ORIENTED FLAT SUBMANIFOLDS IN AN AFFINE SPACE

M. J. ENGLEFIELD

Introduction. The simplest examples of figures which possess inner orientation are a sensed line, and a plane in which a sense of rotation is specified. Suppose two sensed lines, which intersect in a finite point, are given in a definite order. Then there is only one way in which the first can be rotated to coincide with the second without passing through the second. In this way, two ordered, sensed lines determine a sense of rotation in the plane which contains them. The theorems proved below are essentially generalizations of this result to spaces of higher dimension, and the corresponding results concerning outer orientation. This concept is also simply illustrated in two dimensions: a line divides the plane into two parts; it has outer orientation if these two parts are given a definite order. In three dimensions, a line is given outer orientation by specifying a sense of rotation around it, while a plane is given outer orientation by assigning an order to the two parts into which it divides the space.

This paper is concerned with the section E_s (of dimension s), and the join E_t, of two flat submanifolds E_p and E_q of an n-dimensional affine space. I shall demonstrate the following results:

(i) *Suppose the section is not null or improper. If $pq + st$ is even, then* (a) *an inner orientation in the join E_t is determined if inner orientations are given in E_p, E_q, and their section E_s; and* (b) *an outer orientation around the section E_s is determined if outer orientations are given around E_p, E_q, and their join E_t. If $pq + st$ is odd, the results are true if E_p and E_q are ordered.*

(ii) *The corresponding results when the section is null or improper are obtained by replacing $pq + st$ by $pq + st + p + q$.*

The notation used follows the kernel-index method described by Schouten in his recent books **(1**; **2)**, which also provide most of the terminology. The only geometric objects appearing in this paper are vectors, each of which has a kernel consisting of a capital Latin letter, usually with an index below or above it. The initial letters of the alphabet are reserved for contravariant vectors, and the last few letters for covariant vectors. The components of a vector in a given co-ordinate system are denoted by its kernel with the indices of the co-ordinate system on the right. Thus

$$\overset{z}{Z}_{\lambda'} \qquad (\lambda' = 1', 2', \ldots, n')$$

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are the components of the covariant vector $\overset{z}{Z}$ in the co-ordinate system (κ'). Like κ', λ' is one of the running indices of the system, while $1', \ldots, n'$ are the fixed indices. Each co-ordinate system has its own running and fixed indices.

Sections 1 and 3 discuss flat submanifolds, giving some familiar results which are required. Outer and inner orientation are defined in §2 with reference to the covariant and contravariant domains belonging to a submanifold. The result (i)(a), which was given by Schouten, is proved in §4. Finally, the remaining results are obtained and illustrated with simple examples.

1. The domains of a flat submanifold. The set of points with co-ordinates satisfying a number of linear equations is called *a flat submanifold.* If $n - p$ of these equations are independent, the submanifold can be considered as an affine space of dimension p, or E_p. When all the equations, say $x^\kappa U_\kappa^{\ u} + P^u = 0 (u = p + 1, \ldots, n)$, are independent, they are *a null form* of the E_p. For each u, the $U_\kappa^{\ u}$ are the components in (κ) of a covariant vector; these $n - p$ linearly independent vectors $\overset{u}{U}$ span a covariant domain. The vectors of this domain will be said to belong to the E_p.

If $B_b^\kappa U_\kappa^{\ u} = 0 (b = 1, \ldots, p)$, and $K^\kappa U_\kappa^{\ u} + P^u = 0$, then $x^\kappa = \eta^b B_b^\kappa + K^\kappa$ belongs to the submanifold for all η^b. If the matrix $[B_b^\kappa]$ has rank p, $x^\kappa = \eta^b B_b^d + K^\kappa$ is *a parametric form* of the E_p, and the sets η^b may be used as co-ordinates. The contravariant vectors $\underset{b}{B}$ with components B_b^κ in (κ) span a contravariant domain which will be said to belong to the submanifold.

