3

Relativistic Electrodynamics

3.1 Lagrangian Formulation

3.1.1 The Free Maxwell Field

The Lagrangian for a free (noninteracting) electromagnetic field is

$$\mathcal{L} = -\tfrac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right). \tag{3.1}$$

Varying the fields and combining terms (Problem 3.1),

$$\begin{aligned}\delta I &= -\int d^4x\, \partial_\mu \left(\delta A_\nu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \\ &= \int d^4x\, \left(\delta A_\nu\right) \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = 0,\end{aligned} \tag{3.2}$$

so the equations of motion are

$$\partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = 0. \tag{3.3}$$

It is convenient to introduce the antisymmetric tensor[1]

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \tag{3.4}$$

in terms of which the equations of motion read, more compactly,

$$\partial_\mu F^{\mu\nu} = 0. \tag{3.5}$$

Because it is antisymmetric, the diagonal elements F^{00} and F^{ii} are trivially zero. Under 3-dimensional rotations the elements F^{0i} transform as the components of a 3-vector (nothing happens to the 0), so we write

$$F^{i0} \equiv E_i. \tag{3.6}$$

[1] Note that $F^{\mu\nu}$ is *not* an independent dynamical variable, just shorthand notation. In this formulation the (vector) field is A^μ (which we would ordinarily call the "potential"), and the electromagnetic "fields" **E** and **B** are, as we'll see in a moment, auxiliary constructs.

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This defines **E**, the **electric field**. Similarly,

$$F^{ij} \equiv -\epsilon_{ijk} B_k, \tag{3.7}$$

which defines the **magnetic field**, **B**.[2] Meanwhile, the ordinary scalar and vector potentials (V and **A**) are embedded in A^μ:[3]

$$A^\mu = (V, \mathbf{A}). \tag{3.8}$$

What are the field equations in this 3-dimensional notation? For $\nu = 0$ Eq. 3.5 says

$$\partial_i(E_i) = 0, \quad \text{or} \quad \boxed{\nabla \cdot \mathbf{E} = 0.} \tag{3.9}$$

For $\nu = i$ we get

$$\partial_0 F^{0i} + \partial_j F^{ji} = 0 \quad \Rightarrow \quad \frac{\partial(-E_i)}{\partial t} + \frac{\partial}{\partial x^j}(-\epsilon_{jik} B_k) = 0, \tag{3.10}$$

or

$$\boxed{\frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = \mathbf{0}.} \tag{3.11}$$

Meanwhile, Eq. 3.4, with $\mu = i$ and $\nu = j$, says $-\epsilon_{ijk} B_k = \partial^i A^j - \partial^j A^i$. Multiplying both sides by ϵ_{ijm} (and summing on i and j),[4]

$$-2B_m = -\epsilon_{ijm}(\partial_i A^j - \partial_j A^i) = -2(\nabla \times \mathbf{A})_m. \tag{3.12}$$

Thus

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \text{and therefore} \quad \boxed{\nabla \cdot \mathbf{B} = 0.} \tag{3.13}$$

If $\mu = 0$ and $\nu = i$,

$$-E_i = \frac{\partial A^i}{\partial t} + \frac{\partial A^0}{\partial x^i} \quad \Rightarrow \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V, \tag{3.14}$$

or, taking the curl,

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0}.} \tag{3.15}$$

[2] Here ϵ_{ijk} is the 3-dimensional **Levi-Civita symbol**, with $\epsilon_{123} \equiv 1$; **E** and **B** are *not* elements of 4-vectors, and for them there is no covariant/contravariant distinction, so we will always write the indices *down*. We do, however, copy the Einstein summation convention, with the stipulation that repeated Roman letters are summed from 1 to 3.

[3] Eds. Yes, this leaves the expression A_i ambiguous: Is it the ith component of the covariant 4-vector A_μ, or the ith component of the 3-vector **A** (they differ by a sign)? Coleman finesses this sort of problem by being cheerfully inconsistent. We will reserve the boldface **A** for the ordinary 3-vector potential.

[4] Note that $\epsilon_{ijm}\epsilon_{ijk} = 2\delta_{mk}$.

The boxed results are Maxwell's equations in regions free of charge and current. Because they come from a Lorentz-invariant Lagrangian, they are (collectively) Lorentz invariant, by construction.

Although A^μ is the fundamental field, it is not directly measurable—rather, **E** and **B** are the measurable quantities. Indeed, a **gauge transformation**

$$A^\mu \to A^\mu + \partial^\mu \chi \tag{3.16}$$

(for an arbitrary scalar function χ) leaves $F^{\mu\nu}$ unchanged, and the equations of motion unaltered. We consider two solutions *physically equivalent* if they differ only by such a gauge transformation.[5] In the **Lorenz gauge**[6] we choose χ so that A^μ is divergenceless:[7]

$$\partial_\mu A^\mu = 0. \tag{3.17}$$

In the Lorenz gauge the equation of motion simplifies to

$$\Box^2 A^\mu = 0, \tag{3.18}$$

where

$$\Box^2 \equiv \partial_\mu \partial^\mu = \partial_0^2 - \nabla^2 \tag{3.19}$$

is the **d'Alembertian** (4-dimensional generalization of the **Laplacian**, ∇^2). Thus the components of A^μ satisfy the homogeneous wave equation.

The electromagnetic stress tensor (see Problem 3.2) is

$$T^{\mu\nu} = F^{\mu\lambda} F_\lambda{}^\nu + \tfrac{1}{4} g^{\mu\nu} F_{\lambda\sigma} F^{\lambda\sigma}. \tag{3.20}$$

In terms of **E** and **B**,

$$\begin{aligned} F_{\lambda\sigma} F^{\lambda\sigma} &= F_{0i} F^{0i} + F_{i0} F^{i0} + F_{ij} F^{ij} \\ &= (E_i)(-E_i) + (-E_i)(E_i) + (-\epsilon_{ijk} B_k)(-\epsilon_{ijl} B_l) \\ &= -2\mathbf{E}^2 + 2\mathbf{B}^2, \end{aligned} \tag{3.21}$$

[5] This is a stronger statement than **gauge invariance** alone. The latter says that a gauge transformation carries one solution to the equations of motion into another solution, but regarding the two solutions as physically equivalent implies that all the physical information is contained in $F^{\mu\nu}$.

[6] Eds. Coleman calls it the "Lorentz" gauge, but it is now more commonly attributed to Ludvig Lorenz, rather than Hendrik Lorentz. See J. D. Jackson and L. B. Okun, *Rev. Mod. Phys.* **73**, 663 (2001).

[7] The Lorenz condition does not uniquely specify the gauge, since one can still perform a transformation such that

$$\Box^2 \chi = 0.$$

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so the energy density is[8]

$$T^{00} = F^{0i} F_i{}^0 + \tfrac{1}{4} g^{00} F_{\lambda\sigma} F^{\lambda\sigma} = \mathbf{E}^2 + \tfrac{1}{2}(-\mathbf{E}^2 + \mathbf{B}^2) = \tfrac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2), \tag{3.22}$$

and the momentum density is

$$T_0{}^i = F_{0j} F^{ji} = (E_j)(-\epsilon_{jik} B_k) = \epsilon_{ijk} E_j B_k = (\mathbf{E} \times \mathbf{B})_i \tag{3.23}$$

(the **Poynting vector**).

Problem 3.1

In case the passage to Eq. 3.2 was not clear to you (what happened to the 4?), try isolating a particular entry in 3.1—say, $\partial_0 A_1$. Show that

$$\mathcal{L} = \tfrac{1}{2}\left[(\partial_0 A_1)(\partial_0 A_1) - 2(\partial_0 A_1)(\partial_1 A_0)\right] + \cdots,$$

where the dots indicate terms *not* involving $\partial_0 A_1$. Now do the variation of $\partial_0 A_1$.

Problem 3.2

(a) Find the canonical stress tensor (2.133) for the Lagrangian (3.1).
(b) Symmetrize it (2.152) to confirm Eq. 3.20. [Use the full machinery of Section 2.4.3 if you like, but you may be able to guess the perfect divergence you need to add.]
(c) Check that $T^{\mu\nu}$ is divergenceless.

3.1.2 Maxwell Field with Source

Suppose the electromagnetic field is coupled to an external source,

$$J^\mu = (\rho, \mathbf{J}), \tag{3.24}$$

where ρ is the charge density, $\mathbf{J}$ is the current density, and they satisfy the continuity equation (local conservation of charge),

$$\partial_\mu J^\mu = 0. \tag{3.25}$$

The Lagrangian is now

$$\mathcal{L} = -\tfrac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)(\partial^\mu A^\nu - \partial^\nu A^\mu) - A_\mu J^\mu. \tag{3.26}$$

Varying the field,

$$\delta I = \int d^4x \left(\partial_\mu F^{\mu\nu}\, \delta A_\nu - J^\nu\, \delta A_\nu\right) = 0 \quad \Rightarrow \quad \partial_\mu F^{\mu\nu} - J^\nu = 0. \tag{3.27}$$

[8] Eds. Coleman uses **Heaviside–Lorentz units**, with $\epsilon_0 \to 1$ and $\mu_0 \to 1$ (and, of course, $c = 1$).

