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QUOTIENT SPACES WITHOUT BASES IN NUCLEAR FRECHET SPACES

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The first example of a nuclear Fréchet space without a basis was given by B. S. Mitiagin and N. M. Zobin [**9**; **10**]. The question of existence of subspaces without bases in nuclear Fréchet spaces was recently settled in papers by P. Djakov and B. S. Mitiagin [**2**] and Ed Dubinsky [**5**]. In this paper we consider the analogous question for quotient spaces. As in the case of subspaces we obtain a complete solution to the problem.

THEOREM. *Every nuclear Fréchet space not isomorphic to ω has a quotient space which has no basis.*

The proof is very similar to the argument in [**5**]. The specific techniques for replacing subspace constructions with quotient space constructions were worked out in [**6**]. The basic embedding in [**2**] is replaced with a method relative to quotients. Finally, the fact that our quotient space has no basis is based on the results in [**2**].

In addition we use a fact about common quotient spaces (see proposition below) which may be of independent interest.

With the results of this paper we now have a complete set of information about subspaces and quotient spaces with or without bases in nuclear Fréchet spaces. Specifically, every nuclear Fréchet space not isomorphic to ω has:

a subspace with a basis,
a subspace without a basis,
a quotient space with a basis and a continuous norm,
a quotient space without a basis but with a continuous norm,
a quotient space without a basis and without a continuous norm, and
a quotient space isomorphic to ω.

The proofs of these facts are contained in this paper and [**1**; **5**; **7**; **8**]. We do not have the exact reference for the third statement but it is an easy extension of the proof of Theorem IV.1 in [**7**].

1. Notation and terminology. We take as known the elementary theory of bases (including unconditional and absolute bases) in nuclear Fréchet spaces. A Fréchet space admits a *continuous norm* if and only if its topology is

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determined by a sequence of norms. It is *normable* if this can be done in such a way that each norm is dominated by an appropriate scalar multiple of one of them.

Our field of scalars will be the real numbers $\mathbf{R}$. We shall denote by $\mathbf{N}$ the set of positive integers. By ω we mean the nuclear Fréchet space consisting of all sequences of real numbers with the usual Cartesian product topology.

By the term *subspace* we will always mean a closed subspace and *quotient space* will mean a quotient by a closed subspace. F is a *complemented subspace* in E if it is a subspace and there is another subspace G such that E is isomorphic to the topological direct sum of F, G. We shall indicate this situation by writing $E = F \oplus G$.

Let $(a_n^k)_{n,k}$ be an infinite matrix satisfying

$$0 < a_n^k < a_n^{k+1} \quad \text{for all } n, k \in \mathbf{N}.$$

The Köthe space K determined by the matrix (a_n^k) is the Fréchet space of all scalar sequences $\xi = (\xi_n)$ such that

$$p_k(\xi) = \sum_n |\xi_n| a_n^k < \infty \quad \text{for all } k \in \mathbf{N},$$

with topology determined by the sequence of norms (p_k). As is well known, K is nuclear if and only if for each $k \in \mathbf{N}$ there is $j \in \mathbf{N}$ such that

$$\sum_n \frac{a_n^k}{a_n^j} < \infty .$$

2. Known results. We shall make use of several previously established results. For clarity we state them in full detail and give references for the proofs.

(2.1) Every Fréchet space with an unconditional basis and not isomorphic to ω has a complemented subspace which has a basis and a continuous norm. This result is an immediate consequence of [**3**, Theorem 7].

(2.2) Let $(a_n^k)_{n,k}$ be an infinite matrix of positive numbers satisfying

$$(1) \quad \frac{a_{n+k}^k}{a_n^k} < \frac{a_{n+k}^{k+1}}{a_n^{k+1}} \; n, k \in \mathbf{N}.$$

Given scalars $t_1, \ldots, t_p$ not all 0 and $k \in \mathbf{N}$ we define

$$\rho^k(t_1, \ldots, t_p) = \min \left\{ \rho : \min_{1 \leqq i \leqq p} \frac{a_i^k}{|t_i|} = \frac{a_\rho^k}{|t_\rho|} \right\} .$$

It then follows that if $0 < \rho^m < \rho^{m-1} < \ldots < \rho^1 < p$ and $0 = l_0 < l_1 < \ldots < l_m$ are integers, we can choose $t_1, \ldots, t_p$ such that $t_j = 0$ for $j \neq \rho^1, \ldots, \rho^m$, $t_{\rho^1} \neq 0$ but otherwise arbitrary and

$$(2) \quad \frac{a_{\rho^i}^{l_i}}{a_{\rho^{i+1}}^{l_i}} < \frac{|t_{\rho^i}|}{|t_{\rho^{i+1}}|} < \frac{a_{\rho^i}^{l_i+1}}{a_{\rho^{i+1}}^{l_i+1}}, \quad i = 1, \ldots, m-1.$$

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Moreover, if any such choice is made then

$$\rho^k(t_1, \ldots, t_p) = \rho^i \quad \text{for } l_{i-1} < k \leqq l_i, i = 1, \ldots, m.$$

This result is proved in [**6**, Lemma 1.2]. Note that the matrix (a_n^k) need only be defined for $n \leqq \rho^1$ and $k \leqq l_m$.