A p-direction, or improper E_{p-1}, may be regarded either as the set of all submanifolds parallel to a given E_p, or as their common "points at infinity." Since parallel submanifolds have the same covariant and contravariant domains, we may also say that these domains belong to the p-direction, which is called their *support.*

2. Inner and outer orientation. Two sets of ordered, linearly independent contravariant vectors $\underset{i}{G}$ and

$$\underset{j}{H} = \overset{i}{\underset{j}{A}} \underset{i}{G} \qquad (i, j = 1, \ldots, n)$$

are said to have the same *screw-sense* if the determinant

$$\left|\overset{i}{\underset{j}{A}}\right| > 0,$$

and opposite screw-senses if

$$\left|\overset{i}{\underset{j}{A}}\right| < 0.$$

The screw-sense of the set $\underset{i}{G}$ may be identified with the sign of the determinant

$$\left|\underset{i}{G}^{\kappa}\right|$$

if the allowed co-ordinate transformations

$$x^{\kappa'} = x^{\kappa}A_{\kappa}^{\kappa'} + A^{\kappa'} \qquad (\kappa = 1, \ldots, n; \kappa' = 1', \ldots, n')$$

are restricted to those with determinant $|A_{\kappa}^{\kappa'}|$ positive. The space is then *oriented*, and is said to have *an inner orientation* (or screw-sense)—that fixed by the contravariant basis (or measuring) vectors (**1**, p. 7; **2**, p. 12; **3**, p. 5) of an allowed co-ordinate system.

Orientation in any flat submanifold E_p is defined in the same way: if η^b $(b = 1, \ldots, p)$ are rectilinear co-ordinates for the E_p, a screw-sense is fixed in the E_p by an ordered set of p linearly independent quantities

$$\underset{\beta}{B} \qquad (\beta = 1, \ldots, p)$$

which are contravariant vectors with respect to transformations of the η^b. These will be called contravariant vectors in the E_p, and may be represented by pairs of points of the E_p. Any such point-pair also represents a contravariant vector in the E_n, and in this way an isomorphic correspondence can be set up between the contravariant vectors in the E_p and the contravariant vectors in the E_n belonging to the E_p(**3**, p. 20). For example, if

$$x^{\kappa} = \eta^{b'}B_{b'}^{\kappa} + K^{\kappa} \qquad (b' = 1', \ldots, p')$$

is a parametric form for the submanifold, the contravariant vectors with components $B_{b'}^{\kappa}$ in (κ) correspond to the contravariant basis (or measuring) vectors $\underset{b'}{e}$ of the co-ordinate system (b') in the E_p. Through this correspondence, an ordered set of vectors

$$\underset{b}{B} \qquad (b = 1, \ldots, p)$$

spanning the domain belonging to the E_p fixes a screw-sense in the E_p. The set

$$\underset{c}{C} = \overset{b}{\underset{c}{A}}\,\underset{b}{B} \qquad (b, c = 1, \ldots, p)$$

determines the same screw-sense if the determinant

$$\left|\overset{b}{\underset{c}{A}}\right| > 0.$$

Inner orientation in a submanifold E_p could therefore be defined in terms of the contravariant vectors belonging to it. The equivalent considerations for covariant vectors suggest the following definition: *an outer orientation* around an E_p is defined by an ordered set $\overset{u}{U}$ of $n - p$ covariant vectors spanning the covariant domain belonging to the E_p. If

$$\overset{y}{Y} = \overset{u}{U}\,\overset{y}{\underset{u}{A}} \qquad (u, y = 1, \ldots, n - p),$$

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the outer orientations determined by the $\overset{y}{Y}$ and the $\overset{u}{U}$ are defined to be the same or opposite according as the determinant $|\underset{u}{\overset{y}{A}}|$ is positive or negative. With these definitions, it is clear that orientation of an E_p is essentially a property of its p-direction.

To each set of n independent contravariant vectors $\underset{i}{G}$ corresponds a *reciprocal* set of covariant vectors $\overset{j}{W}$ satisfying

$$\underset{i}{G}^{\kappa}\overset{j}{W}_{\kappa} = \delta_i^j.$$

By this correspondence, an ordered set of n linearly independent covariant vectors fixes an inner orientation in the E_n.