For $\nu = 0$ we get

$$\partial_i F^{i0} - J^0 = 0, \quad \text{or} \quad \boxed{\nabla \cdot \mathbf{E} = \rho;} \tag{3.28}$$

for $\nu = i$,

$$\partial_0 F^{0i} + \partial_j F^{ji} - J^i = 0 \quad \Rightarrow \quad -\frac{\partial E_i}{\partial t} + \partial_j(-\epsilon_{jik} B_k) - J^i = 0, \tag{3.29}$$

or

$$\boxed{\frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -\mathbf{J}.} \tag{3.30}$$

The homogeneous Maxwell equations (3.13 and 3.15) are unchanged, since they follow from Eq. 3.4:

$$\boxed{\nabla \cdot \mathbf{B} = 0} \quad \text{and} \quad \boxed{\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0}.} \tag{3.31}$$

The boxed results are Maxwell's equations with sources. In the Lorenz gauge (3.17) the equation of motion (3.27) simplifies:[9]

$$\Box^2 A^\mu = J^\mu. \tag{3.32}$$

In the absence of sources, the stress tensor (3.20) is divergenceless (Problem 3.2(c)), but this is no longer true in the presence of J^μ (the external source acts on the fields, and *vice versa*, so it is not surprising that field energy and field momentum are no longer conserved):

$$\begin{aligned}
\partial_\mu T^{\mu\nu} &= \partial_\mu \left[F^{\mu\lambda} F_\lambda{}^\nu + \tfrac{1}{4} g^{\mu\nu} (F_{\lambda\sigma} F^{\lambda\sigma}) \right] \\
&= J^\lambda F_\lambda{}^\nu + F^{\mu\lambda} \partial_\mu F_\lambda{}^\nu + \tfrac{1}{4} \partial^\nu (F_{\lambda\sigma} F^{\lambda\sigma}) \\
&= J^\lambda F_\lambda{}^\nu + F^{\mu\lambda} \left[\partial_\mu F_\lambda{}^\nu + \tfrac{1}{2} \partial^\nu F_{\mu\lambda} \right].
\end{aligned} \tag{3.33}$$

Raising and lowering the dummy indices,

$$\begin{aligned}
F^{\mu\lambda} \left[\partial_\mu F_\lambda{}^\nu + \tfrac{1}{2} \partial^\nu F_{\mu\lambda} \right] &= F_{\mu\lambda} \left[\partial^\mu F^{\lambda\nu} + \tfrac{1}{2} \partial^\nu F^{\mu\lambda} \right] \\
&= F_{\mu\lambda} \left[\partial^\mu \partial^\lambda A^\nu - \partial^\mu \partial^\nu A^\lambda + \tfrac{1}{2} \partial^\nu \partial^\mu A^\lambda - \tfrac{1}{2} \partial^\nu \partial^\lambda A^\mu \right].
\end{aligned} \tag{3.34}$$

Because $F_{\mu\lambda}$ is antisymmetric,

$$F_{\mu\lambda} \left[\partial^\mu \partial^\lambda A^\nu \right] = 0 \quad \text{and} \quad F_{\mu\lambda} \left[-\tfrac{1}{2} \partial^\nu \partial^\lambda A^\mu - \tfrac{1}{2} \partial^\nu \partial^\mu A^\lambda \right] = 0, \tag{3.35}$$

[9] Equation 3.32 looks like four equations in four unknowns, which sounds OK; on the other hand, we are ultimately hoping to solve for *six* unknowns (three components each of **E** and **B**). The "extra" two equations are implicit in 3.4. To put it another way, Maxwell's equations are ostensibly *eight* equations for six unknowns—too many. The point is that the components of **E** and **B** are not all independent. There are really just *four* independent variables in electrodynamics, and A^μ encodes these most efficiently.

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so the last part of Eq. 3.33 is in fact zero, and we are left with

$$\partial_\mu T^{\mu\nu} = J_\lambda F^{\lambda\nu}. \tag{3.36}$$

For example, with $\nu = 0$,

$$\partial_\mu T^{\mu 0} = J_i F^{i0} = -J^i F^{i0} = -\mathbf{J}\cdot\mathbf{E} = \partial_0 T^{00} + \partial_i T^{i0}. \tag{3.37}$$

Integrating over all space, at time t,

$$\frac{d}{dt}\int d^3\mathbf{x}\, T^{00} = -\int d^3\mathbf{x}\,\mathbf{J}\cdot\mathbf{E}. \tag{3.38}$$

In words, the rate of change of the energy stored in the fields is minus the work done by the fields on the charges.

3.2 Potentials and Fields of a Point Charge

3.2.1 The Action for a Point Charge

Next we consider an electromagnetic field interacting with a point charge—combining the results of Section 2.3 (mechanics of a particle) and Section 3.1.2 (electromagnetic fields with a source).[10] The action consists of two parts, the free term and the interaction term:

$$I = I_0 + I'. \tag{3.39}$$

The free term is composed of a matter part (the charged particle) and a field part (the electromagnetic field):

$$I_0 = I_0^m + I_0^e, \tag{3.40}$$

where (Eq. 2.85 and Problem 2.6)

$$I_0^m = -\frac{m_0}{2}\int d\tau\, \dot{y}_\mu\, \dot{y}^\mu. \tag{3.41}$$

Here $y^\mu(\tau)$ is the world line of the particle, parameterized by its proper time, and the dot denotes the τ derivative; m_0 is the **bare mass**—the mass the particle would have in the absence of any interaction (as we shall see, interaction with the

[10] Eds. This is known as **classical electron theory**. It doesn't have to be an *electron*, of course—any structureless point charge would do. Coleman wrote a famous treatise on the subject in the summer of 1960, while still a graduate student. It was finally published many years later in *Electromagnetism: Paths to Research*, D. Teplitz, ed., Plenum, New York (1982), Chapter 6. It is also available at www.rand.org/pubs/research_memoranda/RM2820.html. In the remainder of this chapter Coleman follows that treatment closely.

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electromagnetic field modifies the effective mass of the particle). The field part is (Eq. 3.1)

$$I_0^e = -\frac{1}{4}\int d^4x\, \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)(\partial^\mu A^\nu - \partial^\nu A^\mu)\,. \tag{3.42}$$

Finally, the interaction term is

$$I' = -e\int d\tau \int d^4x\, A_\mu(x)\,\delta^4(x - y(\tau))\,\dot{y}^\mu, \tag{3.43}$$

where e is the electric charge of the particle. Or, if we define the current

$$J^\mu(x) \equiv e\int d\tau\,\delta^4(x - y(\tau))\,\dot{y}^\mu, \tag{3.44}$$

then

$$I' = -\int d^4x\, J^\mu A_\mu. \tag{3.45}$$

This is from the perspective of the *field* (Eq. 3.26); alternatively (from the point of view of the *particle*), we could do the x integral first:

$$I' = -e\int d\tau\,\dot{y}^\mu\, A_\mu(y) = -e\int dy^\mu\, A_\mu(y). \tag{3.46}$$

The independent dynamical variables here are $y^\mu(\tau)$ and $A_\mu(x)$. Varying A_μ we get (as before, Eq. 3.27)

$$\partial_\mu F^{\mu\nu} - J^\nu = 0. \tag{3.47}$$

For the variation of y^μ, the relevant part of the action is

$$I_p = -\frac{m_0}{2}\int d\tau\,\dot{y}_\mu\,\dot{y}^\mu - e\int d\tau\,\dot{y}^\mu\, A_\mu(y). \tag{3.48}$$

Then

$$\delta I_p = -m_0\int d\tau\,\dot{y}_\mu\,\delta\dot{y}^\mu - e\int d\tau(\delta\dot{y}_\mu)A^\mu(y) - e\int d\tau\,\dot{y}_\mu(\partial_\nu A^\mu(y))\delta y^\nu = 0. \tag{3.49}$$

The resulting equation of motion is

$$m_0\,\ddot{y}_\mu + e\frac{d}{d\tau}A_\mu - e\dot{y}_\nu\,\partial_\mu A^\nu = 0, \quad m_0\,\ddot{y}^\mu = -e(\partial^\nu A^\mu)\dot{y}_\nu + e(\partial^\mu A^\nu)\dot{y}_\nu, \tag{3.50}$$

or

$$m_0\,\ddot{y}^\mu = -eF^{\nu\mu}\,\dot{y}_\nu. \tag{3.51}$$

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Notice that neither 3.47 nor 3.51 involves the potential A_μ directly—only the field tensor $F^{\mu\nu}$—so the theory remains fully gauge invariant. In the customary 3-dimensional notation, Eq. 3.51 reads

$$m_0\, \ddot{y}_0 = e\mathbf{E}\cdot\dot{\mathbf{y}}, \quad m_0\ddot{\mathbf{y}} = e\left[\dot{y}_0\,\mathbf{E} + (\dot{\mathbf{y}}\times\mathbf{B})\right]. \tag{3.52}$$

The former (the $\mu = 0$ component) tells you the power delivered to the charge by the fields; the latter ($\mu = i$) is the **Lorentz force law**[11] telling you the force on the charge.

I won't work out the stress tensor in this theory. It turns out that there is no interaction term[12]—$T_{\mu\nu}$ is the sum of the free particle tensor (2.165) and the free field tensor (3.20),

$$T_{\mu\nu} = T^m_{\mu\nu} + T^e_{\mu\nu}. \tag{3.53}$$

You can check this by showing that $T_{\mu\nu}$ is conserved (Problem 3.3).

Problem 3.3

Check that the stress tensor 3.53 is divergenceless.

3.2.2 *Green's Function for the Wave Equation*

In the Lorenz gauge the field equation reduces to the **inhomogeneous wave equation** (or rather, *four* of them, one for each component):

$$\Box^2 A^\mu = J^\mu \tag{3.54}$$

(Eq. 3.32). I propose to solve it by the Green's function method. Consider first the *scalar* wave equation,

$$\Box^2\varphi = \rho. \tag{3.55}$$

Note that the solution is not going to be unique; you can add any solution to the *homogeneous* equation (in practice boundary conditions will select the right one

[11] This is the **Minkowski force**, $dp^\mu/d\tau$, not the "ordinary" force $\mathbf{F} = d\mathbf{p}/dt$; recall that the proper velocity $\dot{y}^\mu = \gamma(1, \boldsymbol{\beta})$ and $d\tau = (1/\gamma)dt$, so

$$\mathbf{F} = \frac{d\tau}{dt} e\left[\gamma\mathbf{E} + \gamma(\boldsymbol{\beta}\times\mathbf{B})\right] = e\left[\mathbf{E} + (\boldsymbol{\beta}\times\mathbf{B})\right].$$

[12] The absence of a field–particle cross term in $T^{\mu\nu}$ is a peculiarity of electromagnetism, not shared by other theories of field–particle interactions.

