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VARIETIES OF NILPOTENT GROUPS OF CLASS FOUR (I)

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Abstract

This is the first of three papers (the others by the first author alone) which determine all varieties of nilpotent groups of class (at most) four. The initial step is to reduce the problem to two cases: varieties whose free groups have no elements of order 2, and varieties whose free groups have no nontrivial elements of odd order. The varieties of the first kind form a distributive lattice with respect to order by inclusion (which is not a sublattice in the lattice of all group varieties). We give an embedding of this lattice in the direct product of six copies of the lattice which consist of 0 (as largest element) and the odd positive integers ordered by divisibility. The six integer parameters so associated with a variety directly match a (finite) defining set of laws for the variety. We also show that the varieties of the second kind do form a sublattice in the lattice of all varieties. That (nondistributive) sublattice will be treated, in a similarly conclusive manner, in the subsequent papers of this series.

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1. Introduction

This is a report on the first and easy half of a project aimed at determining all varieties of nilpotent groups of class (at most) four. The initial step is to reduce the problem to two cases: varieties whose free groups have no nontrivial elements of odd order, dealt with in the first author's thesis [3] and in subsequent papers of this series, and varieties whose free groups have no elements of order 2, determined here. The main result is that the latter varieties form a distributive lattice (with respect to order by inclusion: this is *not* a sublattice of the lattice of all varieties of nilpotent groups of class at most 4) which may be given as follows. Let Ω denote the set consisting of 0 and the odd positive integers partially ordered by divisibility (with the convention that 0 is the *largest* element of Ω). Clearly, Ω is a distributive lattice, with joins and meets being least common multiples and greatest common divisors. Consider the sublattice of the direct product of six copies of Ω which consists of all the (a, b, c, d, e, f) such that

b divides a,
c is d or $3d$ and divides b,
d is a common multiple of e and f, and
if 3 divides a then $3d$ also divides a.

This sublattice is isomorphic to the lattice of all varieties of nilpotent groups of class at most 4 whose free groups have no elements of order 2; namely, (a, b, c, d, e, f) corresponds to the variety defined by the following laws:

$$\begin{aligned} x_1^a &= [x_1, x_2]^b = [x_1, x_2, x_3]^c = [x_1, x_2, x_1]^d \\ &= [x_1, x_2, x_2, x_1]^e = [[x_1, x_2], [x_3, x_4]]^f \\ &= [x_1, x_2, x_3, x_4]^{ef} = [x_1, x_2, x_3, x_4, x_5] = 1. \end{aligned}$$

(All incompletely bracketed commutators are to be read as "left-normed": that is, $[x_1, x_2, x_3] = [[x_1, x_2], x_3]$, and so on.)

All varieties of nilpotent groups of class at most 3 were known at least fifteen years ago (Jónsson [9], Remeslennikov [15]). A particularly sharp result of Gupta and Newman [4] on commutator laws led then, among other things, to Brisley's conclusive work [1], [2] on varieties of metabelian p-groups of class at most $p + 1$, from which we derive the metabelian part of our result. On the other hand, varieties of nilpotent p-groups of class at most $p - 1$ have an elaborate theory, with the first significant result of Thrall [16] almost forty years old. The first comprehensive treatment in print is Kljačko's [10]. (He also asserted, without proof, the distributivity of the lattice of all varieties of 3-groups of class at most 4: this is, of course, confirmed by our present results.) Only a small part of this theory is relevant in detail here, although that is used rather heavily: Section 2 of the exposition [11].

There is also a parallel theory for torsionfree varieties of nilpotent groups (that is, varieties whose free groups are torsionfree), developed by Newman and the second author in 1968 but not published until recently [11], [12]. We need the fact, which must have been widely known for quite some time though the only reference seems to be [12], that there are precisely seven torsionfree varieties of nilpotent groups of class at most 4. Six of them are obvious to pick: in the notation of Hanna Neumann's book [13], they are $\mathfrak{E}$, $\mathfrak{A}$, $\mathfrak{N}_2$, $\mathfrak{N}_3$, $\mathfrak{A}^2 \cap \mathfrak{N}_4$, and $\mathfrak{N}_4$ itself. The seventh was called $\mathfrak{E}_3$ but left without defining laws in [12]; it was (also) identified there as the variety generated by the torsionfree groups of $\mathfrak{N}_3^{(2)} \cap \mathfrak{N}_4$. Since then, it has come to our attention that the unpublished thesis

[14] of Pentony contains a statement (pages 45–46, proof largely suppressed) to the effect that this variety is defined by the laws corresponding to our $(0,0,0,0,1,0)$. Let $\mathfrak{M}$, say, denote the variety defined by these laws. Clearly, $\mathfrak{N}_4 > \mathfrak{M} \geqslant \mathfrak{N}_3^{(2)} \cap \mathfrak{N}_4 \geqslant \mathfrak{E}_3$, so one can indeed conclude that $\mathfrak{M} = \mathfrak{N}_3^{(2)} \cap \mathfrak{N}_4 = \mathfrak{E}_3$ provided one knows that $\mathfrak{M}$ is torsionfree: but it is just this point which Pentony left without any hint of a proof. We show here (as 2.5) that the Gupta-Newman result (*loc. cit.*) quickly yields that the free groups of $\mathfrak{M}$ have no nontrivial elements of odd order; then (*cf.* Lemma 5.1 in the second paper of this series) Lemma 3.2 of [3] gives (via the appropriate version of the Magnus-Witt argument elaborated in Section 3 of [11]) that these groups have no elements of order 2 either. This (confirms Pentony's claim and) establishes that $\mathfrak{N}_3^{(2)} \cap \mathfrak{N}_4$ is torsionfree and is defined by the laws corresponding to $(0,0,0,0,1,0)$: a much more satisfactory identification of the seventh torsionfree subvariety of $\mathfrak{N}_4$ than those given in [12].