Given an E_p, let $x^{\kappa} = \eta^y B_y^{\kappa} + K^{\kappa}$ $(y = p+1, \ldots, n)$ be a parametric form of an E_{n-p} with no direction in common with the E_p. This means that a contravariant vector cannot belong to both the E_p and the E_{n-p}, and so their contravariant domains span the contravariant domain of the E_n. If a covariant vector X is represented (**1**, p. 7; **2**, p. 10; **3**, p. 5) by the parallel hyperplanes $x^{\kappa}X_{\kappa} = c$, $x^{\kappa}X_{\kappa} = c+1$, the figure intersects the E_{n-p} in the points $\eta^y(B_y^{\kappa}X_{\kappa}) = c - K^{\kappa}X_{\kappa}$, $\eta^y(B_y^{\kappa}X_{\kappa}) = c - K^{\kappa}X_{\kappa} + 1$, which represent a covariant vector in the E_{n-p} with components $B_y^{\kappa}X_{\kappa}$ in (y). If X belongs to the E_p, $B_y^{\kappa}X_{\kappa}$ cannot vanish for all y, for X would then have zero transvection with every contravariant vector belonging to the E_p or the E_{n-p}, and hence to every contravariant vector in E_n. Thus an isomorphic correspondence can be set up between the covariant vectors belonging to E_p, and the covariant vectors in the E_{n-p}. Since an ordered set of $n - p$ linearly independent covariant vectors in the E_{n-p} fixes a screw-sense in it, an outer orientation of the E_p determines an inner orientation in the E_{n-p}.[1] The latter can be any E_{n-p} with no direction in common with the E_p.[2]

This result is illustrated by the example of §6, where the line α plays the role of the E_p, and the plane σ that of the E_{n-p}. An outer orientation around α is fixed by the covariant vectors U and V, which are respectively represented by the ordered pairs of parallel planes π_1, π_2 and ρ_1, ρ_2. Their intersections with σ, the parallel lines shown in Figure 3, are the representations of the corresponding covariant vectors in σ; B and C are the reciprocal set of contravariant vectors in σ.

3. The sections and joins of flat submanifolds. Consider an E_p with a null form $x^{\kappa}U_{\kappa}^{u} + P^u = 0$ $(u = p+1, \ldots, n)$, and a parametric form $x^{\kappa} = \eta^b B_b^{\kappa} + K^{\kappa}$ $(b = 1, \ldots, p)$. Consider also an E_q with a null form $x^{\kappa}V_{\kappa}^{v} + Q^v = 0$ $(v = q+1, \ldots, n)$, and a parametric form $x^{\kappa} = \xi^c C_c^{\kappa} + L^{\kappa}$

[1]Schouten used this property to introduce outer orientation (**1**, p. 5; **2**, p. 7; **3**, p. 4), and then gave my definition as a property (**3**, p. 6).

[2]The E_{n-p} may also be considered as that arising by reduction of the space with respect to the p-direction of the E_p (**3**, p. 21).

$(c = 1, \ldots, q)$. Then $B_b^\kappa U_\kappa^u = 0$ for each u and b, $C_c^\kappa V_\kappa^v = 0$ for each v and c, $K^\kappa U_\kappa^u + P^u = 0$ for each u, and $L^\kappa V_\kappa^v + Q^v = 0$ for each v. From these conditions, the following lemma may be proved: *if the* $n \times (2n - p - q)$ *matrix*

$$\Xi \equiv [U_\kappa^u | V_\kappa^v]$$

has rank $n - s_1$, *then* (i) $0 \leqslant s_1 \leqslant p$, $p + q - n \leqslant s_1 \leqslant q$; (ii) the $(p + q) \times n$ *matrix*

$$\Gamma \equiv \begin{bmatrix} B_b^\kappa \\ \hline C_c^\kappa \end{bmatrix}$$

has rank $t_1 = p + q - s_1$; (iii) *there exist numbers*

$$\underset{d}{\chi}^b, \underset{d}{\zeta}^c$$

$(d = 1, \ldots, s_1)$ *such that*

$$\underset{d}{D}^\kappa = \underset{d}{\chi}^b B_b^\kappa = \underset{d}{\zeta}^c C_c^\kappa$$

are a complete set of solutions of the equations $D^\kappa \Xi = 0$.

