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A NOTE ON SIMULTANEOUS AND MULTIPLICATIVE DIOPHANTINE APPROXIMATION ON PLANAR CURVES

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Abstract. Let $\mathcal{C}$ be a non-degenerate planar curve. We show that the curve is of Khintchine-type for convergence in the case of simultaneous approximation in $\mathbb{R}^2$ with two independent approximation functions; that is if a certain sum converges then the set of all points (x, y) on the curve which satisfy simultaneously the inequalities $\|qx\| < \psi_1(q)$ and $\|qy\| < \psi_2(q)$ infinitely often has induced measure 0. This completes the metric theory for the Lebesgue case. Further, for multiplicative approximation $\|qx\|\,\|qy\| < \psi(q)$ we establish a Hausdorff measure convergence result for the same class of curves, the first such result for a general class of manifolds in this particular setup.

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1. Introduction. Before we state the main results of this paper we fix notation and define some elementary, but essential, concepts.

Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a real, decreasing function. Throughout we shall refer to ψ as an approximating function. Let $x \in \mathbb{R}$ and $\|x\|$ be the distance of x from $\mathbb{Z}$. That is, $\|x\| = \inf\{|x - z| : z \in \mathbb{Z}\}$. Further, if S is a (Lebesgue) measurable set in $\mathbb{R}^n$ then we shall denote the Lebesgue measure, or more simply the measure, of S by $|S|_{\mathbb{R}^n}$.

Consider now the following system of n Diophantine inequalities

$$\|qx_i\| < \psi_i(q), \tag{1}$$

where $x_i \in \mathbb{R}$, $p_i \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\psi_1, \psi_2, \ldots, \psi_n$ are approximation functions. Then a point $x \in \mathbb{R}^n$ is simultaneously $(\psi_1, \psi_2, \ldots, \psi_n)$-approximable if there are infinitely many q satisfying (1). The set of all such points $x \in \mathbb{R}^n$, will be denoted by $\mathcal{S}_n(\psi_1, \psi_2, \ldots, \psi_n)$.

Simultaneous approximation has another variant in the guise of multiplicative approximation; let $x \in \mathbb{R}^n$, then x is multiplicatively ψ-approximable if the inequality

$$\prod_{i=1}^{n} \|qx_i\| < \psi(q) \tag{2}$$

holds for infinitely many $q \in \mathbb{N}$. By analogy with the previous notion of simultaneous approximation, we shall denote by $\mathcal{S}_n^*(\psi)$ the set of all multiplicatively ψ-approximable points x in $\mathbb{R}^n$.

Let $C^{(m)}(U)$ be the space of all m-continuously differentiable functions f where $f : U \to \mathbb{R}$ with U being an open set in $\mathbb{R}^n$. A map $g : U \to \mathbb{R}$ is said to be non-degenerate at $u \in U$ if there exists some $l \in \mathbb{N}$ such that $g \in C^{(l)}(B(u, \delta))$ for some sufficiently small $\delta > 0$ with $B(u, \delta) \subset U$ and the partial derivatives of g, evaluated at u, span $\mathbb{R}^n$. The map g is non-degenerate if it is non-degenerate at almost all points $u \in U$. Let $\mathcal{M}$ be a sub-manifold of $\mathbb{R}^n$. Then $\mathcal{M}$ is said to be non-degenerate if $\mathcal{M} = g(U)$ where g is non-degenerate. The geometric interpretation of non-degeneracy is that the manifold is sufficiently curved that it deviates from any hyperplane. Non-degeneracy is not a particularly restrictive condition and a large class of manifolds satisfy this condition.

Note that if the topological dimension, $\dim \mathcal{M}$, of the manifold is strictly less than n then $|\mathcal{M}|_{\mathbb{R}^n} = 0$. As we wish to make measure theoretic statements about points that lie on a manifold we work with the induced measure, $|\cdot|_{\mathcal{M}}$. All measure statements made below are made with respect to this induced measure.

2. Statement of results. Let I be some open interval in $\mathbb{R}$ and $f \in C^{(3)}(I)$ such that for almost all $x \in I$:

(1) there exist constants $c_1 > c_2 > 0$ with $c_1 > f'(x) > c_2$,
(2) $f'' \neq 0$.

Under these assumptions it is readily verified that the curve C_f, where

$$C_f := \{(x, f(x)) : x \in I\},$$

is non-degenerate. We shall be assuming these conditions throughout the remainder of this article.