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for the context you have in mind). Suppose you could solve Eq. 3.55 for a delta-function source:

$$\Box^2 D(x) = \delta^4(x); \tag{3.56}$$

then

$$\varphi(x) = \int d^4y\, D(x-y)\,\rho(y). \tag{3.57}$$

So our problem is to find the **Green's function**, D. This we will do by **Fourier transform**:

$$\delta^4(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x}, \quad D(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x}\,\widetilde{D}(k). \tag{3.58}$$

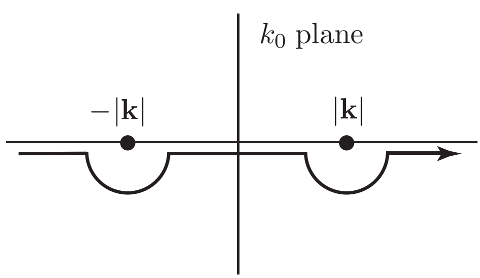
Putting these into Eq. 3.56,

$$-k^2\,\widetilde{D}(k) = 1 \quad \Rightarrow \quad \widetilde{D}(k) = -\frac{1}{k^2}, \tag{3.59}$$

so

$$D(x) = -\int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x}\,\frac{1}{k^2} = -\frac{1}{(2\pi)^4}\int d^3\mathbf{k}\, e^{-i\mathbf{k}\cdot\mathbf{x}} \left\{ \int dk_0\, \frac{e^{ik_0 t}}{k_0^2 - \mathbf{k}^2} \right\}. \tag{3.60}$$

Let's do the k_0 integral first. We need to decide how to skirt the poles (at $k_0 = \pm|\mathbf{k}|$) in the complex k_0 plane. Any prescription will do, because the Green's function is not unique anyhow (different choices will differ by solutions to the homogeneous equation). I'll use the contour that runs *below* both poles:



If $t < 0$ we close the contour below (so as to kill the contribution from the semicircle at negative imaginary k_0), and $D = 0$. Thus only sources at *earlier* times can contribute to φ; we call this the **retarded Green's function**, D_R.[13]

[13] To get the **advanced Green's function**, D_A, let the contour go *over* both poles. But D_A is acausal: sources at *later* times affect $\varphi(x)$. For this reason it is ordinarily rejected.

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For $t > 0$ we close the contour *above*; letting $\kappa \equiv |\mathbf{k}|$, and using Cauchy's integral formula,

$$\int dk_0 \, \frac{e^{ik_0 t}}{(k_0 - \kappa)(k_0 + \kappa)} = 2\pi i \frac{1}{2\kappa} \left[e^{i\kappa t} - e^{-i\kappa t}\right]. \tag{3.61}$$

There remains the $\mathbf{k}$ integral:

$$D_R(x) = -\frac{i}{16\pi^3} \int d^3\mathbf{k} \, \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{\kappa} \left[e^{i\kappa t} - e^{-i\kappa t}\right]. \tag{3.62}$$

Adopting spherical coordinates, with the polar axis along $\mathbf{x}$, $|\mathbf{x}| = r$, $\mathbf{k}\cdot\mathbf{x} = \kappa r \cos\theta$, $d^3\mathbf{k} = \kappa^2 \sin\theta \, d\kappa \, d\theta \, d\phi$,

$$D_R(x) = -\frac{i}{16\pi^3} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin\theta \left\{ \int_0^{\infty} d\kappa \, \kappa \, e^{-i\kappa r \cos\theta} \left[e^{i\kappa t} - e^{-i\kappa t}\right] \right\}. \tag{3.63}$$

Changing variables from θ to $\mu \equiv \cos\theta$,

$$D_R(x) = -\frac{i}{8\pi^2} \int_{-1}^{1} d\mu \left\{ \int_0^{\infty} d\kappa \, \kappa \, e^{-i\kappa r \mu} \left[e^{i\kappa t} - e^{-i\kappa t}\right] \right\}. \tag{3.64}$$

The μ integral is easy:

$$\int_{-1}^{1} d\mu \, e^{-i\kappa r \mu} = \frac{1}{i\kappa r} \left(e^{i\kappa r} - e^{-i\kappa r}\right), \tag{3.65}$$

so

$$\begin{aligned} D_R(x) &= -\frac{1}{8\pi^2 r} \int_0^{\infty} d\kappa \left(e^{i\kappa r} - e^{-i\kappa r}\right)\left(e^{i\kappa t} - e^{-i\kappa t}\right) \\ &= -\frac{1}{8\pi^2 r} \int_{-\infty}^{\infty} d\kappa \left[e^{i\kappa(r+t)} - e^{i\kappa(r-t)}\right] = -\frac{2\pi}{8\pi^2 r} \left[\delta(r+t) - \delta(r-t)\right] \\ &= \frac{1}{4\pi r} \delta(r - t) \end{aligned} \tag{3.66}$$

(the other delta function dies because r and t are both positive). Remember, this was for $t > 0$; if $t < 0$ then $D_R(x) = 0$. In nicer notation,

$$D_R(x) = \frac{1}{2\pi} \delta(r^2 - t^2)\,\theta(t) = \frac{1}{2\pi} \delta(x_\mu x^\mu)\,\theta(x^0). \tag{3.67}$$

Let's explore this Green's function in the vicinity of $t = 0^+$:

$$D_R(\mathbf{x}, 0) = \frac{1}{4\pi r} \delta(r). \tag{3.68}$$

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It's zero for $r > 0$, but could there be a delta function lurking at the origin? For any test function $f(\mathbf{x})$ that is not *itself* singular at $r = 0$,

$$\int d^3\mathbf{x}\, D_R(\mathbf{x},0)\, f(\mathbf{x}) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dr\, r^2 \frac{1}{4\pi r}\, \delta(r)\, f(r,\theta,\phi) = 0 \tag{3.69}$$

(the r integral is zero). So, in the language of distributions,

$$D_R(\mathbf{x},0) = 0. \tag{3.70}$$

How about its time derivative? From Eq. 3.66,

$$\frac{\partial}{\partial t} D_R(\mathbf{x},t)\bigg|_{t=0} = -\frac{1}{4\pi r}\delta'(r). \tag{3.71}$$

Again, multiplying by a test function and integrating,

$$\int d^3\mathbf{x}\, \frac{\partial}{\partial t} D_R(\mathbf{x},t)\bigg|_{t=0} f(\mathbf{x}) \\ = -\frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta \int_0^{\infty} dr\, r\, \delta'(r)\, f(r,\theta,\phi). \tag{3.72}$$

Integrating by parts,

$$\int_0^{\infty} dr\, r\, \delta'(r)\, f(r,\theta,\phi) = r\, \delta(r)\, f(r,\theta,\phi)\Big|_0^{\infty} - \int_0^{\infty} dr\, \delta(r)\, \frac{\partial}{\partial r}(r\, f) \\ = 0 - \int_0^{\infty} dr\, \delta(r) \left(f + r\frac{\partial f}{\partial r} \right) = -f(\mathbf{0}). \tag{3.73}$$

Thus

$$\frac{\partial}{\partial t} D_R(\mathbf{x},t)\bigg|_{t=0} = \delta^3(\mathbf{x}). \tag{3.74}$$

We can use $D_R(x)$ to solve the *inhomogeneous* wave equation (Eq. 3.57); it can *also* be used to solve the *homogeneous* wave equation with specified initial conditions.

Example 3.1

Find $f(\mathbf{x},t)$ if

$$\Box^2 f(\mathbf{x},t) = 0, \quad f(\mathbf{x},0) = h(\mathbf{x}), \quad \frac{\partial}{\partial t} f(\mathbf{x},t)\bigg|_{t=0} = g(\mathbf{x}) \tag{3.75}$$

for given functions $h(\mathbf{x})$ and $g(\mathbf{x})$.

Solution:

$$f(\mathbf{x},t) = \int d^3\mathbf{x}' \left\{ D_R(\mathbf{x}-\mathbf{x}',t)\, g(\mathbf{x}') + \left[\frac{\partial}{\partial t} D_R(\mathbf{x}-\mathbf{x}',t) \right] h(\mathbf{x}') \right\}. \tag{3.76}$$

Proof: In view of Eq. 3.56,

$$\Box^2 D_R(\mathbf{x}-\mathbf{x}',t) = \delta(t)\,\delta^3(\mathbf{x}-\mathbf{x}'), \tag{3.77}$$

so

$$\begin{aligned}\Box^2 f(\mathbf{x},t) &= \int d^3\mathbf{x}'\left\{\delta(t)\delta^3(\mathbf{x}-\mathbf{x}')\,g(\mathbf{x}') + \frac{\partial}{\partial t}\left[\delta(t)\delta^3(\mathbf{x}-\mathbf{x}')\right]h(\mathbf{x}')\right\}\\ &= \delta(t)g(\mathbf{x}) + \delta'(t)\,h(\mathbf{x}),\end{aligned} \tag{3.78}$$

so it satisfies the differential equation (3.75) for $t > 0$. In view of Eqs. 3.70 and 3.74,

$$f(\mathbf{x},0) = \int d^3\mathbf{x}'\left\{(0)\,g(\mathbf{x}') + \delta^3(\mathbf{x}-\mathbf{x}')h(\mathbf{x}')\right\} = h(\mathbf{x}), \tag{3.79}$$

so it satisfies the first boundary condition. And

$$\frac{\partial}{\partial t}f(\mathbf{x},t)\Big|_{t=0} = \int d^3\mathbf{x}'\left\{\delta^3(\mathbf{x}-\mathbf{x}')g(\mathbf{x}') + \left[\frac{\partial^2}{\partial t^2}D_R(\mathbf{x}-\mathbf{x}',t)\Big|_{t=0}\right]h(\mathbf{x}')\right\}. \tag{3.80}$$