(2.3) Let E be a nuclear Fréchet space with a fundamental system of norms $(|\cdot|_k)$ and a basis (x_i). Let $0 = p_0 < p_{n-1} < p_n$, $n \in \mathbf{N}$ be integers and (t_i) a sequence of scalaras such that for each $n \in \mathbf{N}$ there exists $i(n)$ with $p_{n-1} < i(n) \leqq p_n$ and $t_{i(n)} \neq 0$. Set $a_i^k = |x_i|_k$ and, corresponding to this matrix, set $\rho_n^k = \rho^k(t_{p_{n-1}+1}, \ldots, t_{p_n})$. Finally let K be the Köthe space determined by the matrix

$$\left[\frac{a_{\rho_n^k}{}^k}{|t_{\rho_n^k}|}\right] \quad n, k \in \mathbf{N}.$$

Then K is isomorphic to a quotient space of E. This result is contained in [**6**, Theorem 1.3].

(2.4) Every basis in a nuclear Fréchet space has a subsequence (x_n) which generates a complemented subspace E_0 which has a fundamental system of norms $(||\cdot||_k)$ such that

$$\frac{||x_{n+1}||_k}{||x_n||_k} < \frac{||x_{n+1}||_{k+1}}{||x_n||_{k+1}} \quad \text{for all } n, k \in \mathbf{N}.$$

The proof of this fact is contained in [**4**, pp. 211–212].

3. Main results. Our first result is analogous to [**4**, Theorem 3] which deals with subspaces and does not require the assumption of bases. It should be noted that the present result is different from the standard fact that ω is a quotient space of every non-normable Fréchet space with a continuous norm [**8**, § 31, 4(1)].

PROPOSITION. *If E, F are nuclear Fréchet spaces with bases and neither is isomorphic to ω, then there is a nuclear Fréchet space G with a basis and a continuous norm such that G is isomorphic to a quotient space of E and also to a quotient space of F.*

Proof. In view of (2.1) and (2.4) we may assume without loss of generality that E, F have bases (u_j), (v_j) respectively and fundamental sequences of norms $(||\cdot||_p)$, $(|\cdot|_p)$ such that (1) holds for each of the matrices $(||u_n||_k)$, $(|v_n|_k)$ and

$$(3) \quad \lim_j \frac{||u_j||_p}{||u_j||_{p+1}} = \lim_j \frac{|v_j|_p}{|v_j|_{p+1}} = 0.$$

For each n we will select finite subsets $u_1^n, \ldots, u_n^n$ of (u_j) and $v_1^n, \ldots, v_n^n$ of (v_j) keeping the original order and never using the same vector twice. We will also select scalars $s_1^n, \ldots, s_n^n$ and $t_1^n, \ldots, t_n^n$ such that

$$\rho^k(s_1^n, \ldots, s_n^n) = \rho^k(t_1^n, \ldots, t_n^n) = n - k + 1, \quad k = 1, \ldots, n$$

where the first ρ^k is defined as in (2.2) relative to the matrix $(||u_j||_p)$ and the second ρ^k relative to the matrix $(|v_j|_p)$. Finally we shall make our selection so that

$$(4) \quad \frac{|v_{j+1}^n|_{n-j}}{t_{j+1}^n} \leqq \frac{||u_{j+1}^n||_{n-j}}{s_{j+2}^n} \leqq \frac{|v_j^n|_{n-j+1}}{t_j^n} \leqq \frac{||u_j^n||_{n-j+1}}{s_j^n} \quad j = 1, \ldots, n-1.$$