The method of proof was outlined by Schouten (**2**, p. 8, Examples I.2, I.3).

In the same way, *if the* $(n + 1) \times (2n - p - q)$ *matrix*

$$\Omega \equiv \begin{bmatrix} U_\kappa^u | V_\kappa^v \\ \hline P^u | Q^v \end{bmatrix}$$

has rank $n - s$, *then* (i) $-1 \leqslant s \leqslant p, p + q - n \leqslant s \leqslant q$; (ii) *the* $(p + q + 1) \times n$ *matrix*

$$\Delta \equiv \begin{bmatrix} \Gamma \\ \hline L^\kappa - K^\kappa \end{bmatrix}$$

has rank $t = p + q - s$; (iii) *there are numbers* $\overset{w}{\psi}_u, \overset{w}{\phi}_v$ $(w = 1, \ldots, n - t)$ *such that*

$$P^u \overset{w}{\psi}_u = Q^v \overset{w}{\phi}_v,$$

and

$$\overset{w}{W}_\kappa = U_\kappa^u \overset{w}{\psi}_u = V_\kappa^v \overset{w}{\phi}_v$$

are a complete set of solutions of the equations $\Delta W_\kappa = 0$.

There are three possibilities regarding the section of E_p and E_q: (i) they have a finite point in common; (ii) they have no finite point in common, but there is a common direction; (iii) they have no common point or direction. These cases correspond to different values of s_1 and s.

(i) $s_1 = s$, implying $s \neq -1$. If

$$\begin{bmatrix} Z_\kappa^z \\ \hline R^z \end{bmatrix}$$

$(z = s+1, \ldots, n)$ are any $n-s$ linearly independent columns of Ω, then $x^\kappa Z^z_\kappa + R^z = 0$ is a null form of the section, which is therefore an E_s. If M^κ are the co-ordinates of any point of the section, and

$$D^\kappa_d = \underset{d}{D^\kappa},$$

then $x^\kappa = w^d D^\kappa_d + M^\kappa$ is a parametric form of the section. So the contravariant domain of the section is the set of vectors which belong to both E_p and E_q.

Defining

$$W^w_\kappa = \overset{w}{W}_\kappa$$

and

$$S^w = P^u \overset{w}{\psi}_u = Q^v \overset{w}{\phi}_v,$$

$x^\kappa W_\kappa{}^w + S^w = 0$ is a null form of the join, which is therefore an E_t. If N^κ are the co-ordinates of any point of this E_t, and $F^\kappa_f (f = 1, \ldots, t)$ any t independent linear combinations of the rows of Δ(or Γ), then $x^\kappa = \theta^f F^\kappa_f + N^\kappa$ is a parametric form of the join. The set of covariant vectors which belong to both E_p and E_q is the covariant domain of the join.

(ii) $s_1 = s+1,\ s \neq -1$.

Since the equations

$$[x^\kappa | 1]\Omega = 0$$

are inconsistent, E_p and E_q have no finite point in common. However, the vectors with components $\underset{d}{D^\kappa}$ belong to E_p and E_q, and span a contravariant domain of dimension s_1, so it is natural to consider the support of this domain as the section of E_p and E_q. This is consistent with our usual conception of parallel manifolds, for if E_p is parallel to a submanifold contained in E_q, the section will be the p-direction of the E_p. In general only a submanifold of E_p will be parallel to a submanifold of E_q: for instance, two planes in an E_4 may have a single common direction. The covariant domain of the section is the set of vectors which have zero transvection with every vector of the contravariant domain. If Z^z_κ are any $n - s_1 = n-s-1$ linearly independent columns of Ξ, the vectors with these components in (κ) span the covariant domain.

Null and parametric forms of the join, and hence bases for its domains, are obtained exactly as in (i). There, however, t linearly independent rows of Δ may be chosen from Γ, but in this case the last row of Δ must be used.

(iii) $s_1 = 0,\ s = -1$.