But from Eq. 3.70,

$$\frac{\partial^2}{\partial t^2}D_R(\mathbf{x},t)\Big|_{t=0} = \nabla^2 D_R(\mathbf{x},t)\Big|_{t=0} = \nabla^2 D_R(\mathbf{x},0) = \nabla^2\,[0] = 0, \tag{3.81}$$

so

$$\frac{\partial}{\partial t}f(\mathbf{x},t)\Big|_{t=0} = g(\mathbf{x}), \tag{3.82}$$

and therefore Eq. 3.76 also satisfies the second boundary condition. QED

Equation 3.76 can be written in a more concise way by introducing the **left–right derivative**,

$$f\overleftrightarrow{\partial} g \equiv f\,(\partial g) - (\partial f)\,g. \tag{3.83}$$

Thus

$$\begin{aligned}f(\mathbf{x},t) &= \int d^3\mathbf{x}'\left\{D_R(\mathbf{x}-\mathbf{x}',t-t')\partial_0' f(\mathbf{x}',t') - \left[\partial_0' D_R(\mathbf{x}-\mathbf{x}',t-t')\right]f(\mathbf{x}',t')\right\}\Big|_{t'=0}\\ &= \int d^3\mathbf{x}'\left\{D_R(x-x')\overleftrightarrow{\partial}'_0 f(\mathbf{x}',t')\right\}\Big|_{t'=0}.\end{aligned} \tag{3.84}$$

More generally, if the boundary conditions are not specified on a plane,

$$f(x) = \int_\sigma d\sigma^{\mu'}\left\{D_R(x-x')\overleftrightarrow{\partial}'_\mu f(x')\right\}\Big|_{(x'\text{ on }\sigma)} \tag{3.85}$$

(this is in essence **Huygens' principle**: you build up the new wave front from the old wave front using each point as a source).

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Problem 3.4

Show that $D_A(x) = D_R(-x)$ (footnote 13).

3.2.3 "In" and "Out" Fields

In Eq. 3.32 we reduced Maxwell's electrodynamics to the inhomogeneous wave equation with a specified source:

$$\Box^2 A_\mu = J_\mu. \tag{3.86}$$

In Eq. 3.57 we constructed the general solution to this differential equation, using the retarded Green's function:

$$A_\mu(x) = A_\mu^{\text{in}}(x) + \int d^4x'\, D_R(x - x')\, J_\mu(x'), \tag{3.87}$$

where $A_\mu^{\text{in}}(x)$ is a solution to the *homogeneous* wave equation:

$$\Box^2 A_\mu^{\text{in}}(x) = 0. \tag{3.88}$$

We call A_μ^{in} the "in" (or "incoming") field. The second term (in 3.87) tells us the field generated by the current J_μ; the in field is something that would have been there *even if there had been no source at all*. Informally, we think of it as a kind of "primordial" field, coming in from the distant past, before the current was turned on.[14] In classical electrodynamics we normally[15] assume that

$$\boxed{\textbf{all electromagnetic fields are due to charges and currents.}} \tag{3.89}$$

If there's no charge or current anywhere, ever, then the field would be zero. Accordingly, we will stipulate that

$$A_\mu^{\text{in}}(x) = 0. \tag{3.90}$$

You can also solve Eq. 3.86 using the *advanced* Green's function (Problem 3.4)—or for that matter with any linear combination of the two, such as the average:

$$A_\mu(x) = A_\mu^{\text{out}}(x) + \int d^4x'\, D_A(x - x')\, J_\mu(x') \tag{3.91}$$

$$= \left(\frac{A_\mu^{\text{in}} + A_\mu^{\text{out}}}{2}\right) + \int d^4x' \left(\frac{D_R + D_A}{2}\right) J_\mu. \tag{3.92}$$

[14] Because of charge conservation, it may not be *possible* to turn the current on and off. For instance, it is nonsense to ask what the field would be for a point charge that suddenly materializes from nothing at time $t = 0$; such a source would be incompatible with Maxwell's equations. For more on "in" and "out" fields, see Coleman's treatise, Section 3 (details in footnote 10).

[15] Eds. An exception is **stochastic electrodynamics**, which proposes that the universe is permeated by random sourceless electromagnetic radiation.

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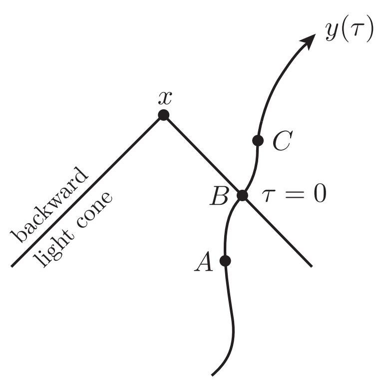
The “out” (or “outgoing”) field again satisfies the homogeneous wave equation. Informally it is the field that remains after all currents have been turned off[14] (and the integral in 3.91 does not contribute); unlike A_μ^{in}, it is certainly *not* zero. This introduces a time asymmetry into classical electrodynamics: Maxwell’s equations are time-reversal invariant, but the standard boundary conditions (in the form of the in and out fields) are *not*: time reversal would take $A^{\text{in}} \leftrightarrow A^{\text{out}}$, but the first is zero and the second is not.

If you pose the question “What is the source of the observed time irreversibility of the universe?” two possible answers come to mind: (a) statistical mechanics (the second law of thermodynamics) and (b) electrodynamics (an accelerating electron radiates into the future, not into the past). **Feynman–Wheeler electrodynamics**[16] challenges option (b). In their formulation, Maxwellian electrodynamics is correct, but the universe is surrounded by a perfect absorber, so that A^{in} and A^{out} are both zero. The cosmic absorber turns all radiation into heat. Electrons radiate both forward and backward in time (Eq. 3.92); the forward radiation gets to the absorber, which reradiates both forward and backward in time—the backward radiation from the absorber returns to the electron with just the right phase so as to cancel the backward-going radiation from the electron. That’s why we don’t see the electron radiating backward; according to Feynman–Wheeler, you *can’t tell* whether an accelerating electron radiates backward, because—with the absorber out there—it gets canceled anyhow. In Feynman–Wheeler electrodynamics, time asymmetry is due entirely to statistical mechanics.[17]

3.3 Radiation from a Point Charge

3.3.1 The Liénard–Wiechert Potential

Imagine a charged particle moving along its world line, $y(\tau)$:



[16] J. A. Wheeler and R. P. Feynman, *Rev. Mod. Phys.* **17**, 157 (1945).
[17] Note that the absorber is not just a mathematical boundary condition, but an extra dynamical system; this is, for most people, the implausible part of the story.

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The field it produces at x is determined by the particle's position/velocity/acceleration at the point B, where the backward light cone (of x) intersects the world line of the particle (note that there can be at most *one* such intersection, because the particle cannot travel at or above the speed of light). For simplicity I'll set $\tau = 0$ at point B. In Maxwellian electrodynamics ($A_\mu^{\text{in}} = 0$), Eq. 3.87 says

$$A_\mu(x) = \int d^4x' \, D_R(x - x') \, J_\mu(x'), \tag{3.93}$$

where (3.67)

$$D_R(x) = \frac{1}{2\pi} \, \delta(x_\mu x^\mu) \, \theta(x^0) \tag{3.94}$$

(the theta function prevents any influence on *earlier* times—the "message" must arrive at x *after* it left B; the delta function confines any influence to the light cone). Finally, the particle current is given by 3.44:

$$J_\mu(x) \equiv e \int d\tau \, \delta^4(x - y(\tau)) \, \dot{y}_\mu(\tau). \tag{3.95}$$

Putting all of this together,

$$A_\mu(x) = \frac{e}{2\pi} \int\int d^4x' \, d\tau \, \delta\left((x - x')^2\right) \delta^4(x' - y(\tau))\theta(x^0 - x^{0'}) \, \dot{y}_\mu(\tau). \tag{3.96}$$

First do the x' integral; this simply replaces x' by $y(\tau)$:

$$A_\mu(x) = \frac{e}{2\pi} \int d\tau \, \delta(z^2)\theta(z^0) \, \dot{y}_\mu(\tau), \tag{3.97}$$

where

$$z^\mu(\tau) \equiv x^\mu - y^\mu(\tau), \quad \text{so} \quad \delta(z^2) = \frac{1}{|\partial z^2/\partial\tau|}\delta(\tau). \tag{3.98}$$

Now,

$$\begin{aligned} \frac{\partial z^2}{\partial \tau} &= \frac{\partial}{\partial \tau} \left(x_\nu - y_\nu(\tau)\right)\left(x^\nu - y^\nu(\tau)\right) \\ &= \left(x_\nu - y_\nu(\tau)\right)\left(-\dot{y}^\nu(\tau)\right) + \left(-\dot{y}_\nu(\tau)\right)\left(x^\nu - y^\nu(\tau)\right) = -2z_\nu \, \dot{y}^\nu(\tau). \end{aligned} \tag{3.99}$$

What is its sign? At B (light-like separated from x), $z^2 = 0$; at C the separation is *space-like*, so $z^2 < 0$; at A the separation is *time-like*, so $z^2 > 0$. Evidently z^2 is a *decreasing* function of τ, so

$$\left|\frac{\partial z^2}{\partial \tau}\right| = -\frac{\partial z^2}{\partial \tau} = 2z_\nu \, \dot{y}^\nu \quad \text{and} \quad \delta(z^2) = \frac{1}{2z_\nu \dot{y}^\nu} \, \delta(\tau). \tag{3.100}$$

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Therefore

$$A_\mu(x) = \frac{e}{4\pi}\frac{1}{z_\nu \dot{y}^\nu}\,\dot{y}_\mu\bigg|_{(z^2=0,\; z^0>0)}. \tag{3.101}$$

This is the **Liénard–Wiechert potential** for a point charge in arbitrary motion.