The selection is made inductively. Set $s_1^n = 1$, choose u_1^n, v_1^n arbitrarily and set

$$t_1^n = \frac{|v_1^n|_n}{||u_1^n||_n}.$$

Suppose that $s_j^n, u_j^n, t_j^n, v_j^n$ have been chosen. From (3) we can select u_{j+1}^n different from all other choices such that

$$\frac{||u_{j+1}^n||_{n-j}}{||u_{j+1}^n||_{n-j+1}} < \frac{s_j^n|v_j^n|_{n-j+1}}{t_j^n||u_j^n||_{n-j+1}}$$

and then select s_{j+1}^n so that

$$\max\left\{t_j^n \frac{||u_{j+1}^n||_{n-j}}{|v_j^n|_{n-j+1}}, s_j^n \frac{||u_{j+1}^n||_{n-j}}{||u_j^n||_{n-j}}\right\} < s_{j+1}^n < s_j^n \frac{||u_{j+1}^n||_{n-j+1}}{||u_j^n||_{n-j+1}}.$$

This is possible because of the previous inequality and because in choosing u_{j+1}^n so that $u_1^n, \ldots, u_{j+1}^n$ remains in the original order we still have (1).

In a similar manner we choose v_{j+1}^n and then t_{j+1}^n so that

$$\frac{|v_{j+1}^n|_{n-j}}{|v_{j+1}^n|_{n-j+1}} < \frac{t_j^n||u_{j+1}^n||_{n-j}}{s_{j+1}^n|v_j^n|_{n-j+1}}$$

$$\max\left\{s_{j+1}^n \frac{|v_{j+1}^n|_{n-j}}{||u_{j+1}^n||_{n-j}}, t_j^n \frac{|v_{j+1}^n|_{n-j}}{|v_j^n|_{n-j}}\right\} < t_{j+1}^n < t_j^n \frac{|v_{j+1}^n|_{n-j+1}}{|v_j^n|_{n-j+1}}.$$

This completes the selection. We have

$$\frac{||u_{j+1}^n||_{n-j}}{||u_j^n||_{n-j}} < \frac{s_{j+1}^n}{s_j^n} < \frac{||u_{j+1}^n||_{n-j+1}}{||u_j^n||_{n-j+1}}.$$

We also have (1) for the matrix $(||u_j^n||_{n-j})_{j,n-j}$ since we did not change the order. Hence, we may apply (2.2) to conclude that $\rho^j(s_1^n, \ldots, s_n^n) = n - j + 1$, $j = 1, \ldots, n$. Similarly we conclude that $\rho^j(t_1^n, \ldots, t_n^n) = n - j + 1$, $j = 1, \ldots, n$. The first two inequalities in (4) can be read off of the above relations and the last comes from replacing $j + 1$ with j.

From (2.3) we conclude that E has a quotient space isomorphic to the Köthe space determined by the matrix (a_n^k) where

$$a_n^k = \frac{||u_{n-k+1}^n||_k}{s_{n-k+1}^n}, \quad k = 1, \ldots, n$$

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and F has a quotient space isomorphic to the Köthe space determined by the matrix (b_n^k) where

$$b_n^k = \frac{|v_{n-k+1}^n|_k}{t_{n-k+1}^n}, \quad k = 1, \ldots, n.$$

From the relations (4) we obtain

$$b_n^k \leqq a_n^k \leqq b_n^{k+1} \leqq a_n^{k+1} \quad k = 1, \ldots, n-1$$

which implies that the two Köthe spaces are identical. This completes the proof.

We are now ready to give the proof of the theorem. Our construction is in two parts. First, given a nuclear Fréchet space E with a basis and not isomorphic to ω we will construct a quotient space Y with a basis and continuous norm such that certain technical inequalities hold. The argument here is dual to that given in [**5**] for subspaces. The second part is to show that a space X of type constructed in [**2**] so as to have no basis can be obtained as a quotient space of Y.

We begin with the construction of Y. By (2.1) we may write E as a direct sum of three infinite dimensional subspaces none of which is isomorphic to ω. To these subspaces we apply the proposition twice along with (2.4) and then put it all back together to obtain a quotient space Z of E with an increasing fundamental sequence of norms $(||\cdot||_p)$ and a basis (z^j) which is the union of three disjoint subsequences each of which may be subdivided into infinitely many pairwise disjoint subsequences so that we have sequences $(u^{j,n})$, $(v^{j,n})$, $(w^{j,n})$ such that

$$(5) \quad \frac{||z^{j+1}||_p}{||z^{j+1}||_{p+1}} < \frac{||z^j||_p}{||z^j||_{p+1}} \quad \text{for all } j, p$$

$$(6) \quad \lim_{j\to\infty} \frac{||z^j||_p}{||z^j||_{p+1}} = 0 \quad \text{for all } p$$

$$(7) \quad ||u^{j,n}||_p = ||v^{j,n}||_p = ||w^{j,n}||_p \quad \text{for all } p, j, n.$$

Let $\mathcal{N} = \{(p_1, p_2, p_3) \in \mathbf{N}^3 : p_1 < p_2 < p_3\}$ and let $\sigma : \mathbf{N} \to \mathcal{N}$ be an infinity-to-one surjection.