We are greatly indebted to Dr M. F. Newman for a continuing exchange of ideas, over many years, on the background to this work.

2. Sylow decomposition

It is well known that each subvariety of $\mathfrak{N}_4$ is defined by its 4-variable laws (see 34.15 and 34.34 in [13]), and that therefore our task is equivalent to finding all fully invariant subgroups in the rank 4 free group F of $\mathfrak{N}_4$. This is the setting we shall work in throughout the paper.

For each fully invariant subgroup U of F, write

U_0/U for the set of elements of finite order in F/U,
U_2/U for the set of elements of 2-power order in F/U, and
$U_{2'}/U$ for the set of elements of odd order in F/U.

As F/U is finitely generated and nilpotent, U_i/U is a finite subgroup for each i in $\{0, 2, 2'\}$, and U_i is obviously fully invariant in F. It is immediate that

2.1. *$U_2 \cap U_{2'} = U$ and $U_2U_{2'} = U_0$,*

while

2.2. *$(U_i)_i = U_i$ and $(U_i)_j = U_0$ whenever $i \neq j$.*

Moreover,

2.3. *$(U \cap V)_i = U_i \cap V_i$; in particular, if $U \leqslant V$ then $U_i \leqslant V_i$.*

Here $(U \cap V)_i \leqslant U_i \cap V_i$ is obvious; the converse inclusion holds because $w \in U_i \cap V_i$ means that $w^m \in U$ and $w^n \in V$ for suitable integers m, n, and then $w^{mn} \in U \cap V$. We also need

2.4. $(UV)_i = (U_iV_i)_i$.

Again, $(UV)_i \leqslant (U_iV_i)_i$ is obvious. To see the converse, note that U_iV_i/UV is a subgroup generated by elements of finite (or 2-power, or odd) orders in the nilpotent group F/UV, and hence consists of such elements.

Now let Λ denote the lattice of all fully invariant subgroups of F, and put $\Lambda_i = \{U \in \Lambda \mid U_i = U\}$. Thus for instance Λ_2 consists of the 2-isolated fully invariant subgroups: that is, of the fully invariant U such that F/U has no elements of order 2. Each Λ_i is partially ordered by inclusion, and is a lattice with respect to this partial order: by 2.2 and 2.3, the meet of U and V in Λ_i is just $U \cap V$, while their join in Λ_i is $(UV)_i$. Thus Λ_i is a sublattice of Λ if and only if $(UV)_i = UV$ for all U, V in Λ_i: we shall see in 2.6 that this is the case when i is $2'$ but not when i is 0 or 2.

Consider the following diagram of maps.

$$\begin{tikzcd}
& U \arrow[r, mapsto] & U_2 & \\
U \arrow[d, mapsto] & \Lambda \arrow[r] \arrow[d] & \Lambda_2 \arrow[d] & V \arrow[d, mapsto] \\
U_{2'} & \Lambda_{2'} \arrow[r] & \Lambda_0 & V_0 \\
& W \arrow[r, mapsto] & W_0 &
\end{tikzcd}$$

By 2.2, the diagram commutes and all four maps are surjective. By 2.3 and 2.4, all the maps are lattice-homomorphisms. (Consequently, the Λ_i are modular, because Λ is.) We therefore also have a lattice-homomorphism of Λ into the direct product lattice $\Lambda_2 \times \Lambda_{2'}$ given by $U \mapsto (U_2, U_{2'})$. The first statement of 2.1 implies that this homomorphism is an embedding. If (V, W) lies in its image then $V_0 = W_0$ by the commutativity of the diagram; conversely, if $(V, W) \in \Lambda_2 \times \Lambda_{2'}$ and $V_0 = W_0$ then 2.2 and 2.3 show that (V, W) is the image of $V \cap W$ and hence lies in the image of Λ. (In technical terms: Λ is the subdirect product of Λ_2 and $\Lambda_{2'}$ defined by the pullback diagram above.) Thus if we know $\Lambda_2 \to \Lambda_0$ and $\Lambda_{2'} \to \Lambda_0$, we can reconstruct Λ. In this sense, the study of all fully invariant subgroups is reduced to the separate studies of the 2-isolated fully invariant subgroups and the $2'$-isolated fully invariant subgroups.