In [**7**], the authors conjectured that

$$|\mathcal{C}_f \cap \mathcal{S}_2(\psi_1, \psi_2)|_{\mathcal{C}_f} = 0 \quad \text{if} \quad \sum_{h=1}^{\infty} \psi_1(h)\psi_2(h) < \infty \tag{3}$$

and

$$|\mathcal{C}_f \cap \mathcal{S}_2^*(\psi)|_{\mathcal{C}_f} = 0 \quad \text{if} \quad \sum_{h=1}^{\infty} \psi(h) \log h < \infty. \tag{4}$$

The conjecture stated in (4) is a special case of Theorem 1 below. Before stating the theorem we briefly define Hausdorff s-measures. For a more detailed exposition of the theory of Hausdorff measures and its many applications in mathematics, see either of the excellent books by Falconer ([**4**] or [**5**]).

Let $X \subset \mathbb{R}^n$ and $s \geq 0$. For any $\delta > 0$, a δ–cover, $\mathcal{C}_\delta(X)$, of X is a countable collection of balls B_i such that $X \subset \bigcup B_i$ and $\operatorname{diam} B_i \leq \delta$. The set function $\mathcal{H}^s_\delta(\cdot)$, where

$$\mathcal{H}^s_\delta(X) := \inf \left\{ \sum \operatorname{diam}^s B_i \right\}$$

with the infimum taken over all δ-covers of X, is an outer measure. Taking the limit of this quantity as $\delta \to 0$ gives the Hausdorff s-measure of X. That is,

$$\mathcal{H}^s(X) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(X).$$

When s takes on values in $\mathbb{N} \cup \{0\}$ then $\mathcal{H}^s$ coincides with s-dimensional Lebesgue measure.

THEOREM 1. *Let ψ be an approximating function and* $0 < s \leq 1$. *Then*

$$\mathcal{H}^s(C_f \cap S_2^*(\psi)) = 0 \quad \textit{if} \quad \sum_{h=1}^{\infty} h^{1-s}(\log^s h)\psi^s(h) < \infty.$$

Note that as $\mathcal{H}^1$ coincides with 1-dimensional Lebesgue measure and as we are working with the induced measure on the manifold, in this case a 1-dimensional manifold, Conjecture 4 follows immediately as a special case of Theorem 1. For the cases when $0 < s < 1$ Theorem 1 appeared as a conjecture in [**7**].

Furthermore, the proof of Theorem 1 can be adapted to settle claim (3) and it is exactly this result that we present as Theorem 2 below.

THEOREM 2. *Let ψ_1, ψ_2 be approximating functions. Then*

$$|C_f \cap S_2(\psi_1, \psi_2)|_{C_f} = 0 \quad \textit{if} \quad \sum_{h=1}^{\infty} \psi_1(h)\psi_2(h) < \infty.$$

For the background to these problems, including a detailed survey of the classical theory of simultaneous Diophantine approximation, we refer the reader to the articles [**2**], [**3**] and [**7**]. Indeed, it is precisely the ideas of these papers, most notably those of [**7**], that we use to prove Theorem 2 and Theorem 1.

3. Proof of Theorem 1. We are given that

$$\sum_{h=1}^{\infty} h^{1-s} \log^s h \cdot \psi^s(h) < \infty. \tag{5}$$

Therefore without loss of generality we can assume that

$$q^{1-\frac{2}{s}}(\log q)^{-2-\frac{1}{s}} < \psi(q) \tag{6}$$

for sufficiently large q. To see why, suppose that (6) is not satisfied. Then we replace ψ with the auxiliary function

$$\widetilde{\psi} : q \mapsto \widetilde{\psi} := \max\left\{\psi(q), q^{1-\frac{2}{s}}(\log q)^{-2-\frac{1}{s}}\right\}.$$

Clearly, $\widetilde{\psi}$ is an approximation function. One can easily check that (5) and (6) are satisfied with ψ replaced by $\widetilde{\psi}$. Furthermore,

$$S_2^*(\widetilde{\psi}) \supset S_2^*(\psi).$$

Thus it suffices to prove the theorem with ψ replaced by $\widetilde{\psi}$ and (6) can be assumed.

As $\{(x, f(x)) : x \in \mathbb{Q}\}$ is countable and therefore of 0 Hausdorff s-measure, the value of $\mathcal{H}^s(C_f \cap S_2^*(\psi))$ will be unaltered if we omit such points. Hence we assume without any loss of generality, that $x \notin \mathbb{Q}$.