Example 3.2

Coulomb's law. For a particle at rest,

$$y^\nu(\tau) = (\tau, \mathbf{y}), \quad \dot{y}^\nu = (1, \mathbf{0}); \quad A_\mu(x) = \frac{e}{4\pi}\frac{1}{z_0}(1,\mathbf{0})\bigg|_{(z^2=0,\; z^0>0)}. \tag{3.102}$$

At $z^2 = 0$, $z_0 = \pm|\mathbf{z}| = \pm|\mathbf{x}-\mathbf{y}| = \pm\,\mathscr{r}$ (where $\boldsymbol{\mathscr{r}} \equiv \mathbf{x}-\mathbf{y}$), and because of the theta function we want the *positive* root, $\mathscr{r}$, the (spatial) distance from the charge to the point $\mathbf{x}$, so

$$A_\mu(x) = \frac{e}{4\pi\,\mathscr{r}}(1,\mathbf{0}) \tag{3.103}$$

(the ordinary Coulomb potential, in Heaviside–Lorentz units).

3.3.2 The Fields of a Point Charge

Differentiating Eq. 3.97 (and suppressing the θ-function, which we can take care of at the end),

$$\begin{aligned}
\partial_\nu A_\mu(x) &= \frac{e}{2\pi}\int d\tau\,\delta'(z^2)\,2z_\nu\,\dot{y}_\mu(\tau) \\
&= \frac{e}{\pi}\int d\tau\,\delta'(z^2)\left(\frac{\partial z^2}{\partial\tau}\right)\left(\frac{\partial z^2}{\partial\tau}\right)^{-1} z_\nu(\tau)\,\dot{y}_\mu(\tau) \\
&= \frac{e}{\pi}\int d\tau\,\frac{\partial}{\partial\tau}\left[\delta\left(z^2\right)\right]\left(\frac{\partial z^2}{\partial\tau}\right)^{-1} z_\nu(\tau)\,\dot{y}_\mu(\tau) \\
&= -\frac{e}{\pi}\int d\tau\,\delta\left(z^2\right)\frac{\partial}{\partial\tau}\left[\left(\frac{\partial z^2}{\partial\tau}\right)^{-1} z_\nu(\tau)\,\dot{y}_\mu(\tau)\right].
\end{aligned} \tag{3.104}$$

Now

$$\frac{\partial}{\partial\tau}\left[\left(\frac{\partial z^2}{\partial\tau}\right)^{-1} z_\nu\,\dot{y}_\mu\right] = \left(\frac{\partial z^2}{\partial\tau}\right)^{-1}\left(-\dot{y}_\nu\,\dot{y}_\mu + z_\nu\,\ddot{y}_\mu\right) - \left(\frac{\partial z^2}{\partial\tau}\right)^{-2}\frac{\partial^2 z^2}{\partial\tau^2}z_\nu\,\dot{y}_\mu, \tag{3.105}$$

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but we are not interested in anything symmetric in $\nu \leftrightarrow \mu$, because these drop out when we calculate $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. So ignore the $\dot{y}_\nu \dot{y}_\mu$ term. From Eq. 3.99,

$$\frac{\partial z^2}{\partial \tau} = -2\, z_\nu\, \dot{y}^\nu \quad \Rightarrow \quad \frac{\partial^2 z^2}{\partial \tau^2} = 2\dot{y}_\nu\, \dot{y}^\nu - 2\, z_\nu\, \ddot{y}^\nu = 2\,(1 - z_\nu\, \ddot{y}^\nu)\,. \tag{3.106}$$

Except for the explicitly symmetric term, then,

$$\partial_\nu A_\mu = -\frac{e}{\pi} \int d\tau\, \delta(z^2) \left[\frac{1}{(-2z_\lambda \dot{y}^\lambda)} z_\nu\, \ddot{y}_\mu - \frac{1}{(-2z_\lambda \dot{y}^\lambda)^2} 2(1 - z_\lambda\, \ddot{y}^\lambda) z_\nu\, \dot{y}_\mu \right]. \tag{3.107}$$

But (3.100)

$$\delta(z^2) = \frac{1}{2z_\lambda \dot{y}^\lambda}\, \delta(\tau), \tag{3.108}$$

so

$$\begin{aligned} \partial_\nu A_\mu &= \frac{e}{4\pi} \int d\tau\, \delta(\tau) \left[\frac{1}{(z_\lambda \dot{y}^\lambda)^2} z_\nu\, \ddot{y}_\mu + \frac{1}{(z_\lambda \dot{y}^\lambda)^3} (1 - z_\lambda\, \ddot{y}^\lambda) z_\nu\, \dot{y}_\mu \right] \\ &= \frac{e}{4\pi} \left[\frac{1}{(z_\lambda \dot{y}^\lambda)^2} z_\nu\, \ddot{y}_\mu + \frac{1}{(z_\lambda \dot{y}^\lambda)^3} (1 - z_\lambda\, \ddot{y}^\lambda) z_\nu\, \dot{y}_\mu \right], \end{aligned} \tag{3.109}$$

with everything evaluated at $\tau = 0$ (which is to say, at the **retarded time** when the "message" left: $z^2 = 0,\ z^0 > 0$). Finally,

$$\boxed{F_{\mu\nu} = \frac{e}{4\pi} \left[\frac{1}{(z_\lambda \dot{y}^\lambda)^2} z_\mu\, \ddot{y}_\nu + \frac{1}{(z_\lambda \dot{y}^\lambda)^3} (1 - z_\lambda\, \ddot{y}^\lambda) z_\mu\, \dot{y}_\nu \right] - (\mu \leftrightarrow \nu).} \tag{3.110}$$

This is the electromagnetic field produced by a point charge in arbitrary motion; it includes a part that goes like $1/\mathscr{r}^2$ (the second term), known as the **induction field**,[18] and others that go like $1/\mathscr{r}$, (the first and third), the **radiation field**.[19] Note that if $\ddot{y}_\mu = 0$ (no acceleration), the radiation field vanishes.

Problem 3.5

Use Eq. 3.110 to find **E** and **B** for a point charge at rest.

Incidentally, we could use Eq. 3.91 to determine the A^{out} that corresponds to $A^{\text{in}} = 0$:

$$A_\mu = \int D_R\, J_\mu = A^{\text{out}}_\mu + \int D_A\, J_\mu \quad \Rightarrow \quad A^{\text{out}}_\mu = \int [D_R - D_A]\, J_\mu, \tag{3.111}$$

[18] Eds. Coleman's word is unusual; more common is the **generalized Coulomb field** or **velocity field**.
[19] Eds. The radiation field is also called the **acceleration field**.

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and interpret this as the "radiation" emitted. Provided the particle acceleration is limited (so it doesn't radiate an infinite amount), the total energy and momentum carried off by the asymptotic field can be calculated[20] from A^{out}. The result is

$$p_\mu^{\text{radiated}} = -\frac{e^2}{6\pi} \int_{-\infty}^{\infty} d\tau\, (\ddot{y}_\lambda\, \ddot{y}^\lambda)\, \dot{y}_\mu. \tag{3.112}$$

It is a miracle that this can be expressed as a single integral; if the photon had mass, it would be a *double* integral.

3.4 Regularization and Renormalization

In principle, our job is done: just plug in the appropriate initial conditions, and solve for the motion of the electron and the fields it generates. But there's a hitch: How does a point charge move under the influence of its *own* field? If you insert $F^{\mu\nu}$ (3.110) into the equation of motion (3.51),

$$m_0 \ddot{y}_\mu = e F_{\mu\nu}\, \dot{y}^\nu, \tag{3.113}$$

you get infinity, because $F_{\mu\nu}$ blows up at the location of the charge.[21] To get around this difficulty we invoke a procedure known as **mass renormalization**.

First we **regularize** the theory: modify it so as to eliminate the infinities. The source of the problem is the interaction Lagrangian (Eq. 3.43):

$$I' = -e \int d\tau \int d^4x\, A_\mu(x)\, \delta^4\, (x - y(\tau))\;\; \dot{y}^\mu(\tau). \tag{3.114}$$

The delta function (representing the point charge) is the culprit. We want to *smear out* the source, but in a Lorentz-invariant way, so the theory remains consistent with relativity (it wouldn't do to make it a 3-dimensional spherical shell, for example). In place of $\delta^4(x - y)$ we'll use $f(x - y)$, where $f \to \delta$ in some suitable limit. Specifically, we start with any old function $F(x)$ such that

$$\int d^4x\, F(x) = 1, \tag{3.115}$$

and let

$$f(x) \equiv \lambda^4\, F(\lambda x). \tag{3.116}$$

Then as $\lambda \to \infty$, $f \to \delta$.

[20] Eds. This is done in Coleman's treatise, Section 4 (details in footnote 10).
[21] The same thing occurs in quantum electrodynamics, but there the divergence is softer: logarithmic, instead of $1/\mathscr{r}$ (or $1/\mathscr{r}^2$).