The role a fully invariant subgroup U plays in the lattice Λ is not the only thing, perhaps not even the most important thing, we want to know about it. We are certainly interested, for instance, in finding a finite defining set for U (that is, a finite subset of which it is the fully invariant subgroup closure), for such a set (with the class 4 law adjoined) will give a finite basis for the laws of the corresponding variety. Our reduction gives U in terms of U_2 and $U_{2'}$, as $U_2 \cap U_{2'}$; and, in general, there is no known procedure for obtaining a defining set for the intersection $V \cap W$ of two fully invariant subgroups from defining sets of V and W. So it is relevant to observe that there *is* such a procedure when $(V, W) \in \Lambda_2 \times \Lambda_{2'}$, $V_0 = W_0$, provided we have upper estimates for the (odd) exponent of V_0/V and the (2-power) exponent of W_0/W. Namely, suppose that the subsets R and S define v and W, respectively, and that $V_0/V \in \mathfrak{B}_n$ (with n odd) and $W_0/W \in \mathfrak{B}_{2^k}$. Let U be the fully invariant subgroup closure of the set T defined by

$$T = \left\{r^{2^k} \mid r \in R\right\} \cup \{s^n \mid s \in S\}.$$

As $V_{2'} = (V_2)_{2'} = V_0 = W_0 = (W_{2'})_2 = W_2$ (by 2.2 and the assumptions on V, W), the indices of V and W in this subgroup are coprime, so $VW = V_0 = W_0$. It follows that $V_0/V \cap W = (V/V \cap W) \times (W/V \cap W)$ and $V/V \cap W \cong W_0/W \in \mathfrak{B}_{2^k}$, $W/V \cap W \cong V_0/V \in \mathfrak{B}_n$. Hence $T \subseteq V \cap W$, so $U \leqslant V \cap W$. On the other hand, the elements of R have 2-power orders modulo U, so V/U is generated by (endomorphic images of) elements of 2-power order in the finitely generated nilpotent group F/U, and therefore V/U has 2-power order. Similarly, W/U has odd order. Thus $(V/U) \cap (W/U) = 1$, that is, $V \cap W = U$. This proves that T defines $V \cap W$. Note that if R and S are finite, so is T.

Our aim in the rest of the paper is therefore to determine the lattice Λ_2 of all 2-isolated fully invariant subgroups V of F; to identify, for each V, its "isolator" V_0; to give a finite defining set for each V and an upper estimate for the exponent of V_0/V.

Before we embark on this task, there are two other points to settle: the claim made in the introduction concerning Λ_0, and the assertion earlier in this section that of the Λ_i only $\Lambda_{2'}$ is a sublattice of Λ. Six members of Λ_0 are well known: F itself, the commutator subgroup F', the other nontrivial terms of the lower central series, namely $\mathfrak{N}_2(F)$ which we rarely have to refer to, and $\mathfrak{N}_3(F)$ which we need frequently and denote by N to save writing too much; the second commutator subgroup F''; and the trivial subgroup 1. Let $\{x, y, z, t\}$ be a free generating set of F, and M the fully invariant subgroup defined by the (left-normed) commutator $[y, x, x, y]$; we know from [12] that M_0 is the seventh and last member of Λ_0, with $M_0 < N$ and $M_0 \cap F'' = 1$. A result of Gupta and Newman [4] may be applied to F/MF'', and yields that N/MF'' has exponent dividing 4: that is, $N^4 \leqslant MF''$ (where N^4 is the subgroup of the abelian group N consisting of the fourth powers of the elements of N). Thus $M_0^4 \leqslant MF'' \cap M_0 = M(F'' \cap M_0) = M$ (where we have used the modularity of Λ). It follows that $M_{2'} = M$. By (*cf.* Lemma 5.1 in the second paper of this series) Lemma 3.2 of [3], read via an obvious adaptation of Section 3 of [11], we have that $M_{2'}$ is isolated: thus

$$M = M_2 = M_{2'} = M_0.$$

This settles the first point.

From this discussion, we also need $(MF'')_0 = (MF'')_2 = N$ and $(MF'')_{2'} = MF''$ towards the second point. As further preparation, we establish that $MF'' \neq N$. The (standard) wreath product of a group of order 2 by an elementary abelian group of order 8 is a well-known example: a 4-generator metabelian group which is nilpotent of class precisely 4, in which all 2-generator subgroups have class at most 3 (compare 34.54 of [13]). Thus F does have homomorphisms onto this group, and the kernel of such a homomorphism must contain MF'' but cannot contain N. We are now ready to prove the following.

2.5. *For $U, V \in \Lambda_i$ we have $(UV)_i \neq UV$ if and only if $i \neq 2'$ and either $U_0 = M$, $V_0 = F''$ or $U_0 = F''$, $V_0 = M$.*

Suppose first that U_0 and V_0 satisfy one of the alternative conditions: then $UV \leqslant U_0V_0 = MF''$. By 2.4, we have $(UV)_0 = (U_0V_0)_0 = N$. As the nontrivial 2-group N/MF'' is a factor group of $(UV)_0/UV$, neither $(UV)_0/UV$ nor $(UV)_2/UV$ can be trivial. On the other hand, if also $U, V \in \Lambda_{2'}$ then by 2.2 we have $U_0 = U_2$, $V_0 = V_2$, so 2.4 yields $(UV)_2 = (U_2V_2)_2 = (U_0V_0)_2 = (MF'')_2 = N = (UV)_0$; thus $(UV)_0/UV$ is a 2-group and $(UV)_{2'} = UV$. If U_0 and V_0 do not satisfy either condition, then they are comparable; for, on inspecting the seven elements of Λ_0 one finds that M, F'' is the only incomparable pair. Say, $U_0 \leqslant V_0$. Then $(UV)_0 = (U_0V_0)_0 = V_0$ by 2.4 and 2.2; thus $(UV)_0/UV$ is a factor group of V_0/V. Now $V \in \Lambda_i$ gives that $(UV)_0/UV$ is trivial or a $2'$-group or a 2-group, according as i is 0 or 2 or $2'$, so that also $UV \in \Lambda_i$, that is, $(UV)_i = UV$. This completes the proof of 2.5.