The set $C_f \cap S_2^*(\psi)$ is a lim sup-set with the following natural representation:

$$C_f \cap S_2^*(\psi) = \bigcap_{n=1}^{\infty} \bigcup_{q=n}^{\infty} \bigcup_{(p_1,p_2)\in\mathbb{Z}^2} S^*(p_1, p_2, q)$$

where

$$S^*(p_1, p_2, q) := \left\{(x, y) \in C_f \ : \ \left|x - \frac{p_1}{q}\right| \cdot \left|y - \frac{p_2}{q}\right| < \frac{\psi(q)}{q^2}\right\}.$$

Using the fact that ψ is decreasing, we have that for any n

$$C_f \cap S_2^*(\psi) \subset \bigcup_{t=n}^{\infty} \bigcup_{2^t \leqslant q < 2^{t+1}} \bigcup_{(p_1,p_2)\in\mathbb{Z}^2} S^*(p_1, p_2, q, t) \tag{7}$$

where

$$S^*(p_1, p_2, q, t) := \left\{(x, y) \in C_f \ : \ \left|x - \frac{p_1}{q}\right| \cdot \left|y - \frac{p_2}{q}\right| < \frac{\psi(2^t)}{(2^t)^2}\right\}.$$

If $t \in \mathbb{N}$, $(x, y) \in C_f$, $q \in \mathbb{N}$ with $2^t \leqslant q < 2^{t+1}$ and

$$\left|x - \frac{p_1}{q}\right| \cdot \left|y - \frac{p_2}{q}\right| < \frac{\psi(2^t)}{(2^t)^2}$$

for some $(p_1, p_2) \in \mathbb{Z}^2$, then there is a unique integer m such that

$$2^{m-1}\frac{\sqrt{2\psi(2^t)}}{2^t} \leqslant \left|x - \frac{p_1}{q}\right| < 2^m\frac{\sqrt{2\psi(2^t)}}{2^t}.$$

For this m, it follows that

$$\left|y - \frac{p_2}{q}\right| < \frac{\psi(2^t)}{(2^t)^2}\frac{2^t}{2^{m-1}\sqrt{2\psi(2^t)}} = 2^{-m}\frac{\sqrt{2\psi(2^t)}}{2^t}.$$

Then,

$$C_f \cap S_2^*(\psi) \subset \bigcup_{t=n}^{\infty} \bigcup_{2^t \leqslant q < 2^{t+1}} \bigcup_{(p_1,p_2)\in\mathbb{Z}^2} \bigcup_{m=-\infty}^{+\infty} C_f \cap S(q, p_1, p_2, m) \tag{8}$$

where

$$S(q, p_1, p_2, m) = \left\{(x, y) \in \mathbb{R}^2 \ : \ \left|x - \frac{p_1}{q}\right| < 2^m\frac{\sqrt{2\psi(2^t)}}{2^t}, \ \left|y - \frac{p_2}{q}\right| < 2^{-m}\frac{\sqrt{2\psi(2^t)}}{2^t}\right\}.$$

Thus, we have constructed a sequence of coverings of $C_f \cap S_2^*(\psi)$. The aim is now to show that if, for a given s, (5) holds then the associated sequence of Hausdorff s-measures for these coverings tends to 0 as $n \to \infty$. It then follows that we have $\mathcal{H}^s(C_f \cap S_2^*(\psi)) = 0$, as required.

To proceed we consider two separate cases. For a fixed t, Case (a): $m \in \mathbb{Z}$ such that

$$2^{-|m|} \geqslant t\sqrt{\psi(2^t)}. \tag{9}$$

and Case (b): $m \in \mathbb{Z}$ such that

$$2^{-|m|} \leqslant t\sqrt{\psi(2^t)}. \tag{10}$$

***Case* (a).** First, observe that (9) together with (6) implies that

$$2^{-|m|} \geqslant t\sqrt{2^{t(1-\frac{2}{s})} \cdot t^{-2-\frac{1}{s}}} \text{ and so } 2^{|m|} \leqslant t^{\frac{1}{2s}} 2^{t\left(\frac{1}{s}-\frac{1}{2}\right)}.$$

Upon taking logarithms (to the base 2) of both sides of the above inequality, we arrive at

$$|m| \leqslant t\left(\frac{2-s}{2s}\right) + \frac{1}{2s}\log_2 t \ll t. \tag{11}$$

As $c_2 > f'(x) > c_1$ for all $x \in I$ it follows that

$$\operatorname{diam}(C_f \cap S(q, p_1, p_2, m)) \ll 2^{-|m|}\frac{\sqrt{\psi(2^t)}}{2^t}. \tag{12}$$

The implied constant depends only on c_1 and is irrelevant to the remainder of the argument.