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What's the effect of replacing δ by f? It is still true (of course) that $\Box^2 A_\mu = J_\mu$ (that's just Maxwell's equations in the Lorenz gauge, and we're not touching them—only smearing out the source). But J^μ is no longer given by Eq. 3.44,

$$J^\mu = e \int d\tau \, \delta^4(x - y) \, \dot{y}^\mu, \tag{3.117}$$

but rather by

$$\bar{J}^\mu = e \int d\tau \, f(x - y) \, \dot{y}^\mu \tag{3.118}$$

(I'll use an overbar to denote regularized quantities). Thus

$$\begin{aligned} \bar{J}^\mu &= e \int \int d\tau \, d^4x' \, f(x - x') \, \delta^4(x' - y) \, \dot{y}^\mu(\tau) \\ &= \int d^4x' \, f(x - x') \, J^\mu(x') \equiv f * J^\mu \end{aligned} \tag{3.119}$$

(the **convolution integral**). This modified current generates a field $\bar{F}_{\mu\nu}$ given by

$$\partial^\mu \bar{F}_{\mu\nu} = \bar{J}_\nu. \tag{3.120}$$

How is it related to the old (*un*regularized) field? We need to solve Eq. 3.120, given that $\bar{J}^\mu = f * J^\mu$ and $\partial^\mu F_{\mu\nu} = J_\nu$: What is $\bar{F}_{\mu\nu}$, in terms of $F_{\mu\nu}$? I claim that

$$\bar{F}_{\mu\nu} = f * F_{\mu\nu}. \tag{3.121}$$

Proof: If this is right then

$$\bar{F}_{\mu\nu} = \int d^4x' \, f(x - x') \, F_{\mu\nu}(x'), \tag{3.122}$$

so

$$\begin{aligned} \partial^\mu \bar{F}_{\mu\nu}(x) &= \int d^4x' \left[\partial^\mu_{(x)} f(x - x')\right] F_{\mu\nu}(x') \\ &= \int d^4x' \left[-\partial^\mu_{(x')} f(x - x')\right] F_{\mu\nu}(x') \\ &= \int d^4x' \, f(x - x') \left[\partial^\mu F_{\mu\nu}(x')\right] = \int d^4x' \, f(x - x') \, J_\nu(x') \\ &= \bar{J}_\nu(x). \quad \text{QED} \end{aligned} \tag{3.123}$$

The idea now is to plug 3.110 into 3.122, and expand in inverse powers of λ (the regularization parameter—remember, we want the limit as $\lambda \to \infty$), for x in the

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immediate vicinity of the particle. I shall not go through the details,[22] but merely quote the answer:

$$\begin{aligned}\bar{F}_{\mu\nu} &= A\lambda + B + C\frac{1}{\lambda} + D\frac{1}{\lambda^2} + \cdots \\ &= \frac{e}{8\pi}(\dot{y}_\mu \ddot{y}_\nu - \dot{y}_\nu \ddot{y}_\mu)\lambda - \frac{2}{3}\frac{e}{4\pi}(\dot{y}_\mu \dddot{y}_\nu - \dot{y}_\nu \dddot{y}_\mu) + \mathcal{O}\left(\frac{1}{\lambda^2}\right).\end{aligned} \tag{3.124}$$

The electron's equation of motion (3.113) is

$$m_0 \ddot{y}_\mu = e\bar{F}_{\mu\nu}\dot{y}^\nu + F_\mu^{\text{ext}}. \tag{3.125}$$

The first term (on the right) represents the **self-force** on the charge, due to its *own* electromagnetic field, and F_μ^{ext} stands for any external forces that may be acting on it. Putting in Eq. 3.124, and recalling that $\dot{y}_\nu \dot{y}^\nu = 1$ (1.70) and $\dot{y}_\nu \ddot{y}^\nu = 0$ (1.80),

$$m_0 \ddot{y}_\mu = -\frac{e^2\lambda}{8\pi}\ddot{y}_\mu + \frac{2}{3}\frac{e^2}{4\pi}(\dddot{y}_\mu - \dot{y}_\mu \dddot{y}_\nu \dot{y}^\nu) + F_\mu^{\text{ext}}. \tag{3.126}$$

(Because $\dot{y}_\nu \ddot{y}^\nu = 0$, $\ddot{y}_\nu \ddot{y}^\nu = -\dot{y}_\nu \dddot{y}^\nu$, so the second term in parentheses can also be written as $+\dot{y}_\mu \ddot{y}_\nu \ddot{y}^\nu$.)

Of course, the self-force still blows up (as $\lambda \to \infty$), but the infinity has been isolated in a term that multiplies $\ddot{y}_\mu$ and can be combined with the (bare) mass m_0 to define the **renormalized mass**:

$$m \equiv m_0 + \frac{e^2\lambda}{8\pi}. \tag{3.127}$$

Thus

$$\boxed{m\ddot{y}_\mu = \frac{2}{3}\frac{e^2}{4\pi}(\dddot{y}_\mu + \dot{y}_\mu \ddot{y}_\nu \ddot{y}^\nu) + F_\mu^{\text{ext}},} \tag{3.128}$$

and we identify m as the true **physical mass** of the particle. It is awkward that the electromagnetic contribution is infinite, but—who knows?—maybe the bare mass is *minus* infinity (it is not, after all, a measurable quantity, since we cannot strip the electron of its charge). In any case, the remainder of the self-force (the so-called **radiation reaction**) is perfectly finite:

$$F_\mu^{\text{rad}} = \frac{2}{3}\frac{e^2}{4\pi}(\dddot{y}_\mu + \dot{y}_\mu \ddot{y}_\nu \ddot{y}^\nu). \tag{3.129}$$

This formula was first obtained by Dirac, though the nonrelativistic version goes back to Abraham and Lorentz, at the turn of the 20th century.[23]

[22] Eds. In Coleman (footnote 10) he calls the calculation "tedious and uninstructive," and relegates it to an appendix.

[23] P. A. M. Dirac, *Proc. Roy. Soc. (London)* **A167**, 148 (1938); H. A. Lorentz, *The Theory of Electrons*, Dover, New York (1952—based on his 1906 lectures at Columbia University).

Feynman–Wheeler electrodynamics revisited: Maxwell's electrodynamics is not time-reversal invariant, because we stipulate that the "in" fields are zero. Restoring them for a moment (and setting aside the external forces), the equation of motion (3.128) becomes

$$m\ddot{y}_\mu = eF^{\text{in}}_{\mu\nu}\,\dot{y}^\nu + \frac{2}{3}\frac{e^2}{4\pi}(\dddot{y}_\mu + \dot{y}_\mu\ddot{y}_\nu\ddot{y}^\nu). \tag{3.130}$$

Because the advanced formulation is the time-reversed version of the retarded formulation, and time reversal switches the sign of the term in parentheses (it's odd in τ), we could as well write

$$m\ddot{y}_\mu = eF^{\text{out}}_{\mu\nu}\,\dot{y}^\nu - \frac{2}{3}\frac{e^2}{4\pi}(\dddot{y}_\mu + \dot{y}_\mu\ddot{y}_\nu\ddot{y}^\nu). \tag{3.131}$$

Averaging these two expressions,

$$m\ddot{y}_\mu = \frac{e}{2}\left(F^{\text{out}}_{\mu\nu} + F^{\text{in}}_{\mu\nu}\right)\dot{y}^\nu. \tag{3.132}$$

But the sum of F^{in} and F^{out} is zero, in the Feynman–Wheeler formulation, and the self-force term cancels out entirely! (A radiating charge does slow down, in the Feynman–Wheeler approach, but that is due to the backward radiation from the absorber, not the influence of the particle's own field.)

Indeed, you can construct an action in which the fields never appear:

$$I = \sum_n I_0^{(n)} + \sum_{n>m}\int d\tau^{(n)}\,d\tau^{(m)}\,\dot{y}^{(n)}_\mu\,\left(\dot{y}^{(n)}\right)^\mu \bar{D}(y^{(n)} - y^{(m)}), \tag{3.133}$$

where the parenthetical superscripts denote different interacting particles,

$$\bar{D} \equiv \frac{D_A + D_R}{2}, \tag{3.134}$$

and the free particle action is

$$I_0^{(n)} = \frac{m^{(n)}}{2}\int d\tau^{(n)}\,\dot{y}^{(n)}_\mu\,\left(\dot{y}^{(n)}\right)^\mu. \tag{3.135}$$

This is a retarded action-at-a-distance theory; the charges interact directly, without mediating fields, and with no need for mass renormalization (the only mass that ever appears is the physical mass). Feynman and Wheeler hoped to quantize this system, avoiding the infinities that plague quantum electrodynamics by removing them already at the classical level. Unfortunately, no one has succeeded in quantizing Feynman–Wheeler electrodynamics.

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3.4.1 Particle Motion with Radiation Reaction

In the nonrelativistic régime ($|d\mathbf{y}/dt| \ll 1$), the Abraham–Lorentz–Dirac equation (3.128) reduces to

$$m\frac{d^2\mathbf{y}}{dt^2} = \frac{2}{3}\frac{e^2}{4\pi}\frac{d^3\mathbf{y}}{dt^3} + \mathbf{F}^{\text{ext}}. \tag{3.136}$$

Suppose there is a harmonic binding force, $\mathbf{F}^{\text{ext}} = -k\mathbf{y}$:

$$m\frac{d^2\mathbf{y}}{dt^2} = \frac{2}{3}\frac{e^2}{4\pi}\frac{d^3\mathbf{y}}{dt^3} - k\mathbf{y}. \tag{3.137}$$

We'll look for oscillatory solutions:

$$\mathbf{y}(t) = e^{i\omega t}\mathbf{a}. \tag{3.138}$$

Putting this in,

$$m\omega^2 = \frac{2}{3}\frac{e^2}{4\pi}i\omega^3 + k. \tag{3.139}$$

Define

$$\omega_0 \equiv \sqrt{\frac{k}{m}}, \qquad \lambda \equiv \frac{2}{3}\frac{e^2}{4\pi m} \tag{3.140}$$

(not to be confused with the λ in (3.116)!); then

$$\omega^2 = i\lambda\omega^3 + \omega_0^2. \tag{3.141}$$

In the *absence* of any radiation reaction ($\lambda = 0$), $\omega = \pm\omega_0$. If λ is *small*, we can treat it as a perturbation:

$$\omega = \pm\omega_0 + \delta\omega, \tag{3.142}$$

and (dropping terms quadratic and higher in $\delta\omega$)

$$\omega^2 = \omega_0^2 \pm 2\omega_0\,\delta\omega, \qquad \omega^3 = \pm\omega_0^3 + 3\omega_0^2\,\delta\omega. \tag{3.143}$$

So Eq. 3.141 becomes

$$\omega_0^2 \pm 2\omega_0\,\delta\omega = i\lambda(\pm\omega_0^3 + 3\omega_0^2\,\delta\omega) + \omega_0^2, \tag{3.144}$$

or

$$\delta\omega = \frac{i\lambda\omega_0^2}{2 \mp 3i\omega_0\lambda}. \tag{3.145}$$

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But we are taking λ to be infinitesimal, so these two solutions coincide (to lowest order):

$$\delta\omega = \frac{i\lambda\omega_0^2}{2}, \tag{3.146}$$

and the motion becomes

$$\mathbf{y}(t) = e^{i(\pm\omega_0+\delta\omega)t}\mathbf{a} = e^{-\lambda\omega_0^2 t/2}e^{\pm i\omega_0 t}\mathbf{a}. \tag{3.147}$$

Not surprisingly, the emission of radiation damps the oscillations (in this context the radiation reaction is known as **radiation damping**).