A moment's reflection shows that this, with $(MF'')_0 = N = (MF'')_2$, settles everything: $\Lambda_{2'}$ is, but Λ_2 is not, a sublattice of Λ; while Λ_0 is a sublattice of Λ_2, but not of Λ, nor of $\Lambda_{2'}$.

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3. Distributivity

The aim of this section is to prove that Λ_2 is distributive. It is this fact, more than anything else, which makes the description of Λ_2 so much easier than the case of $\Lambda_{2'}$ dealt with in [3].

Since F' is of class 2, it is easy to see that for each odd prime power p^k the $\mathfrak{V}_{p^k}$-subgroup of any subgroup A of F' is just the set of all p^kth powers of elements of A; accordingly, we shall denote it by A^{p^k} instead of the more cumbersome $\mathfrak{V}_{p^k}(A)$. The situation for F itself is not quite so simple: there we do, emphatically, distinguish between the *set* of p^kth powers denoted by F^{p^k} and the *subgroup* $\mathfrak{V}_{p^k}(F)$ they generate. Our first preliminary result shows that even this distinction is irrelevant when the odd prime p is not 3, and provides a (necessarily) weaker but for our purposes adequate variant when $p = 3$.

3.1. *If p is a prime and $p > 3$, then $\mathfrak{V}_{p^k}(F) = F^{p^k}$, while $\mathfrak{V}_{3^{k+1}}(F) \subseteq F^{3^k}$.*

PROOF. The first statement holds not only for F but for every nilpotent group of class less than p, and is familiar in the context of regular p-groups. We shall only sketch a proof for the less familiar second claim. To this end we temporarily abandon F and work in an infinite rank free group G of $\mathfrak{N}_4$, freely generated by $x_1, x_2, \ldots$. For $k \geqslant 0$, put

$$u_2 = x_1 x_2 [x_1, x_2]^{(3^{k+1}-1)/2}.$$

The Hall-Petrescu Identities (III.9.4 in Huppert [8]) readily yield that there is an element v_2 in $\mathfrak{N}_2(G)$ for which

$$x_1^{3^{k+1}} x_2^{3^{k+1}} = u_2^{3^{k+1}} v_2^{3^k}.$$

Induction on n rapidly establishes the existence of elements u_n in G and v_n in $\mathfrak{N}_2(G)$ such that

$$x_1^{3^{k+1}} x_2^{3^{k+1}} \cdots x_n^{3^{k+1}} = u_n^{3^{k+1}} v_n^{3^k}.$$

As the subgroup of G generated by u_n^3 and v_n has class at most 2, a straightforward calculation within that subgroup then yields

$$x_1^{3^{k+1}} x_2^{3^{k+1}} \cdots x_n^{3^{k+1}} = \left(u_n^3 v_n [u_n^3, v_n]^{(3^k-1)/2} \right)^{3^k},$$

and this proves our claim.

One more piece of folklore before we can start in earnest: if p is an odd prime then there is no fully invariant subgroup of F strictly between F'' and $(F'')^p$. This is proved for $p = 3$ in the (unpublished part of the) thesis [5] of Harris (pages 73–74) by an argument which works equally well when $p > 3$. For $p > 3$ it can, of course, also be extracted from the classification of varieties of groups of exponent p and class less than p, in which context one relies on the fact that even the automorphism group of F acts irreducibly on $F''/(F'')^p$, as a quotient of $\mathrm{GL}(4, p)$.

It follows then that if f is an odd integer and p is a prime divisor of f, there is no fully invariant subgroup of F strictly between $(F'')^{f/p}$ and $(F'')^f$. We take this one step further.

3.2. *If $V \in \Lambda_2$ and $1 < V \leqslant F''$ then $V = (F'')^f$ for some odd positive integer f.*

Proof. Since $1 < V \leqslant F''$, also $1 < V_0 \leqslant F''$; as F'' is minimal in Λ_0, we have $V_0 = F''$. By 2.2, the assumption $V \in \Lambda_2$ gives $V_0 = (V_2)_{2'} = V_{2'}$, so V has odd index in F''. Let f denote the (exact) exponent of F''/V. If $V/(F'')^f$ is nontrivial, it has an element of some odd prime order p. As F'' is free abelian, all elements of order p in $F''/(F'')^f$ lie in $(F'')^{f/p}/(F'')^f$, so $(F'')^f < V \cap (F'')^{f/p} \leqslant (F'')^{f/p}$. By the preceding discussion this implies that $V \cap (F'')^{f/p} = (F'')^{f/p}$, so $V \geqslant (F'')^{f/p}$: contrary to the choice of f as the exact exponent of F''/V. Therefore $V/(F'')^f$ must be trivial.

A similar argument will give us the following.