For each fixed $n \in \mathbf{N}$ we will make a certain selection. As long as n remains fixed there will be no ambiguity if we do not indicate it in our notation. Thus, let $(p_1, p_2, p_3) = \sigma(n)$.

We will select three linear combinations of elements of $(u^{j,n})_j$, $(v^{j,n})_j$, $(w^{j,n})_j$ and write them as follows:

$$r_1u_1 + r_2u_2 + r_3u_3$$
$$s_1v_1 + s_2v_2 + s_3v_3$$
$$t_1w_1 + t_2w_2 + t_3w_3.$$

Thus, u_i, $i = 1, 2, 3$ are taken from $(u^{j,n})_j$ without changing the order and r_i are scalars. Similarly for the other two linear combinations.

In view of (7) we can choose u_1, v_1, w_1 such that

$$||u_1||_p = ||v_1||_p = ||w_1||_p \quad \text{for all } p \in \mathbf{N}.$$

Choose $r_1 = t_1 = 1$ and

$$s_1 = n \frac{||v_1||_{p_3}}{||u_1||_{p_2+1}}.$$

In view of (6) we can choose v_2 such that

$$\frac{||v_2||_{p_2}}{||v_2||_{p_2+1}} < \min \left\{ \frac{||v_1||_{p_2}}{||v_1||_{p_2+1}}, \frac{1}{s_1} \frac{||v_1||_{p_2}}{||w_1||_{p_2+1}}, \frac{||w_1||_{p_2}}{||w_1||_{p_2+1}}, s_1 \frac{||w_1||_{p_2}}{||v_1||_{p_2+1}} \right\}.$$

It is then possible to choose s_2 so that

$$\max \left\{ s_1 \frac{||v_2||_{p_2}}{||v_1||_{p_2}}, \frac{||v_2||_{p_2}}{||w_1||_{p_2}} \right\} < s_2 < \min \left\{ s_1 \frac{||v_2||_{p_2+1}}{||v_1||_{p_2+1}}, \frac{||v_2||_{p_2+1}}{||w_1||_{p_2+1}} \right\}.$$

Again we apply (7) to select w_2 such that

$$||w_2||_p = ||v_2||_p \quad \text{for all } p \in \mathbf{N}.$$

Choose $t_2 = s_2$ and apply (6) to choose u_2 such that

$$\frac{||u_2||_{p_2}}{||u_2||_{p_2+1}} < \min \left\{ \frac{||u_1||_{p_2}}{||u_1||_{p_2+1}}, \frac{1}{n\, s_2} \frac{||v_2||_{p_1+1}}{||u_1||_{p_2+1}} \right\}.$$

It is then possible to choose r_2 such that

$$\max \left\{ \frac{||u_2||_{p_2}}{||u_1||_{p_2}}, n\, s_2 \frac{||u_2||_{p_2}}{||v_2||_{p_1+1}} \right\} < r_2 < \frac{||u_2||_{p_2+1}}{||u_1||_{p_2+1}}.$$

Again we apply (6) to select u_3 such that

$$\frac{||u_3||_{p_1}}{||u_3||_{p_1+1}} < \min \left\{ \frac{||u_2||_{p_1}}{||u_2||_{p_1+1}}; \frac{||v_2||_{p_1}}{||v_2||_{p_1+1}}; \frac{s_2}{r_2} \frac{||u_2||_{p_1}}{||v_2||_{p_1+1}}; \frac{r_2}{s_2} \frac{||v_2||_{p_1}}{||u_2||_{p_1+1}} \right\}.$$

It is then possible to select r_3 such that

$$\max \left\{ r_2 \frac{||u_3||_{p_1}}{||u_2||_{p_1}}, s_2 \frac{||u_3||_{p_1}}{||v_2||_{p_1}} \right\} < r_3 < \min \left\{ r_2 \frac{||u_3||_{p_1+1}}{||u_2||_{p_1+1}}, s_2 \frac{||u_3||_{p_1+1}}{||v_2||_{p_1+1}} \right\}.$$

Again we apply (7) to select v_3 such that

$$||v_3||_p = ||u_3||_p \quad \text{for all } p \in \mathbf{N}$$

and we choose $s_3 = r_3$.