But 3.141 is a *cubic* equation: What about the *third* root? This one is *not* approximately ω_0 for small λ, so we're not going to get it as a perturbation, but inspection of 3.141 suggests

$$\omega = \frac{1}{i\lambda}. \tag{3.148}$$

It comes in from infinity (as λ increases from zero), so for small λ the ω_0^2 term is negligible. For this root the motion is

$$\mathbf{y}(t) = e^{i\omega t}\mathbf{a} = e^{t/\lambda}\mathbf{a}. \tag{3.149}$$

This is a catastrophe—an *in*creasing exponential, with no oscillations at all. It is known as a **runaway mode**. In spite of appearances, it doesn't actually violate conservation of energy, as we'll see in a moment. But it *is* a huge embarrassment for classical electron theory.

Even the *free* equation of motion admits runaway modes: the general solution to

$$m\frac{d^2\mathbf{y}}{dt^2} = \frac{2}{3}\frac{e^2}{4\pi}\frac{d^3\mathbf{y}}{dt^3} \tag{3.150}$$

is

$$\mathbf{y}(t) = \mathbf{a} + \mathbf{b}t + \mathbf{c}e^{t/\lambda} \tag{3.151}$$

(for constants **a**, **b**, and **c**), as you can check for yourself.

Are runaways perhaps an artifact of the nonrelativistic approximation? Unfortunately, they are *not*. For if we start with the relativistic equation 3.128,

$$m\ddot{y}_\mu = \frac{2}{3}\frac{e^2}{4\pi}\left(\dddot{y}_\mu + \dot{y}_\mu\,\ddot{y}_\nu\,\ddot{y}^\nu\right), \tag{3.152}$$

and multiply in $\ddot{y}^\mu$ (remembering that $\dot{y}_\mu\ddot{y}^\mu = 0$), we obtain

$$m\ddot{y}^\mu\ddot{y}_\mu = \frac{2}{3}\frac{e^2}{4\pi}\ddot{y}^\mu\dddot{y}_\mu. \tag{3.153}$$

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Define the (real[24]) scalar function

$$f(\tau) \equiv \sqrt{-\ddot{y}_\mu \ddot{y}^\mu}. \tag{3.154}$$

Note that

$$-\frac{d}{d\tau} f^2 = -2 f \dot{f} = 2\ddot{y}_\mu \dddot{y}^\mu, \tag{3.155}$$

so 3.153 says

$$-m\, f^2 = \frac{2}{3}\frac{e^2}{4\pi}(-f\dot{f}) \quad \Rightarrow \quad f = \lambda \frac{df}{d\tau}, \tag{3.156}$$

and therefore

$$f(\tau) = e^{\tau/\lambda} f(0). \tag{3.157}$$

Again, exponential runaway growth (with no external force acting).

How big is λ, for an actual electron? From the definition 3.140 (with three factors of c to give it units of time),

$$\lambda = \frac{e^2}{6\pi m c^3} = \frac{(4.80 \times 10^{-10}\,\text{esu})^2}{(6\pi)(9.11 \times 10^{-28}\,\text{gm})(3.00 \times 10^{10}\,\text{cm/s})^3} \approx 5 \times 10^{-25}\,\text{s}. \tag{3.158}$$

Evidently the runaway is extremely fast. This *cannot* be right. Is it perhaps a premonition of quantum mechanics—classical electrodynamics warning us that it cannot be trusted in the realm of the very small?[25]

Why not use a boundary condition to kill the runaway? After all, 3.136 is a *third*-order differential equation, and there will still be two constants left over, to fit the initial position and velocity.[26] This works for a *free* charge (simply stipulate that $\mathbf{c} = \mathbf{0}$ in the general solution 3.151), but if there is an external force acting, the cure can be worse than the disease. Suppose, for example, that the force is a sharp kick at $t = 0$. The equation of motion 3.136 becomes

$$m\frac{d^2\mathbf{y}}{dt^2} = m\lambda\,\frac{d^3\mathbf{y}}{dt^3} + \mathbf{F}_0\,\delta(t). \tag{3.159}$$

[24] In the instantaneous rest frame of the particle, $\ddot{y}^\mu = (0, d^2\mathbf{y}/dt^2)$, so $\ddot{y}^\mu$ is space-like.

[25] Lorentz modeled the electron as a spherical shell of charge, calculating the net self-force due to different parts acting on one another. In the point limit this reproduces the (nonrelativistic) radiation reaction force (3.136). Lorentz could blame the runaways on the model, but we don't have that option. Eds. Interestingly, for sufficiently large spheres the Lorentz model is free of runaways. See, for example, E. Moniz and D. Sharp, *Phys. Rev. D* **15**, 2850 (1977).

[26] This approach was explored by Dirac (see footnote 23).

When $t \neq 0$ the force is zero, so (letting $\mathbf{v} \equiv d\mathbf{y}/dt$)

$$\frac{d\mathbf{v}}{dt} = \lambda \frac{d^2\mathbf{v}}{dt^2} \quad \Rightarrow \quad \frac{d\mathbf{v}}{dt} = \mathbf{a}\, e^{t/\lambda} \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{a}\,\lambda\, e^{t/\lambda} + \mathbf{b}. \tag{3.160}$$

This holds both *before* the force acts ($t < 0$) and again *afterward* ($t > 0$)—but with different constants **a** and **b**, of course. We might as well assume the particle starts from rest in the distant past, so $\mathbf{b} = \mathbf{0}$ when $t < 0$. And presumably the acceleration, $d\mathbf{v}/dt$, is zero prior to the intervention of **F**, so $\mathbf{a} = \mathbf{0}$. Then

$$\mathbf{v}(t) = \mathbf{0} \quad (t < 0). \tag{3.161}$$

To determine the constants **a** and **b** *after* the force acts, integrate 3.159 across the delta function:

$$\int_{-\epsilon}^{\epsilon} dt\, \frac{d\mathbf{v}}{dt} = \lambda \int_{-\epsilon}^{\epsilon} dt\, \frac{d^2\mathbf{v}}{dt^2} + \frac{\mathbf{F}_0}{m} \int_{-\epsilon}^{\epsilon} dt\, \delta(t), \tag{3.162}$$

or

$$\mathbf{v}(\epsilon) - \mathbf{v}(-\epsilon) = \lambda \left(\left.\frac{d\mathbf{v}}{dt}\right|_{\epsilon} - \left.\frac{d\mathbf{v}}{dt}\right|_{-\epsilon} \right) + \frac{\mathbf{F}_0}{m}. \tag{3.163}$$

Thus

$$\mathbf{v}(\epsilon) = \lambda \left.\frac{d\mathbf{v}}{dt}\right|_{\epsilon} + \frac{\mathbf{F}_0}{m}. \tag{3.164}$$

Putting in the general solution (3.160), and taking the limit $\epsilon \to 0$,

$$\mathbf{a}\lambda + \mathbf{b} = \lambda\mathbf{a} + \frac{\mathbf{F}_0}{m}, \quad \text{so} \quad \mathbf{b} = \frac{\mathbf{F}_0}{m}, \tag{3.165}$$

and hence

$$\mathbf{v}(t) = \mathbf{a}\lambda e^{t/\lambda} + \frac{\mathbf{F}_0}{m}. \tag{3.166}$$

Assuming $\mathbf{v}(t)$ is continuous[27] at $t = 0$, it follows that

$$\mathbf{v}(t) = \left(1 - e^{t/\lambda}\right) \frac{\mathbf{F}_0}{m} \quad (t > 0). \tag{3.167}$$

This is the solution we get by assuming that the velocity and the acceleration are zero prior to the intervention of the force; such boundary conditions inevitably excite the runaway.

[27] Eds. You can confirm this by treating the delta function as a limit of triangles or rectangles, say. If $v(t)$ had a discontinuity ($\theta(t)$), then dv/dt would carry a delta function, and d^2v/dt^2 a δ', but there is no compensating δ' in 3.159.

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To eliminate the runaway, we need $\mathbf{a} = \mathbf{0}$ for $t > 0$. To get the resulting solution we could either run the same argument backward, or (more simply) add a solution $e^{t/\lambda}\,(\mathbf{F}_0/m)$ of the homogeneous equation. Then

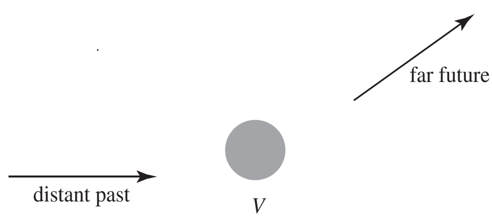
$$\mathbf{v}(t) = \begin{cases} e^{t/\lambda}\,\mathbf{F}_0/m & (t < 0), \\ \mathbf{F}_0/m & (t > 0). \end{cases} \tag{3.168}$$

That kills the runaway, but now the particle starts to move *before the force acts*! This **acausal preacceleration** only jumps the gun very briefly, but still, it is clearly unacceptable.

Question: Is it *always* possible to eliminate the runaways in this way (or does it work only for the delta-function force)? The answer is yes, as long as the equations of motion are linear. It is even true in the *non*linear case if the potential V is sufficiently bounded. The general solution far from the scattering center takes the form (3.151)

$$\mathbf{y}(t) = \begin{cases} \mathbf{a} + \mathbf{b}\,t + \mathbf{c}\,e^{t/\lambda}, & t \to -\infty, \\ \mathbf{a}' + \mathbf{b}' t + \mathbf{c}'\,e^{t/\lambda}, & t \to \infty. \end{cases} \tag{3.169}$$

You can think of the scattering problem as a mapping from $\mathbf{a}, \mathbf{b}, \mathbf{c}$ to $\mathbf{a}', \mathbf{b}', \mathbf{c}'$.