3.3. *If $V \in \Lambda_2$ and $F'' < V \leqslant N$ then $V = N^eF''$ for some odd positive integer e.*

Proof. All we need to establish is that if p is an odd prime then there is no fully invariant subgroup of F strictly between N^pF'' and N. As F/N is torsionfree, 3.1 ensures that $\mathfrak{B}_{p^2}(F) \cap N \leqslant N^p$, so by the modular law $\mathfrak{B}_{p^2}(F)N^pF'' \cap N = N^pF''$. Put $H = F/\mathfrak{B}_{p^2}(F)N^pF''$. The natural homomorphism of F onto H would map a fully invariant subgroup V strictly between N^pF'' and N to a fully invariant subgroup of H strictly between 1 and $\mathfrak{N}_3(H)$. However, H is a free group of a variety of metabelian p-groups of class at most 4, with $\mathfrak{N}_3(H)$ of exponent p, so one can read off Brisley's classification of such varieties (from [1] if $p > 3$, from [2] if $p = 3$) that no fully invariant subgroup of H can lie strictly between 1 and $\mathfrak{N}_3(H)$. This completes the proof.

We shall need much more detail from Brisley in the end, but this much will suffice in this section. Before we start on the proof of the distributivity of Λ_2, we must recall a little more of the structure of N. Of course, N is free abelian on the basis consisting of the basic commutators of weight 4 (formed with respect to the ordered free generating set $\{x, y, z, t\}$ of F, say); by Witt's Formula, there are 60 of these. Direct inspection shows that precisely 15 of them are not left-normed: those lie in F''. In fact they (freely) generate F'': for, by a theorem of Magnus (36.32 in Neumann's [13]), the cosets of the other 45 form a free abelian basis of N/F''.

We are now ready to prove the distributivity of Λ_2, along the lines of Section 2 of [11]. The reader is invited to check that the arguments described there can be adapted to prove that if $U, V \in \Lambda_2$ then the sublattice of Λ_2 generated by U, V, and N, is distributive: so Λ_2 is a subdirect product of its sublattices

$$\{W \in \Lambda_2 \mid W \geqslant N\} \quad \text{and} \quad \{W \in \Lambda_2 \mid W \leqslant N\}.$$

The first of these is also a sublattice of Λ (on account of 2.6), and so dual to a sublattice of the lattice of all varieties of nilpotent groups of class at most 3: hence it is distributive. It remains to prove the distributivity of the second lattice. To this end, it is sufficient to show that if $U, V \in \Lambda_2$ and $U, V \leqslant N$ then the sublattice of Λ_2 generated by U, V, and F'', is distributive. Indeed, once this is established we can argue that $\{W \in \Lambda_2 \mid W \leqslant N\}$ is a subdirect product of $\{W \in \Lambda_2 \mid W \leqslant F''\}$ and $\{W \in \Lambda_2 \mid F'' \leqslant W \leqslant N\}$, and 3.2, 3.3 show that each of these is dual to the distributive lattice Ω described in the introduction. Imitate Section 2 of [11] once more: if the sublattice generated by U, V, F'' in $\{W \in \Lambda_2 \mid W \leqslant N\}$ were not distributive, one could deduce that F'' and N/F'' had (End F)-admissible, nontrivial, 2-torsionfree sections which were (End F)-isomorphic. This is impossible: for, by 3.2 each nontrivial, (End F)-admissible section of F'' is the direct product of 15 pairwise isomorphic cyclic groups, while by 3.3 the same holds for N/F'' with 45 in place of 15. This completes the proof of the distributivity of Λ_2.

4. Meetirreducibles

Since F is a finitely generated nilpotent group, it has no infinite properly ascending chains of subgroups. As in any distributive lattice with such a chain condition, each element of Λ_2 has a unique expression as an irredundant meet of meetirreducible elements; and, indeed, the lattice can be reconstructed from the poset of its meetirreducible elements. The aim of this section is to determine that poset for Λ_2.

If $V \in \Lambda_2$ and V_0/V is not a p-group for any prime p, then V has a proper meet decomposition $V = \bigcap_p V_p$ with V_p/V the nontrivial Sylow p-subgroups of V_0/V. If V_0/V has exponent p^k (> 1) for some (odd) prime p then one sees from 3.1 that $\mathfrak{B}_{p^{k+1}}(F) \cap V_0 \leqslant \mathfrak{B}_{p^k}(V_0) \leqslant V$ and so the modular law gives a meet

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decomposition $V = V_0 \cap \mathfrak{B}_{p^{k+1}}(F)V$ which is proper unless $\mathfrak{B}_{p^{k+1}}(F)V = V$ or, equivalently, $V_0 = F$. Thus the meetirreducibles of Λ_2 outside Λ_0 all have prime-power index in F. Those which contain F'' correspond to joinirreducible varieties of metabelian p-groups of class at most 4, and hence are known from Brisley's work (see especially the summing up in the first paragraph of page 61 of [2], from which it is an elementary exercise to identify them).