Finally, we apply (6) to select w_3 such that

$$\frac{||w_3||_{p_1}}{||w_3||_{p_1+1}} < \min \left\{ \frac{||w_2||_{p_1}}{||w_2||_{p_1+1}}, \frac{t_2}{nr_3} \frac{||u_3||_1}{||w_2||_{p_1+1}} \right\}$$

and it is then possible to select t_3 such that

$$\max\left\{t_2\frac{||w_3||_{p_1}}{||w_2||_{p_1}}, n\, r_3\frac{||w_3||_{p_1}}{||u_3||_1}\right\} < t_3 < t_2\frac{||w_3||_{p_1+1}}{||w_2||_{p_1+1}}.$$

This completes the construction of three linear combinations.

We apply (2.2) to the matrix $(||u_i||_p)_{i,p}$ and the scalars r_1, r_2, r_2. From the above relations we have

$$\frac{||u_2||_{p_2}}{||u_1||_{p_2}} < \frac{r_2}{r_1} < \frac{||u_2||_{p_2+1}}{||u_1||_{p_2+1}}$$

$$\frac{||u_3||_{p_1}}{||u_2||_{p_1}} < \frac{r_2}{r_2} < \frac{||u_3||_{p_1+1}}{||u_2||_{p_1+1}}.$$

Hence, it follows that

$$\rho^p(r_1, r_2, r_3) = \begin{cases} 3 & \text{if } p \leqq p_1 \\ 2 & \text{if } p_1 < p \leqq p_2 \\ 1 & \text{if } p_2 < p. \end{cases}$$

Next we apply (2.3) to the complemented subspace of Z generated by $(u^{j,n})$. Of course, it must be recalled that the above selection is made for each n. This gives us (p_n) and (t_i) so we may conclude that the subspace generated by $(u^{j,n})$ has a quotient space isomorphic to the Köthe space determined by the matrix (a_n^p) where

$$a_n^p = \frac{||u_{\rho^p}||_p}{r_{\rho^p}}, \quad p \in \mathbf{N}$$

with the dependence on n understood and $\rho^p = \rho^p(r_1, r_2, r_3)$.

In a similar manner we construct matrices (b_n^p) and (c_n^p) corresponding to $s_1v_1 + s_2v_2 + s_3v_3$ and $t_1w_1 + t_2w_2 + t_3w_3$ respectively.

Since the basic sequences $(u^{j,n})$, $(v^{j,n})$, $(w^{j,n})$ generate pairwise disjoint complemented subspaces of Z it follows that Z and hence E has a quotient space Y isomorphic to a Köthe space determined by a matrix which consists of three disjoint parts (a_n^p), (b_n^p), (c_n^p) given by

$$a_n^p = \frac{||u_3||_p}{r_3}, \quad b_n^p = \frac{||v_3||_p}{s_3}, \quad c_n^p = \frac{||w_3||_p}{t_3} \quad \text{for } p \leqq p_1;$$

$$a_n^p = \frac{||u_2||_p}{r_2}, \quad b_n^p = \frac{||v_2||_p}{s_2}, \quad c_n^p = \frac{||w_2||_p}{t_2} \quad \text{for } p_1 < p \leqq p_2;$$

$$a_n = \frac{||u_1||_p}{r_1}, \quad b_n = \frac{||v_1||_p}{s_1}, \quad c_n^p = \frac{||w_1||_p}{t_1} \quad \text{for } p_2 < p.$$

(Again some of the dependence on n is not explicit in the notation).

Thus, if we put these equalities together with the relations obtained in the selection we have (and this is what will be used in the sequel) the following facts:

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Each of the three matrices is monotone increasing in p.

$$c_n^p \leqq a_n^p = b_n^p \text{ for } 1 \leqq p \leqq p_1 \quad \text{and} \quad c_n^{p_1} \leqq \frac{1}{n} a_n^1 \text{ for } n \in \mathbf{N}$$

$$a_n^p \leqq c_n^p = b_n^p \text{ for } p_1 < p \leqq p_2 \quad \text{and} \quad a_n^{p_2} \leqq \frac{1}{n} c_n^{p_1+1} \text{ for } n \in \mathbf{N}$$

$$a_n^p = c_n^p \text{ for } p_2 < p \quad \text{and} \quad b_n^{p_3} \leqq \frac{1}{n} a_n^{p_2+1} \text{ for } n \in \mathbf{N}.$$

This completes the construction of the space Y and we turn now to construct a space X which is a quotient of Y (and hence E) and has no basis.

