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# ISOMORPHISMS IN SUBSPACES OF $c_0$

BY
ROBERT H. LOHMAN

A Banach space $X$ is said to be subspace homogeneous if for every two isomorphic closed subspaces $Y$ and $Z$ of $X$, both of infinite codimension, there is an automorphism of $X$ (i.e. a bounded linear bijection of $X$) which carries $Y$ onto $Z$. In [1] Lindenstrauss and Rosenthal showed that $c_0$ is subspace homogeneous, a property also shared by $l_2$, and conjectured that $c_0$ and $l_2$ are the only subspace homogeneous Banach spaces. In that paper no mention was made of subspaces of $c_0$.

In this note we show that a closed, infinite dimensional subspace $X$ of $c_0$ is subspace homogeneous if and only if $X$ is isomorphic to $c_0$. This is also shown to be equivalent to a "$c_0$ sandwich property." Thus in the particular case $X \subset c_0$ our result lends support to the conjecture of Lindenstrauss and Rosenthal. We follow the notation and terminology of [1] and rely heavily upon the following results of [2]:

(1) A closed, complemented subspace $Z$ of $c_0$ is either finite dimensional or isomorphic to $c_0$.

(2) Every closed, infinite dimensional subspace of $c_0$ contains a closed, complemented subspace isomorphic to $c_0$.

THEOREM. *Let $X$ be a closed, infinite dimensional subspace of $c_0$. The following are equivalent*:

(a) $X \approx c_0$

(b) *$X$ is subspace homogeneous*

(c) *$X$ satisfies the "$c_0$ sandwich property": If $Z$ is a closed subspace of $X$ with* $\dim X/Z = \infty$, *there is a subspace $Y$ such that $Z \subset Y \subset X$ and $Y \approx c_0$.*

**Proof.** (a) $\Rightarrow$ (b). This follows immediately from the results of [1].

(b) $\Rightarrow$ (c). Let $Z$ be a closed subspace of $X$ with $\dim X/Z = \infty$. By (2), $X$ contains a subspace $W$ such that $W \approx c_0$. Consequently, there is an isomorphism $T$ from $Z$ into $W$. If $TZ$ is of finite codimension in $W$, then by (1) we have $Z \approx TZ \approx W \approx c_0$. If $TZ$ is of infinite codimension in $W$, then by (b), $T$ has an extension to an automorphism $\tilde{T}$ of $X$ onto $X$. Letting $Y = \tilde{T}^{-1}(W)$, we see that $Z \subset Y$ and $Y \approx c_0$.

(c) $\Rightarrow$ (a). By (2), $X$ contains a closed, complemented subspace $W$ such that $W \approx c_0$. We can write $X = W \oplus Z$, where $Z$ is a closed subspace of $X$. Since $\dim X/Z = \infty$, there is a subspace $Y$ such that $Z \subset Y \subset X$ and $Y \approx c_0$. Since $Z$ is complemented in $X$, $Z$ is complemented in $Y$. By (1), $Z$ is finite dimensional or $Z \approx c_0$. In either case, we have $X \approx c_0$.



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Kent State University,
Kent, Ohio

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