Thus we have narrowed down the real task of this section to the consideration of meetirreducibles V of prime-power index in F, with $V \not\geqslant F''$. For each odd prime power p^k ($\neq 1$), put

$$B(p^k) = \begin{cases} (F')^{3^k}\mathfrak{B}_{3^{k+1}}(F) & \text{if } p = 3, \\ \mathfrak{B}_{p^k}(F) & \text{if } p > 3. \end{cases}$$

We shall prove that

4.1. *each $B(p^k)M$ is meetirreducible, with*

4.2. $B(p^k)M \cap F'' = (F'')^{p^k}$,

and conversely: if V is a meetirreducible with prime-power index in F and $V \not\geqslant F''$, by 3.2 we have $V \cap F'' = (F'')^{p^k}$ for some odd prime power p^k, and then

4.3. $V = B(p^k)M$.

We shall use modularity (Dedekind's Law) so frequently that we must do so without reference. Occasionally we appeal to the distributivity of Λ_2, without formally writing joins in Λ_2: in those cases the relevant joins are simply products, because 2.6 applies favourably.

Let us start with the proof of 4.2. As F/F' is torsionfree, 3.1 yields that $B(p^k)M \cap F' = (F')^{p^k}M$ (regardless of whether $p = 3$ or $p > 3$). Since F'/F'' is torsionfree, $(F')^{p^k} \cap F'' = (F'')^{p^k}$. Using also the distributivity of Λ_2, we can then argue that

$$\begin{aligned} B(p^k)M \cap F'' &= B(p^k)M \cap F' \cap F'' = (F')^{p^k}M \cap F'' \\ &= \left[(F')^{p^k} \cap F''\right][M \cap F''] = (F'')^{p^k}. \end{aligned}$$

Next we prove 4.3, but this takes much longer; for the duration of this proof, write simply B for $B(p^k)$ where p^k is defined by $V \cap F'' = (F'')^{p^k}$. The distributivity of Λ_2, together with 4.2, gives that

$$VBM \cap VF'' = V(BM \cap F'') = V(F'')^{p^k} = V.$$

As $VF'' > V$ and V is meetirreducible, we must have $VBM = V$: that is,

4.4. $V \geqslant BM$.

Assume for the moment that

4.5. $V \leqslant BN$.

We have seen that N/MF'' is a 2-group, while (by the definition of B) F/BMF'' is of odd order, so we must have $N \leqslant BMF''$ and hence $N = (B \cap N)MF''$. Therefore, by 4.4,

$$V \cap N = V \cap (B \cap N)MF'' = (B \cap N)M(V \cap F'') = (B \cap N)M(F'')^{p^k},$$

so 4.2 gives that $V \cap N = (B \cap N)M$. Thus if 4.5 is true then using 4.4 again, we get

$$V = V \cap BN = B(V \cap N) = B(B \cap N)M = BM,$$

and this is what we are trying to establish.

The proof of 4.5 proceeds by contradiction. Suppose it is false; then, by 4.4, we have $VN > BN$. Now, VN and BN are verbal subgroups of F corresponding to varieties of nilpotent groups of class at most 3, and all such varieties are well known. In particular, the variety corresponding to BN is joinirreducible, and its unique maximal subvariety is defined by the extra law $[x_2, x_1, x_1]^{p^{k-1}} = 1$. It follows that $[y, x, x]^{p^{k-1}} = vw$ for some v in V and w in N. Consider the endomorphisms α and δ of F which leave x, z, and t unchanged while $y\alpha = [z, y]$ and $y\delta = 1$. Note that α and δ agree on N: for a basic commutator of weight four is mapped to 1 or left fixed by α depending only on whether y does or does not occur among its entries, and the same is true for δ. We have that $(v\delta)(w\delta) = 1$, for $[y, x, x]\delta = 1$; hence

$$[z, y, x, x]^{p^{k-1}} = (vw)\alpha = (v\alpha)(w\alpha) = (v\alpha)(w\delta) = (v\alpha)(v\delta)^{-1} \in V.$$

On the other hand,

$$[y, x, x, z]^{p^{k-1}} = [vw, z] = [v, z] \in V.$$

Since the fully invariant subgroup closure of $[z, y, x, x]$ and $[y, x, x, z]$ in F is N (Heineken [6]; III.6.9 in Huppert [8]), it follows that $N^{p^{k-1}} \leqslant V$. This contradicts $V \cap F'' = (F'')^{p^k}$, and thereby completes the proofs of 4.5 and 4.3.

We have left the proof of 4.1 to the last. Consider the expression of $B(p^k)M$ as a meet of meetirreducibles $V(1), \ldots, V(n)$. Then $(F'')^{p^k} = B(p^k)M \cap F'' = \bigcap_i (V(i) \cap F'')$. By 3.2, the fully invariant subgroup of F between $(F'')^{p^k}$ and F'' form a chain, so we must be able to choose j so that $V(j) \cap F'' = (F'')^{p^k}$. Then

$V(j)$ is a meetirreducible which is not isolated, hence by the introductory discussion of this section it must have prime-power index in F; as it does not contain F'', 4.3 applies to it: hence $V(j) = B(p^k)M$. This completes the proof of 4.1.