Given t and m, let $N(t, m)$ denote the number of triples (q, p_1, p_2) with $2^t \leqslant q < 2^{t+1}$ such that $C_f \cap S(q, p_1, p_2, m) \neq \emptyset$. Suppose now that $C_f \cap S(q, p_1, p_2, m) \neq \emptyset$. Then for some $(x, y) \in C_f$ and θ_1, θ_2 satisfying $-1 < \theta_1, \theta_2 < 1$, we have that

$$x = \frac{p_1}{q} + \theta_1 2^{|m|}\frac{\sqrt{2\psi(2^t)}}{2^t}, \qquad y = \frac{p_2}{q} + \theta_2 2^{|m|}\frac{\sqrt{2\psi(2^t)}}{2^t}.$$

Thus, it can be shown that

$$\begin{aligned} f\left(\frac{p_1}{q}\right) - \frac{p_2}{q} &= f\left(\frac{p_1}{q}\right) - f(x) + f(x) - y + y - \frac{p_2}{q} \\ &= -\theta_2 f'(\xi) \cdot 2^{|m|}\frac{\sqrt{2\psi(2^t)}}{2^t} + \theta_1 \cdot 2^{|m|}\frac{\sqrt{2\psi(2^t)}}{2^t} \end{aligned}$$

where ξ lies between x and p_1/q. Further, one can easily deduce that

$$\left|f\left(\frac{p_1}{q}\right) - \frac{p_2}{q}\right| \ll 2^{|m|}\frac{\sqrt{\psi(2^t)}}{2^t} \leqslant \frac{1}{t2^t}.$$

Set $Q = 2^{t+1}$. Then we have $q \leqslant Q$ and

$$\left|f\left(\frac{p_1}{q}\right) - \frac{p_2}{q}\right| \ll \frac{1}{Q\log Q}.$$

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A result of Vaughan & Velani is crucial to our argument. We now state this result, which is Theorem 2 from [**7**].

Theorem VV. *Let* $N_f(Q,\psi,I) = \#\{\mathbf{p}/q\ :\ q \leqslant Q, p_1/q \in I, |f(p_1/q) - p_2/q| < \psi(Q)/Q\}$*. Suppose that* ψ *is an approximating function with* $\psi(Q) \geqslant Q^{-\phi}$ *where* ϕ *is any real number with* $\phi \leqslant \frac{2}{3}$*. Then*

$$N_f(q,\psi,I) \ll \psi(Q)Q^2. \tag{13}$$

In our case $\psi(Q) = 2^{|m|}\sqrt{2\psi(2^t)} \asymp \frac{1}{\log Q}$ which satisfies the conditions of Theorem VV. Therefore there exists an absolute constant $c > 0$ such that

$$N(t,m) \leqslant c2^{2t}2^{|m|}\sqrt{\psi(2^t)}. \tag{14}$$

Now using (12) and (14) we can bound the Hausdorff sum associated with the set $C_f \cap S_2^*(\psi)$:

$$\begin{aligned}
\mathcal{H}^s(C_f \cap S_2^*(\psi)) &\ll \sum_{t=n}^{\infty}\sum_{m\in\text{case (a)}}\left(2^{-|m|}\frac{\sqrt{\psi(2^t)}}{2^t}\right)^s \times 2^{|m|}2^{2t}\sqrt{\psi(2^t)}\\
&\ll \sum_{t=n}^{\infty}\sum_{m\in\text{case (a)}} \psi(2^t)^{\frac{1}{2}(1+s)}\cdot 2^{t(2-s)}\cdot 2^{|m|(1-s)}\\
&\overset{(9)}{\ll} \sum_{t=n}^{\infty}\sum_{m\in\text{case (a)}} t^{s-1}\psi(2^t)^s 2^{t(2-s)}\\
&\overset{(11)}{\ll} \sum_{t=n}^{\infty} t^s 2^{t(2-s)}\psi(2^t)^s \asymp \sum_{q=2^n}^{\infty} q^{1-s}\log^s q\cdot\psi(q)^s.
\end{aligned}$$

The above comparability follows from the fact that ψ is an approximating function and therefore decreasing. In view of (5),

$$\sum_{q=2^n}^{\infty} q^{1-s}\log^s q\cdot\psi^s(q) \to 0 \quad\text{as}\quad n\to\infty.$$

Therefore, for Case (a) it follows that $\mathcal{H}^s(C_f \cap S_2^*(\psi)) = 0$ as required.