For convenience of notation we write

$$\alpha_n^p = (a_n^p)^2 \quad \beta_n^p = (b_n^p)^2 \quad \gamma_n^p = (c_n^p)^2.$$

Thus, we may consider that Y consists of all sequences $(\eta_n, \theta_n, \tau_n)_n$ of triples of scalars for which the following norms are finite

$$|(\eta_n, \theta_n, \tau_n)|_p = \left(\sum_n \ (\alpha_n^p(\eta_n)^2 + \beta_n^p(\theta_n)^2 + \gamma_n^p(\tau_n)^2)\right)^{1/2} \quad p \in \mathbf{N}.$$

For $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$ define $\delta_{np}(\xi)$ as follows:

$$\delta_{np}(\xi) = \begin{cases} \dfrac{(\alpha_n^p + \beta_n^p)\gamma_n^p}{\alpha_n^p + \beta_n^p + \gamma_n^p}\xi_1^2 + \dfrac{\alpha_n^p(\beta_n^p + \gamma_n^p)}{\alpha_n^p + \beta_n^p + \gamma_n^p}\xi_2^2 & \text{for } p \leqq p_2 \\ \dfrac{\alpha_n^p\beta_n^p}{2(2\alpha_n^p + \beta_n^p)}(\xi_1 + \xi_2) + \dfrac{\alpha_n^p\beta_n^p + 2\alpha_n^p\gamma_n^p}{2(2\alpha_n^p + \beta_n^p)}(\xi_1 - \xi_2)^2 & \text{for } p > p_2 \end{cases}$$

and define X to be the Fréchet space given by

$$X = \{x = (\xi^n)_n : \xi^n = (\xi_1^n, \xi_2^n) \in \mathbf{R}^2 \text{ and } \delta_p(x) = (\textstyle\sum_n (\delta_{np}(\xi^n)))^{1/2} < \infty \text{ for all } p \in \mathbf{N}\}.$$

Let

$$K = \{(\eta_n, \theta_n, \tau_n)_n \in Y : \eta_n = \theta_n = \tau_n \quad \text{for all } n \in \mathbf{N}\}$$

which is clearly a closed subspace of Y. We will show that X is isomorphic to Y/K.

First we can compute, for each p, the quotient norm q_p corresponding to $|\cdot|_p$. If $(\eta_n, \theta_n, \tau_n)_n \in Y$ then

$$(q_p((\eta_n, \theta_n, \tau_n) + K))^2 = \inf\{\textstyle\sum_n \alpha_n^p(\eta_n + \zeta_n)^2 + \beta_n^p(\theta_n + \zeta_n)^2 + \gamma_n^p(\tau_n + \zeta_n)^2\}$$

where the inf is taken over all sequences (ζ_n) of scalars such that $(\zeta_n, \zeta_n, \zeta_n) \in Y$. For each n, we can compute (e.g. by calculus) the value ζ_n which minimizes the nth term in the summation. It is given by

$$\zeta_n = -\frac{\alpha_n^p\eta_n^p + \beta_n^p\theta_n + \gamma_n^p\tau_n}{\alpha_n^p + \beta_n^p + \gamma_n^p}.$$

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One can check directly that $(\zeta_n, \zeta_n, \zeta_n)_n \in Y$. It also follows from the fact that $|(\eta_n + \zeta_n, \theta_n + \zeta_n, \tau_n + \zeta_n)|_p \leqq |(\eta_n, \theta_n, \tau_n)|_l$. Hence, we have

$$\begin{aligned}
&(q_p((\eta_n, \theta_n, \tau_n) + K))^2 \\
&= \sum_n \frac{1}{(\alpha_n^p + \beta_n^p + \gamma_n^p)^2} (\alpha_n^p(\beta_n^p(\eta_n - \theta_n) + \gamma_n^p(\eta_n - \tau_n))^2 \\
&+ \beta_n^p(\alpha_n^p(\theta_n - \eta_n) + \gamma_n^p(\theta_n - \tau_n))^2 + \gamma_n^p(\alpha_n^p(\tau_n - \eta_n) + \beta_n^p(\tau_n - \theta_n))^2) \\
&= \sum_n \frac{1}{\alpha_n^p + \beta_n^p + \gamma_n^p} (\alpha_n^p(\beta_n^p + \gamma_n^p)(\eta_n)^2 + \beta_n^p(\alpha_n^p + \gamma_n^p)(\theta_n)^2 \\
&+ \gamma_n^p(\alpha_n^p + \beta_n^p)(\tau_n)^2 - 2\alpha_n^p\beta_n^p\eta_n\theta_n - 2\alpha_n^p\gamma_n^p\eta_n\tau_n - 2\beta_n^p\gamma_n^p\theta_n\tau_n).
\end{aligned}$$

Now for $(\eta_n, \theta_n, \tau_n)_n + K \in Y/K$ we define

$$Q((\eta_n, \theta_n, \tau_n)_n + K) = (\theta_n - \tau_n, \theta_n - \eta_n)_n.$$

This clearly defines a linear, $1 - 1$ map on Y/K. We will show that it is an isomorphism onto X.