It can be proved that if $|\mathbf{c}| \gg 1$, then $\mathbf{c} \approx \mathbf{c}'$. (The effect of the potential is roughly proportional to the length of time the particle spends in its vicinity, so for high velocities the effect is negligible.) Thus the mapping takes large $\mathbf{c}$ into large $\mathbf{c}'$; it maps large spheres into themselves. According to the **fixed point theorem**, it must map *something* into *zero*, which is to say that there is always a solution with no runaway—you can always choose the preacceleration $\mathbf{c}$ in the past such that there is no runaway in the future. (However, in the case of electron/positron scattering this argument does not hold, since the force is singular at the origin.) Incidentally, one would never know whether there exist runaway modes in *quantum* electrodynamics, because the only method of analysis available is perturbation theory, where they would never appear anyway.

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3.4.2 Conservation of Energy

Runaway modes occur when the bare mass m_0 is negative (see Problem 3.7). The total (nonrelativistic) energy of a charged particle (kinetic energy plus energy in its electromagnetic fields, Eq. 3.22) is

$$\frac{1}{2}m_0 v^2 + \frac{1}{2}\int d^3\mathbf{x}\,(\mathbf{E}^2 + \mathbf{B}^2). \tag{3.170}$$

Acceleration pours (positive definite) energy into the fields, but it also increases the magnitude of the kinetic energy. The total energy is not measurable, but it *is* conserved, even in the runaway modes; we are saved by the fact that $m_0 < 0$: the kinetic term gets more and more negative, while the field term gets more and more positive. But this doesn't tell us much, because it makes explicit reference to the unmeasurable bare mass.

Is there a *useful* energy conservation law for this case—one expressed in terms of the *physical* mass, m? Yes, there is. Going back to the relativistic equation of motion (3.128),

$$\frac{d}{d\tau}(m\,\dot{y}_\mu) = F^{\text{ext}}_\mu + \frac{2}{3}\frac{e^2}{4\pi}\left[\dddot{y}_\mu + \dot{y}_\mu\,\ddot{y}^\nu\,\ddot{y}_\nu\right]. \tag{3.171}$$

The $\dddot{y}$ term tells us the rate at which energy is pumped into the induction field; the $\dot{y}\,\ddot{y}\,\ddot{y}$ term is (minus) the rate at which energy/momentum is radiated (Eq. 3.112). In the nonrelativistic régime, the zeroth component reads

$$\mathbf{v}\cdot\mathbf{F}^{\text{ext}} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) + \frac{2}{3}\frac{e^2}{4\pi}\mathbf{a}^2 - \frac{d}{dt}\left[\frac{2}{3}\frac{e^2}{4\pi}\mathbf{a}\cdot\mathbf{v}\right], \tag{3.172}$$

where $\mathbf{v}$ is the velocity and $\mathbf{a}$ is the acceleration. On the left we have the power delivered to the electron by the external force; it is equal to the rate of increase of the kinetic energy, plus the power radiated (the so-called **Larmor formula**), plus the rate of change of the energy stored in the induction field. This, then, is the conservation of energy expressed in terms of measurable quantities. For periodic motion, or motions that begin and end with $\mathbf{a} = \mathbf{0}$, the last term integrates to zero, and in this average sense the work done (by the external force) is equal to the increase in kinetic energy plus the energy radiated. But, in general, energy is also exchanged between the particle and the local fields it drags around with it.

Problem 3.6

Derive 3.172 from 3.171.

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Problem 3.7

The physical mass of a charged object at rest is the sum of its bare mass and the energy stored in its electrostatic field (over c^2):

$$m = m_0 + U/c^2. \tag{3.173}$$

Consider the case of a uniformly charged spherical shell, of radius R and charge Q.

(a) What is U in this case?

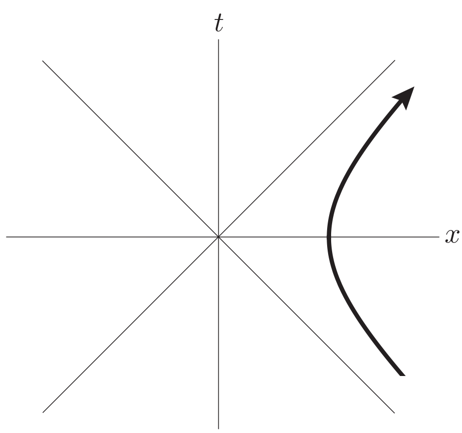
(b) Find m_0, as a function of the measurable quantities m, Q, and R.

(c) Lorentz contemplated a "purely electromagnetic" particle, whose (physical) mass is *entirely* attributable to energy in the fields (i.e. $m_0 = 0$). The so-called **classical radius** of a particle is the radius at which this occurs. Find the classical radius of a spherical shell (mass m and charge Q). What would this be for an electron (in meters)?

Comment: For particles *smaller* than their classical radius (and *a fortiori* for *point* particles) the bare mass runs negative. Runaways and preacceleration do not occur for particles *larger* than their classical radius (Moniz and Sharp; details in footnote 25). Unfortunately, the electron is known to be much smaller than its classical radius.

3.4.3 Hyperbolic Motion

Recall the twin paradox problem (Section 1.4.2): motion under uniform proper acceleration. The world line is hyperbolic:



In hyperbolic motion the radiation reaction vanishes.

Proof: The radiation reaction force (Eq. 3.129, with 3.40) is

$$F_\mu^{\text{rad}} = m\lambda\left[\dddot{y}_\mu + \dot{y}_\mu(\ddot{y}_\nu\,\ddot{y}^\nu)\right]. \tag{3.174}$$

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First of all, note that F_μ^{rad} is always orthogonal to the 4-velocity:

$$\frac{1}{m\lambda}\dot{y}^\mu F_\mu^{\text{rad}} = \dot{y}^\mu \dddot{y}_\mu + \dot{y}^\mu \dot{y}_\mu(\ddot{y}_\nu \ddot{y}^\nu) = \dot{y}^\mu \dddot{y}_\mu + \ddot{y}_\mu \ddot{y}^\mu$$
$$= \frac{d}{d\tau}(\dot{y}^\mu \ddot{y}_\mu) = 0 \qquad (3.175)$$

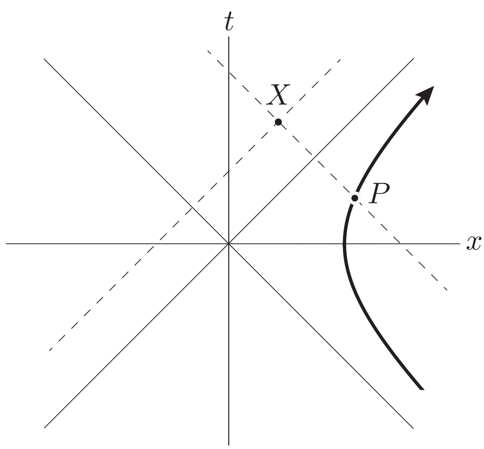
(because $\dot{y}^\mu \ddot{y}_\mu = 0$). In the case of hyperbolic motion it is *also* orthogonal to the acceleration:

$$\frac{1}{m\lambda}\ddot{y}^\mu F_\mu^{\text{rad}} = \ddot{y}^\mu \dddot{y}_\mu + \ddot{y}^\mu \dot{y}_\mu(\ddot{y}_\nu \ddot{y}^\nu) = \ddot{y}^\mu \dddot{y}_\mu = \frac{1}{2}\frac{d}{d\tau}(\ddot{y}^\mu \ddot{y}_\mu). \qquad (3.176)$$

But under *uniform* acceleration $\ddot{y}^\mu \ddot{y}_\mu$ is *constant* (1.82). So (in two dimensions) F_μ^{rad} is orthogonal to two mutually orthogonal vectors, and it must be zero.[28] QED

Does this mean there is no *radiation* from a particle in hyperbolic motion? It does *not* (though Wolfgang Pauli famously drew this erroneous conclusion).[29] According to Eq. 3.111,

$$A_\mu^{\text{out}} = \int [D_R - D_A]\, J_\mu. \qquad (3.177)$$



The radiated field at X is given by the intersection of the particle trajectory with the backward light cone (via D_R) and the intersection with the *forward* light cone (via D_A). But for hyperbolic motion there *is* no intersection with the forward light cone, only with the backward light cone (at P), so

[28] You can check this by the direct "bulldozer" method, of course, since we know the solution to the equations of motion (1.86 and 1.87).

[29] Eds. Part of the problem is simply bad language: it should not be called the "radiation" reaction, but rather the *field* reaction. It is the force exerted by the particle's own fields, acting on the particle itself—not just the fields that ultimately manifest themselves as radiation.

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$$A_\mu^{\text{out}} = \int D_R\, J_\mu \neq 0. \tag{3.178}$$

The charge radiates, but it experiences no radiation reaction. Doesn't this violate conservation of energy? No, for the reason discussed in Section 3.4.2.

In fact, if there *were* a radiation reaction force on a particle in hyperbolic motion, then the **principle of equivalence** would not hold. The principle of equivalence says you cannot distinguish uniform acceleration from a uniform gravitational field (with the particle at rest). But a particle at rest does not radiate. If a particle in hyperbolic motion experienced a radiation reaction you would be able to tell a gravitational field from uniform acceleration. *Objection:* Since a particle in hyperbolic motion *does radiate* (even though there is no accompanying radiation reaction), why not measure that *radiation* to demonstrate violation of the equivalence principle? It turns out that what constitutes "radiation" in one reference frame may *not* be radiation in another: the distinction between radiation fields and induction fields is different for accelerating observers.[30] In particular, according to a *stationary* observer a freely falling charge radiates, but according to an observer who is *also* in free fall it does *not*.

[30] T. Fulton and F. Rohrlich, *Ann. Phys.* **9**, 499 (1960).

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