All that remains is to add in the meetirreducible from Λ_0 and the meetirreducibles one obtains from Brisley (*loc. cit.*). The result is that Λ_2 has precisely the following meetirreducible elements:

$$F, \quad F', \quad \mathfrak{N}_2(F), \quad N, \quad F'', \quad M,$$
$$\mathfrak{B}_{p^k}(F)F', \quad \mathfrak{B}_{p^k}(F)\mathfrak{N}_2(F), \quad \mathfrak{B}_{p^k}(F)N \quad \text{with } p \geqslant 3, k \geqslant 1,$$
$$\mathfrak{B}_{p^k}(F)F'', \quad \mathfrak{B}_{p^k}(F)M \quad \text{with } p > 3, k \geqslant 1, \quad \text{and}$$
$$\mathfrak{B}_{3^{k+1}}(F)\mathfrak{B}_{3^k}(F')N, \quad \mathfrak{B}_{3^{k+1}}(F)\mathfrak{B}_{3^k}(F')F'', \quad \mathfrak{B}_{3^{k+1}}(F)\mathfrak{B}_{3^k}(F')M \quad \text{with } k \geqslant 1.$$

Obviously, two of these subgroups are comparable if and only if that is directly visible from the way we have written them.

5. Conclusion

Our final task is to prove the main result stated in the introduction. Note that this result will achieve the aims we set in Section 2. For, if $V \in \Lambda_2$ and we change to 1 each nonzero entry of the corresponding (a, b, c, d, e, f), we get another admissible set of parameters; the subgroup U corresponding to this lies in Λ_0, and the exponent of U/V divides the product of the original nonzero parameters: so V_0 is this U, and we do have an estimate on the exponent of V_0/V.

We shall make use of a simple fact from lattice theory (see, for instance, Section 21 of Hermes [7]): in a distributive lattice which satisfies the ascending chain condition, each element can be written in one and only one way as the meet of pairwise incomparable meetirreducibles.

To fit our context, let $\mho$ denote the dual of Ω, and Δ the dual of the sublattice of Ω^6 described in the introduction: thus Δ is a sublattice of $\mho^6$, and we shall never refer to Ω or Ω^6 again. Define φ: $\mho^6 \to \Lambda$ by $(a, b, c, d, e, f)\varphi = U$ where U is the fully invariant subgroup of F generated as such by x^a, $[y, x]^b$, $[y, x, z]^c$, $[y, x, x]^d$, $[y, x, x, y]^e$, $[[t, z], [y, x]]^f$, and $[t, x, y, z]^{ef}$. What we have to prove amounts to the claim that restriction of φ yields a lattice-isomorphism $\Delta \to \Lambda_2$. In fact, we shall also obtain the inverse of this lattice-isomorphism. Define ψ: $\Lambda_2 \to \mho^6$, $U \mapsto U\psi = (a, b, c, d, e, f)$ by choosing a as the order of xU in F/U (or as 0 if that order is infinite), b as the order of $[y, x]U$ in F/U, and so on. Our full claim is that ψ maps Λ_2 (lattice) isomorphically onto Δ, and $\psi\varphi$ is the identity map on Λ_2.

The first step of the proof is to note that by its definition φ is a poset-homomorphism, and that ψ is even a meet-homomorphism (as the order of an element of F modulo $U \cap V$ is the least common multiple of its orders modulo U and V).

The second step is to check that $\mho^6\varphi \subseteq \Lambda_2$; that is, that if $(a, b, c, d, e, f)\varphi = U$ then $U_2 = U$. This is done case by case, according to which is the first (if any) nonzero entry in (a, b, c, d, e, f). Take, for instance, the case $a = b = c = 0 \neq d$, when $[y, x, x]^d \in U \leqslant \mathfrak{N}_2(F)$. Let A/U be the fully invariant subgroup of F/U defined by $[y, x, x]U$: since this element has odd order (dividing d), $|A/U|$ is odd. On the other hand, $\mathfrak{N}_2(F)/A$ has exponent dividing 3, so $|\mathfrak{N}_2(F)/U|$ is odd, and hence $U \in \Lambda_2$. The other cases are very much easier; we leave them to the reader.

Henceforth we may, and shall, regard φ as a map from $\mho^6$ to Λ_2.

For the third step, note that $g\varphi\psi \geqslant g$ for all g in $\mho^6$, simply by the definitions of φ and ψ, and that $U\psi\varphi \leqslant U$ for all U in Λ_2. The second claim needs only that $M^e(F'')^f \leqslant U$ implies $N^{ef} \leqslant U$: this holds because $(MF'')^{ef} \leqslant M^e(F'')^f$ so $N/M^e(F'')^f$ has exponent dividing $4ef$ (recall $N^4 \leqslant MF''$) while $N/N \cap U$ has no element of order 2. Thus $(U\psi)\varphi\psi \geqslant U\psi$ by the first comment, while $(U\psi\varphi)\psi \leqslant U\psi$ by the second comment and the order-preserving nature of ψ: so we have that

$$\psi\varphi\psi = \psi.$$

Let Γ denote the set of the meetirreducible elements of Λ_2 (listed at the end of the previous section). Our fourth step is to prove that $\Gamma\psi \subseteq \Delta$ and $\psi\varphi$ acts identically on Γ. For the V in Γ with F/V of prime-power order and $V \geqslant F''$, which we took from Brisley's work, this has (at least implicitly) been done by Brisley. For the V in $\Gamma \cap \Lambda_0$, this is simply a matter of inspection. For the other V in Γ, we know from 4.3 that $V = B(p^k)M$; put $V\psi = (a, b, c, d, e, f)$. Direct from the definitions of $B(p^k)$ and ψ, we see that

$$V\psi \geqslant \begin{cases} (3^{k+1}, 3^k, 3^k, 3^k, 1, 3^k) & \text{if } p = 3, \\ (p^k, p^k, p^k, p^k, 1, p^k) & \text{if } p > 3. \end{cases}$$

Thus $e = 1$; from 4.2, we know that $f = p^k$. As to the other parameters, use that ψ respects order, that $B(p^k)N \geqslant V$, and that $[B(p^k)N]\psi$ is known from Brisley (or indeed from the facts on varieties of nilpotent groups of class at most 3); and conclude that the inequality displayed above is in fact an equality. It is then immediate that $V\psi \in \Delta$ and $V\psi\varphi = V$.