The equations $[x^\kappa|1]\Omega = 0$ are inconsistent, and the equations $D^\kappa \Xi = 0$ have no non-zero solution. This means that E_p and E_q have no point or direction in common. The previous procedure gives null and parametric forms of the join, which is thus an E_{p+q+1}.

4. Orientation in the join when the section is a proper E_s. The simplest case occurs when the section is a single point, so that $s_1 = s = 0$, and $t = p + q$. An inner orientation for the join is fixed by an ordered set of contravariant vectors spanning the domain of the join. Any parametric form of the join specifies such a set; hence, from the previous section, the rows of Γ may be taken. We must remember, however, that E_p and E_q do not uniquely specify the rows of Γ. On the other hand, the set of contravariant vectors does not need to be unique, provided that the ambiguity does not affect the screw-sense determined.

Suppose an inner orientation is specified in both E_p and E_q. For the ordered set of t contravariant vectors, take any set fixing the given orientation in E_p, followed by any set fixing the orientation given in E_q:

$$\underset{1}{B}, \ldots, \underset{p}{B}; \underset{1}{C}, \ldots, \underset{q}{C}.$$

Any two sets chosen in this way are related by a transformation with positive determinant, and so specify the same orientation in the join. This selection implies the ordering of E_p and E_q. If pq is even, this ordering is irrelevant, because the set

$$\underset{1}{C}, \ldots, \underset{q}{C}; \underset{1}{B}, \ldots, \underset{p}{B}$$

is changed into the above set by a permutation with the same parity as pq; the determinant of the corresponding transformation is $(-1)^{pq}$.

We are now ready to consider the more general case in which $s_1 = s > 0$. The choice of contravariant vectors belonging to the join must be modified because the rows of Γ are not linearly independent. *When $pq + st$ is even, an orientation in the join of E_p and E_q is determined by specifying orientations in E_p, E_q, and their section E_s; this is true when $pq + st$ is odd if E_p and E_q are ordered* (**2**, p. 8, Example I. 5).

Choose first any set of s_1 vectors $\underset{d}{D}$ which determines the given orientation in the section; then choose any set of $p - s_1$ vectors

$$\underset{g}{G} \qquad (g = s_1 + 1, \ldots, p)$$

such that the set $\underset{d}{D}; \underset{g}{G}$ determines the given orientation in E_p; finally choose any set of $q - s_1$ vectors

$$\underset{h}{H} \qquad (h = s_1 + 1, \ldots, q)$$

such that the set $\underset{d}{D}; \underset{h}{H}$ determines the given orientation in E_q. The set $\underset{d}{D}; \underset{g}{G}; \underset{h}{H}$ is changed into the set $\underset{d}{D}; \underset{h}{H}; \underset{g}{G}$ by a transformation with determinant $(-1)^{(p-s)(q-s)} = (-1)^{pq-st}$. So E_p and E_q need not be ordered if $pq + st$ is even. The necessity of fixing a screw-sense in the section is seen by the effect of interchanging two of the vectors $\underset{d}{D}$. To preserve the given orientations in

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E_p and E_q, we could, for example, simply interchange two of the $\underset{g}{G}$, and interchange two of the $\underset{h}{H}$. The resulting three interchanges in the whole set give a set specifying the opposite orientation in the join.

If $p = s_1 = s$, giving an orientation in E_s also gives an orientation in E_p. In all the results, it is to be understood that when p or q equals s_1, the chosen orientations are consistent with each other. When the section is proper, the theorems are then trivial.

5. Orientation in the join when the section is improper or null. The appropriate changes when the section is improper or null lead to the following theorem: *if $pq + st + p + q$ is even, an orientation in the join of E_p and E_q is determined by specifying orientations in E_p, E_q, and their section E_s (unless this is null); but if $pq + st + p + q$ is odd, E_p and E_q must be ordered.* When p (or q) equals $s_1 = s + 1$, it is sufficient to give an orientation in E_q (or E_p) and to order E_p and E_q.