Case **(b).** In view of (10), we have that

$$S(q,p_1,p_2,m) \subset (S'(q,p_1)\times[0,1]) \cup ([0,1]\times S'(q,p_2))$$

where

$$S'(q,p) = \left\{y\in[0,1]\ :\ \left|y-\frac{p}{q}\right| < \frac{2t\psi(2^t)}{2^t}\right\}.$$

Thus, the set of (8) is a subset of

$$\bigcup_{t=n}^{\infty}\ \bigcup_{2^t\leqslant q<2^{t+1}}\ \bigcup_{(p_1,p_2)\in\mathbb{Z}^2} C_f\cap((S'(q,p_1)\times[0,1])\cup([0,1]\times S'(q,p_2))). \tag{15}$$

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As $c_1 > f'(x) > c_2 > 0$, for any choice of p_1, p_2 and q appearing in (15),

$$\operatorname{diam}(C_f \cap (S'(q, p_1) \times [0, 1])) \ll t\frac{\psi(2^t)}{2^t}$$

and

$$\operatorname{diam}(C_f \cap ([0, 1] \times S'(q, p_2)) \ll t\frac{\psi(2^t)}{2^t}.$$

The implied constants depends only on c_1 and c_2 and are irrelevant in the context of the rest of the proof. Furthermore, for a fixed t and q in (15), there are $\ll q$ pairs $(p_1, p_2) \in \mathbb{Z}^2$ for which the sets

$$C_f \cap ((S'(q, p_1) \times [0, 1]) \cup ([0, 1] \times S'(q, p_2)))$$

cover

$$\bigcup_{(p_1,p_2)\in\mathbb{Z}^2} C_f \cap ((S'(q, p_1) \times [0, 1]) \cup ([0, 1] \times S'(q, p_2))).$$

Drawing all the above considerations together it follows that the Hausdorff s-sum for this covering of the set $C_f \cap S_2^*(\psi)$, as defined in (15), is bounded above by

$$\sum_{t=n}^{\infty}\left(t\frac{\psi(2^t)}{2^t}\right)^s 2^{2t} \asymp \sum_{q=2^n}^{\infty} q^{1-s}\log^s q \cdot \psi^s(q) \to 0 \ \text{ as } n \to \infty.$$

As in the previous case, Case (a), by letting $n \to \infty$ we conclude that

$$\mathcal{H}^s(C_f \cap S_2^*(\psi)) = 0.$$

This completes the proof of Theorem 1.

4. Proof of Theorem 2. We now proceed with establishing Theorem 2. The proof is somewhat analogous to that of Theorem 1. Many of the estimates required to prove Theorem 2 follow in exactly the same way as, or at worst require minor modifications to the arguments used above. For this reason, and for the purpose of brevity, we leave some of the technical details to the reader.

For the sake of convenience, let $\psi = \psi_1$ and $\phi = \psi_2$. It is clear that

$$S_2(\phi, \psi) \subset S_2(\psi^*, \psi_*) \cup S_2(\psi_*, \psi^*)$$

where

$$\psi_* = \min\{\psi, \phi\} \quad \text{and} \quad \psi^* = \max\{\psi, \phi\}.$$

Since $\psi^*\psi_* = \psi\phi$, we have that $\sum \psi^*(q)\psi_*(q) < \infty$. Thus to prove Theorem 2 it is sufficient to show that both the sets $C_f \cap S_2(\psi^*, \psi_*)$ and $C_f \cap S_2(\psi_*, \psi^*)$ are of Lebesgue measure zero. We will consider one of these two sets, the other case is similar. Thus, without any loss of generality we assume that $\psi(q) \geqslant \phi(q)$ for all $q \in \mathbb{N}$.

Since $\sum_{q=1}^{\infty} \psi(q)\phi(q) < \infty$ and both ψ, ϕ are decreasing we have that $\psi(q)\phi(q) < q^{-1}$ for all sufficiently large q. Hence, $\phi(q) \leqslant q^{-1/2}$ for sufficiently large q. Further, we

