \to B(E/k)$ are homomorphisms.

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REFERENCES

1. E. Aljadeff and J. Sonn, Projective Schur division algebras are abelian crossed products, *J. Algebra* **163** (1994), 795–805.

2. M. Auslander and O. Goldman, The Brauer group of a commutative ring, *Trans. Amer. Math. Soc.* **97** (1960), 367–409.

3. E. Choi and H. Lee, Projective Schur algebras over a field of positive characteristic, *Bull. Austral. Math. Soc.* **58** (1998), 103–106.

4. F. DeMeyer and E. Ingraham, *Separable algebras over commutative rings* (Springer-Verlag, 1970).

5. G. Janusz, The Schur group of cyclotomic fields, *J. Number Theory* **7** (1975), 345–352.

6. G. Karpilovsky, *Projective representations of finite groups* (Marcel Dekker, New York, 1985).

7. F. Lorenz and H. Opolka, Einfache Algebren und Projektive Darstellungen über Zahlköpern, *Math. Z.* **162** (1978), 175–182.

8. I. Reiner, *Maximal orders* (Academic Press, 1975).

9. E. Weiss, *Cohomology of groups* (Academic Press, 1969).

10. T. Yamada, *The Schur subgroup of the Brauer group* (Springer-Verlag, 1974).