The changes in the proof of the theorem are due to the dimension of the contravariant domain of the section now being $s + 1$, and to the fact that no t rows of Γ are linearly independent. Choose (i) a set of $s + 1$ vectors $\underset{d}{D}$ which fixes the given screw-sense in the section, (ii) a set of $p - s - 1$ vectors $\underset{g}{G}$ such that the set $\underset{d}{D}; \underset{g}{G}$ fixes the given screw-sense in E_p, (iii) a set of $q - s - 1$ vectors $\underset{h}{H}$ such that $\underset{d}{D}; \underset{h}{H}$ fixes the given screw-sense in E_q, and (iv) a vector A which may be represented by a point in E_p and a point in $E_q (d = 1, \ldots, s_1; g = s_1 + 1, \ldots, p; h = s_1 + 1, \ldots, q)$. From cases (ii) and (iii) of section 3, these t vectors span the contravariant domain of the join.

As in the previous case, it is obvious that the arbitrariness in the choice of the $\underset{d}{D}$, $\underset{g}{G}$, and $\underset{h}{H}$ will not affect the orientation determined in the join. Suppose $\underset{1}{A}$ and $\underset{2}{A}$ are two different selections of the last vector, and let

$$(1) \qquad \underset{2}{A} = \overset{d}{\Phi} \underset{d}{D} + \overset{g}{\Theta} \underset{g}{G} + \overset{h}{\Psi} \underset{h}{H} + \Lambda \underset{1}{A}.$$

The determinant of the transformation between the two sets is then Λ. Let $Z_\kappa (\kappa = 1, \ldots, n)$ satisfy $\Gamma Z_\kappa = 0$, but not $\Delta Z_\kappa = 0$. Choose the Z_κ so that $Z_\kappa L^\kappa - Z_\kappa K^\kappa = 1$, and put $b = Z_\kappa K^\kappa$. Then the Z_κ are the components of a covariant vector Z represented by the parallel hyperplanes $Z_\kappa x^\kappa = b$, which contains E_p, and $Z_\kappa x^\kappa = b + 1$, which contains E_q. Taking the transvection of Z and each side of (1) gives $\Lambda = 1$. Thus the choice of A does not affect the orientation given in the join.

E_p and E_q need not be ordered if the sets $\underset{d}{D}; \underset{g}{G}; \underset{h}{H}; A$ and $\underset{d}{D}; \underset{h}{H}; \underset{g}{G}; -A$ define the same orientation in the join. The determinant of the transformation between these sets is $(-1)^{(p-s-1)(q-s-1)+1} = (-1)^{pq+st+p+q}$.

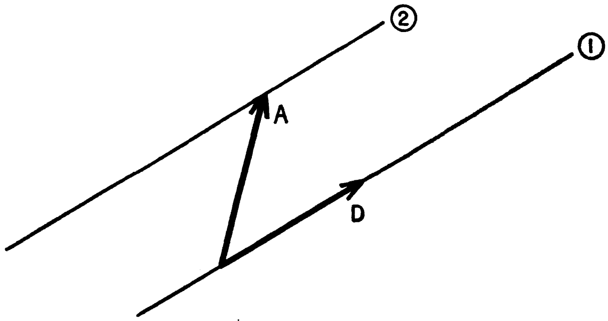

FIGURE 1. ($p = q = 1, s = 0$).