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can assume that

$$\psi(q) \geqslant q^{-2/3} \tag{16}$$

for all $q \in \mathbb{N}$. To see this consider the auxiliary function $\tilde{\psi}$ where

$$\tilde{\psi}(q) = \max\{\psi(q), q^{-2/3}\}.$$

Clearly, $\tilde{\psi}$ is an approximating function. It also satisfies the following set inclusion,

$$S_2(\psi, \phi) \subset S_2(\tilde{\psi}, \phi).$$

Moreover,

$$\begin{aligned}\sum_{q=1}^{\infty} \tilde{\psi}(q)\phi(q) &\leqslant \sum_{q=1}^{\infty} \psi(q)\phi(q) + \sum_{q=1}^{\infty} q^{-2/3}\phi(q)\\ &\ll \sum_{q=1}^{\infty} \psi(q)\phi(q) + \sum_{q=1}^{\infty} q^{-2/3}q^{-1/2} < \infty.\end{aligned}$$

This means that it is sufficient to prove Theorem 2 with ψ replaced by $\tilde{\psi}$ and therefore without any loss of generality, (16) can be assumed.

In a manner analogous to that of (7), it is readily verified that for any $n \geqslant 1$

$$C_f \cap S_2(\psi, \phi) \subset \bigcup_{t=n}^{\infty} \bigcup_{2^t \leqslant q < 2^{t+1}} \bigcup_{(p_1,p_2)\in\mathbb{Z}^2} C_f \cap S_2(p_1, p_2, q) \tag{17}$$

where

$$S_2(p_1, p_2, q) = \left\{(x, y) \in \mathbb{R}^2 : \left|x - \frac{p_1}{q}\right| < \frac{\psi(2^t)}{2^t}, \left|y - \frac{p_2}{q}\right| < \frac{\phi(2^t)}{2^t}\right\}$$

and t is uniquely defined by $2^t \leqslant q < 2^{t+1}$. Next, we can use the same argument as that used in (12) to verify that

$$|C_f \cap S_2(q, p_1, p_2)|_{C_f} \ll \frac{\phi(2^t)}{2^t}. \tag{18}$$

Finally, for fixed t let $N(t)$ be the number of triples (q, p_1, p_2) with $2^t \leqslant q < 2^{t+1}$ such that $C_f \cap S(q, p_1, p_2) \neq \emptyset$. On modifying the argument used to establish (14), one obtains the estimate

$$N(t) \ll 2^{2t}\psi(2^t). \tag{19}$$

The upshot of (17), (18) and (19) is that

$$\begin{aligned}|C_f \cup S_2(\psi, \phi)|_{C_f} &\ll \sum_{t=n}^{\infty} \sum_{2^t \leqslant q < 2^{t+1}} \sum_{(p_1,p_2)\in\mathbb{Z}^2} |C_f \cap S_2(q, p_1, p_2)|_{C_f}\\ &\ll \sum_{t=n}^{\infty} N(t)\frac{\phi(2^t)}{2^t} \ll \sum_{t=n}^{\infty} 2^t\psi(2^t)\phi(2^t) \asymp \sum_{q=2^n}^{\infty} \psi(q)\phi(q).\end{aligned}$$

Since $\sum_{q=1}^{\infty} \psi(q)\phi(q) < \infty$, we have that $\sum_{q=2^n}^{\infty} \psi(q)\phi(q) \to 0$ as $n \to \infty$ and it follows that

$$|C_f \cap S_2(\psi, \phi)|_{C_f} = 0$$

as required.

This completes the proof of Theorem 2.

5. Remarks and possible developments. An obvious next step is to establish the divergence counterpart to Theorem 1. That is, one would like to show that

$$\sum_{h=1}^{\infty} h^{1-s}(\log^s h)\psi^s(h) = \infty \Rightarrow \mathcal{H}^s(C_f \cap S_2^*(\psi)) = \infty.$$

By adapting the arguments in this paper and using the ideas of local ubiquity, as developed in [**1**], it is likely that one could establish a zero-full result for $\mathcal{H}^h(C_f \cap S_2^*(\psi))$ where h is a general dimension function. This would include the above result and Theorem 1 as a special case. A dimension function $h : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing, continuous function such that $h(r) \to 0$ as $r \to 0$. By replacing the quantity $\text{diam}^s(C_i)$ with $h(\text{diam}(C_i))$ in the definition of $\mathcal{H}^s$, one can define the Hausdorff h-measure of a set. For further details see [**5**] or [**6**]. Dimension functions give very precise information about the measure theoretic properties of a set. The convergence part of such a theorem follows almost immediately on from Theorem 1. Most of the estimates obtained in the proof of Theorem 1 remain the same, the generalisation to Hausdorff h-measures affects only the estimates involving the measures of the actual covers defined in the proof. The main task in proving such a theorem would be in the proof of the divergence case.

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