The fifth step is left to the reader: determine all the meetirreducibles in Δ, and verify that the set they form is precisely $\Gamma\psi$.

The proof of the main result can now be completed quickly. Since $\Gamma\psi$ generates Δ as a meet-semilattice and ψ is a meet-homomorphism, $\Lambda_2\psi = \Delta$. Since φ and ψ

are poset-homomorphisms and $\psi\varphi$ is the identity map on Γ, their restrictions to $\Gamma\psi$ and Γ are poset-isomorphisms. If we can establish that $\psi\varphi$ is the identity map, the same argument will now give that ψ and the restriction of φ to Δ are poset-isomorphisms, and it is well known that all poset-isomorphisms of lattices are lattice-isomorphisms.

For the final step, suppose that $U\psi\varphi \neq U$ for some U in Λ_2: all we have to do is to show that this leads to a contradiction. Write $U\psi\varphi = \bigcap_i V(i)$ and $U = \bigcap_j W(j)$ with the $V(i)$ pairwise incomparable elements of Γ, and the $W(j)$ also pairwise incomparable elements of Γ. As $U\psi\varphi \neq U$, the set of the $V(i)$ is not the set of the $W(j)$. Since ψ acts as poset-isomorphism from Γ to the set of meetirreducibles in Δ, the $V(i)\psi$ form a set of pairwise incomparable meetirreducible elements in Δ, and the $W(j)\psi$ form a different set of pairwise incomparable meetirreducible elements. Yet, because ψ is a meet-homomorphism and $\psi\varphi\psi = \psi$,

$$\bigcap_i V(i)\psi = U\psi\varphi\psi = U\psi = \bigcap_j W(j)\psi.$$

This contradicts the uniqueness of expressions as meets of pairwise incomparable meetirreducibles in Δ (which obviously satisfies the ascending chain condition), and so completes the proof.

References

[1] Warren Brisley, 'On varieties of metabelian p-groups and their laws', *J. Austral. Math. Soc.* **7** (1967), 64–80.

[2] Warren Brisley, 'Varieties of metabelian p-groups of class p, $p + 1$', *J. Austral. Math. Soc.* **12** (1971), 53–62.

[3] Patrick Fitzpatrick, *Varieties of nilpotent groups of class at most four* (Ph.D. thesis, Australian National University, Canberra, 1980).

[4] N. D. Gupta and M. F. Newman, 'On metabelian groups', *J. Austral. Math. Soc.* **6** (1966), 362–368.

[5] L. F. Harris, *Varieties and section closed classes of groups* (Ph.D. thesis, Australian National University, Canberra, 1973).

[6] Hermann Heineken, 'Über ein Levisches Nilpotenzkriterium', *Arch. Math.* (*Basel*) **12** (1961), 176–178.

[7] Hans Hermes, *Einführung in die Verbandstheorie* (Die Grundlehren der mathematischen Wissenschaften, 73. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1955).

[8] B. Huppert, *Endliche Gruppen I* (Die Grundlehren der mathematischen Wissenschaften, 134. Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[9] Bjarni Jonsson, 'Varieties of groups of nilpotency three', *Notices Amer. Math. Soc.* **13** (1966), 488.

[10] A. A. Kljačko, 'Varieties of p-groups of small class' (Russian), *Ordered Sets and Lattices* No. 1, 31–42 (Izdat. Saratov. Univ., Saratov, 1971).

[11] L. G. Kovács, 'Varieties of nilpotent groups of small class', *Topics in algebra*, Proc. 18th SRI, [edited by M. F. Newman] (Lecture Notes in Mathematics, 697, pp. 205–229. Springer-Verlag, Berlin, Heidelberg, New York, 1978).

[12] L. G. Kovács, 'The thirty-nine varieties', *The Math. Scientist* **4** (1979), 113–128.

[13] Hanna Neumann, *Varieties of Groups* (Ergebnisse der Mathematik und ihrer Grenzgebiete, 37. Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[14] Paul Pentony, *Laws in torsion free nilpotent varieties* (Ph.D. thesis, Australian National University, Canberra, 1970). See also: Abstract, *Bull. Austral. Math. Soc.* **5** (197), 283–284.

[15] V. N. Remeslennikov, 'Two remarks on 3-step nilpotent groups' (Russian), *Algebra i Logika Sem.* **4** (1965), no. 2, 59–65.

[16] Robert M. Thrall, 'A note on a theorem by Witt', *Bull. Amer. Math. Soc.* **47** (1941), 303–308.

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