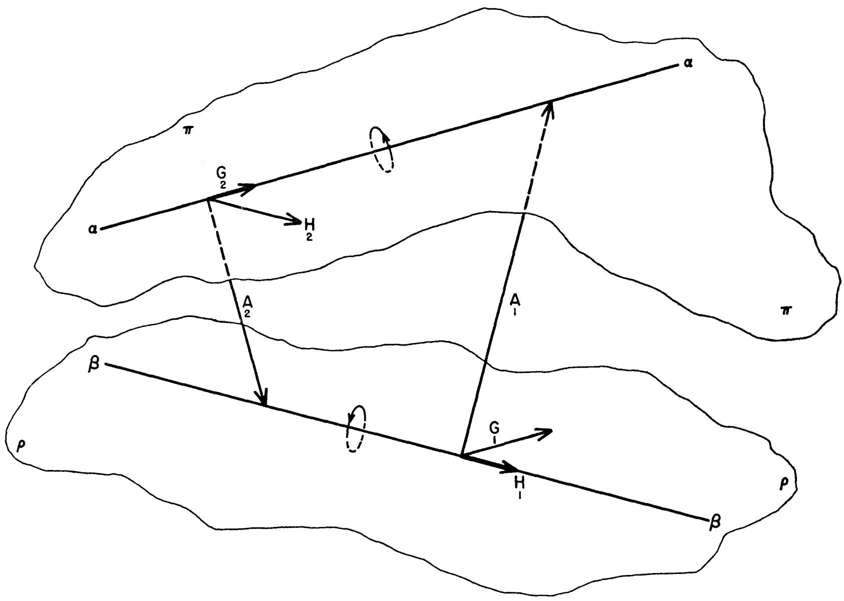

FIGURE 2. ($p = q = 1, s = -1$).

Figures 1 and 2 illustrate simple applications of this result. In the first, the set $D; A$ fixes a screw-sense in the plane which is the join of the parallel lines. The sense of D is determined by a specified orientation in the section, and the sense of A by ordering the lines.

In the second, an orientation is fixed in a 3-dimensional space by specifying senses for the skew lines α and β. The sets of vectors $\underset{1}{H}; \underset{1}{G}; \underset{1}{A}$ and $\underset{2}{G}; \underset{2}{H}; \underset{2}{A}$ are both measuring vectors for co-ordinate systems with right-handed axes, showing that it is unnecessary to order the lines. The parallel planes π and ρ, which are only drawn to help visualize the 3-dimensional figure, are a representation, in this case, of the covariant vector Z used in the proof. Z is uniquely determined when $n = t$.

6. Orientation around a proper section. The analogous considerations for covariant vectors lead to certain theorems on outer orientation. *If the section of E_p and E_q is a proper E_s, an outer orientation around the section is determined if outer orientations are given around E_p, and E_q, and their join E_t, provided that E_p and E_q are ordered when $pq + st$ is odd.* Of course, when the join is the whole space, we do not have to give an outer orientation around it (cf. §4 when $s = 0$).

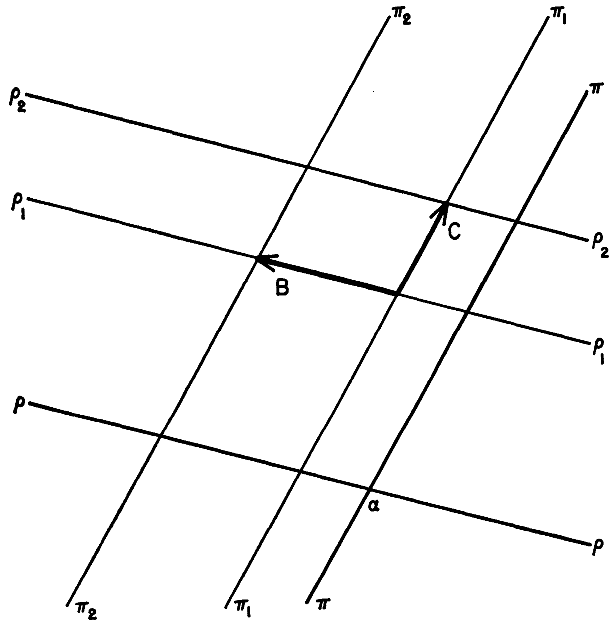

FIGURE 3. ($p = q = 2, s = 1$).

As an example in E_3, consider two planes π and ρ whose section is a line α. Figure 3 shows their intersections with some plane σ that does not contain α. According to the theorem, giving outer orientations to π and ρ, in that order, fixes an outer orientation around α, which in turn fixes an inner orientation in σ. Let the ordered pair of parallel planes π_1 and π_2 represent a covariant vector U which determines the outer orientation around π; and similarly ρ_1 and ρ_2 represent V which determines the orientation around ρ. The screw-sense in σ is that given by the contravariant vectors B and C; the outer orientation around α is thus a clockwise rotation when seen from this side of σ.