Consider the norms on X given by $d_p(x) = (\sum (d_{np}(\xi^n)))^{1/2}$ where for $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$ we have

$$d_{np}(\xi) = \frac{(\alpha_n^p + \beta_n^p)\gamma_n^p}{\alpha_n^p + \beta_n^p + \gamma_n^p}\xi_1^2 - \frac{2\alpha_n^p\gamma_n^p}{\alpha_n^p + \beta_n^p + \gamma_n^p}\xi_1\xi_2 + \frac{\alpha_n^p(\beta_n^p + \gamma_n^p)}{\alpha_n^p + \beta_n^p + \gamma_n^p}\xi_2^2.$$

It follows from the relations obtained in the construction of Y that $d_{np}(\xi) = \delta_{np}(\xi)$ when $p > p_2$. On the other hand, when $p \leqq p_2$, $\alpha_n^p, \gamma_n^p \leqq \beta_n^p$ so, writing $D = \alpha_n^p + \beta_n^p + \gamma_n^p$ we have,

$$\begin{aligned}
D\, d_{np}(\xi) &\geqq (\alpha_n^p + \beta_n^p)\gamma_n^p\xi_1^2 - \alpha_n^p\gamma_n^p(\xi_1^2 + \xi_2^2) + \alpha_n^p(\beta_j^p + \gamma_n^p)\,\xi_2^2 \\
&= \beta_n^p\gamma_n^p\xi_1^2 + \alpha_n^p\beta_n^p\xi_2^2 \\
&\geqq \tfrac{1}{2}((\alpha_n^p + \beta_n^p)\gamma_n^p\xi_1^2 + \alpha_n^p(\beta_n^p + \gamma_n^p)\xi_2^2) = \tfrac{1}{2} D\, \delta_{np}(\xi).
\end{aligned}$$

Finally, a similar argument shows that $d_{np}(\xi) \leqq 2\delta_{np}(\xi)$ when $p \leqq p_2$. Thus, we may conclude that $(d_p)_p$ is a fundamental system of norms for X.

We may then compute for $(\eta_n, \theta_n, \tau_n)_n \in Y$,

$$\begin{aligned}
&(d_p(Q((\eta_n, \theta_n, \tau_n) + K)))^2 = (d_p((\theta_n - \tau_n, \theta_n - \eta_n)_n))^2 \\
&\quad = \sum_n (d_{np}(\theta_n - \tau_n, \theta_n - \eta_n)) \\
&\quad = \sum_n \frac{1}{D}((\alpha_n^p + \beta_n^p)\gamma_n^p(\theta_n - \tau_n)^2 - 2\alpha_n^p\gamma_n^p(\theta_n - \tau_n)(\theta_n - \eta_n) \\
&\qquad\qquad + \alpha_n^p(\beta_n^p + \gamma_n^p)(\theta_n - \eta_n)^2) \\
&\quad = (q_p((\eta_n, \theta_n, \tau_n) + K))^2.
\end{aligned}$$

Hence, Q is an isomorphism into X. Since its range is clearly dense, it is an isomorphism onto.

Finally, we will show that X has no basis (indeed, it is not a complemented subspace of any space with a basis). First observe that the relations used to define $\delta_{np}(\xi)$ are exactly the same as equations (1.4) of [**2**]. Therefore, it is only necessary to verify relations (1.1*) and (1.5) of [**2**]. Now (1.1*) is used only to show that X is nuclear which we already know since X is a quotient of the nuclear space E. Thus, we need only verify the following three inequalities, which will suffice in place of (1.5) in [**2**]:

$$\frac{(\alpha_n^{p_1} + \beta_n^{p_1})\gamma_n^{p_1}}{\alpha_n^{p_1} + \beta_n^{p_1} + \gamma_n^{p_1}} \leqq \frac{1}{n} \frac{\alpha_n^{1}(\beta_n^{1} + \gamma_n^{1})}{\alpha_n^{1} + \beta_n^{1} + \gamma_n^{1}}$$