7. Orientation around an improper or a null section. When the section E_s is improper, or null, its covariant domain has dimension $n - s_1 = n - s - 1$. Suppose E_p and E_q are ordered, and that outer orientations are given around E_p, E_q, and their join. Choose $n - s - 1$ covariant vectors as follows: (i) a set $\overset{w}{W}$ which specifies the outer orientation around the join

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$(w = 1, \dots, n - t)$; (ii) a vector Z, the support of which contains the directions of both E_p and E_q, and the sense of which is determined by the order E_p, E_q; (iii) a set $\overset{x}{X}$ $(x = n - t + 2, \dots, n - p)$ of $t - p - 1$ vectors such that the set $\overset{w}{W}; Z; \overset{x}{X}$ specifies the outer orientation around E_p; and (iv) a set $\overset{y}{Y}(y = n - t + 2, \dots, n - q)$ of $t - q - 1$ vectors such that the set $\overset{w}{W}; Z; \overset{y}{Y}$ specifies the outer orientation around E_q. The vector Z is the same as that used in §5. If $t = p + 1$ (or $q + 1$) the set $\overset{x}{X}$ (or $\overset{y}{Y}$) does not appear; and if $t = n$, the set $\overset{w}{W}$ does not appear. Suitable alterations to the proof in §5 show that this set determines an outer orientation around the section. Changing the order of E_p and E_q gives a set $\overset{w}{W}; -Z; '\overset{y}{Y}; '\overset{x}{X}$, where the sets $'\overset{y}{Y}, '\overset{x}{X}$ are related to the sets $\overset{y}{Y}, \overset{x}{X}$ by transformations with negative determinants. The order of E_p and E_q is therefore irrelevant if $(t - p - 1)$ $(t - q - 1) + 1$ is even, that is, if $pq + st + p + q$ is even.

In general, then, *when* $pq + st + p + q$ *is even, an outer orientation around the (null or improper) section of* E_p *and* E_q *is determined if outer orientations are specified around* E_p*, around* E_q*, and around their join unless this is the whole space; when* $pq + st + p + q$ *is odd, an order must also be assigned to* E_p *and* E_q. If $t = p + 1$ (or $t = q + 1$), an orientation around the section is determined by the orientation around E_q (or E_p). When $t = p + 1 = q + 1$, it is sufficient to give an orientation around the join and to order E_p and E_q.

The covariant domain of a null section has dimension n, so an outer orientation of a null section may be interpreted as an inner orientation for the E_n. Figure 2 illustrates the case of skew lines in an E_3; suppose α and β are given the outer orientations shown by the arrowed circles. For the covariant vectors

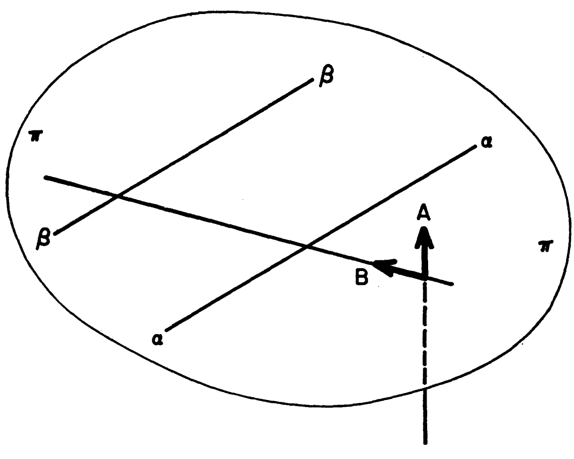

FIGURE 4.

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of the theorem, the set Z, X, Y reciprocal to $\underset{2}{A}$, $\underset{2}{H}$, $\underset{2}{G}$ may be chosen. Alternatively, taking the lines in the opposite order, the set $-Z$, $'Y$, $'X$ reciprocal to $\underset{1}{A}$, $-\underset{1}{G}$, $-\underset{1}{H}$ may be chosen. These sets are the measuring vectors of co-ordinate systems with left-handed axes.

The theorem may also be applied to two parallel lines α and β in a plane π in an E_3 (Figure 4). An outer orientation around the direction of α and β is determined by giving an outer orientation to π and assigning an order to α and β. The conditions respectively determine the senses of the contravariant vectors A and B. Rotating A into B gives an outer orientation for any line parallel to α and β.

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