$$\frac{\alpha_n^{p_2}(\beta_n^{p_2} + \gamma_n^{p_2})}{\alpha_n^{p_2} + \beta_n^{p_2} + \gamma_n^{p_2}} \leqq \frac{1}{n} \frac{(\alpha_n^{p_1+1} + \beta_n^{p_1+1})\gamma_n^{p_1+1}}{\alpha_n^{p_1+1} + \beta_n^{p_1+1} + \gamma_n^{p_1+1}}$$

$$\frac{\alpha_n^{p_3}\beta_n^{p_3}}{2(2\alpha_n^{p_3} + \beta_n^{p_3})} \leqq \frac{1}{n} \frac{\alpha_n^{p_2+1}\beta_n^{p_2+1} + 2\alpha_n^{p_2+1}\gamma_n^{p_2+1}}{2(2\alpha_n^{p_2+1} + \beta_n^{p_2+1})}.$$

Applying the relation obtained in constructing Y we have

$$\frac{(\alpha_n^{p_1} + \beta_n^{p_1})\gamma_n^{p_1}}{\alpha_n^{p_1} + \beta_n^{p_1} + \gamma_n^{p_1}} \leqq \gamma_n^{p_1} \leqq \frac{1}{n^2}\alpha_n^{1} \leqq \frac{3}{n^2} \frac{\alpha_n^{1}(\beta_n^{1} + \gamma_n^{1})}{3\beta_n^{1}} \leqq \frac{1}{n} \frac{\alpha_n^{1}(\beta_n^{1} + \gamma_n^{1})}{\alpha_n^{1} + \beta_n^{1} + \gamma_n^{1}},$$

$$\frac{\alpha_n^{p_2}(\beta_n^{p_2} + \gamma_n^{p_2})}{\alpha_n^{p_2} + \beta_n^{p_2} + \gamma_n^{p_2}} \leqq \alpha_n^{p_2} \leqq \frac{1}{n^2}\gamma_n^{p_1+1} \leqq \frac{3}{n^2} \frac{(\alpha_n^{p_1+1} + \beta_n^{p_1+1})\gamma_n^{p_1+1}}{3\beta_n^{p_1+1}}$$
$$\leqq \frac{1}{n} \frac{(\alpha_n^{p_1+1} + \beta_n^{p_1+1})\gamma_n^{p_1+1}}{\alpha_n^{p_1+1} + \beta_n^{p_1+1} + \gamma_n^{p_1+1}}$$

and finally,

$$\frac{\alpha_n^{p_3}\beta_n^{p_3}}{2(2\alpha_n^{p_3} + \beta_n^{p_3})} \leqq \frac{\beta_n^{p_3}}{2} \leqq \frac{1}{2n^2}\alpha_n^{p_2+1} = \frac{1}{n^2} \frac{\alpha_n^{p_2+1}(\beta_n^{p_2+1} + 2\alpha_n^{p_2+1})}{2(2\alpha_n^{p_2+1} + \beta_n^{p_2+1})}$$
$$\leqq \frac{1}{n} \frac{\alpha_n^{p_2+1}\beta_n^{p_2+1} + 2\alpha_n^{p_2+1}\gamma_n^{p_2+1}}{2(2\alpha_n^{p_2+1} + \beta_n^{p_2+1})}.$$

References

1. Cz. Bessaga and A. Pełczyński, *Własności baz w przestrzeniach typu B*, Prace Matematyczne *3* (1959), 123–142.
2. Plamen Djakov and Boris Mitiagin, *Modified construction of nuclear Fréchet spaces without basis*, J. Functional Anal. *23* (1976), 415–433.
3. Ed Dubinsky, *Perfect Fréchet spaces*, Math. Ann. *174* (1967), 186–194.
4. ——— *Concrete subspaces of nuclear Fréchet spaces*, Studia Math. *52* (1975), 209–219.
5. ——— *Subspaces without bases in nuclear Fréchet spaces*, J. Functional Anal. *26* (1977), 121–130.
6. Ed Dubinsky and W. B. Robinson, *Quotient spaces of (s) with basis*, Studia Math.
7. W. B. Johnson and H. P. Rosenthal, *On ω*-basic sequences and their applications to the study of Banach spaces*, Studia Math. *43* (1972), 77–92.
8. G. Köthe, *Topological vector spaces I* (Springer-Verlag, New York, 1969).
9. B. S. Mitiagin and N. M. Zobin, *Contre-exemple à l'existence d'une base dans un espace de Fréchet nucléaire*, C. R. Acad. Sci. Paris, ser. A *279* (1974).
10. N. M. Zobin and B. S. Mitiagin, *Examples of nuclear Fréchet spaces without basis*, Funkcional. Anal. i Priložen. *8* (1974), 35–47.

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