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MULTIPLICITIES IN THE TENSOR PRODUCT OF FINITE-DIMENSIONAL REPRESENTATIONS OF DISCRETE GROUPS

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Let G be a group and ρ and σ two irreducible unitary representations of G in complex Hilbert spaces and assume that $\dim \rho = n < \infty$. D. Poguntke [2] proved that $\rho \otimes \sigma$ is a sum of at most n^2 irreducible subrepresentations. The case when $\dim \sigma$ is also finite he attributed to R. Howe.

We shall prove analogous results for arbitrary finite-dimensional representations, not necessarily unitary. Thus let F be an algebraically closed field of characteristic 0. We shall use the language of modules and we postulate that *all our modules are finite-dimensional* as F-vector spaces. The field F itself will be considered as a trivial G-module.

If V and W are G-modules then $\mathrm{Hom}_F(V, W)$ is also a G-module. If $f: V \to W$ is a linear map and $a \in G$ then $a \cdot f$ is the linear map $V \to W$ defined by $(a \cdot f)(v) = a \cdot f(a^{-1} \cdot v)$. In particular, when $W = F$ we get a G-module $V^* = \mathrm{Hom}_F(V, F)$.

If V is a G-module then we denote by V^G the submodule of V consisting of G-invariant elements, i.e., elements $v \in V$ such that $a \cdot v = v$ for all $a \in G$. Note that if V and W are G-modules then

$$\mathrm{Hom}_G(V, W) = (\mathrm{Hom}_F(V, W))^G.$$

If V and W are G-modules then we define

$$\langle V, W \rangle = \dim_F \mathrm{Hom}_G(V, W),$$

which is an integer usually called the intertwining number. It is clear that

$$\langle V, W_1 \oplus W_2 \rangle = \langle V, W_1 \rangle + \langle V, W_2 \rangle,$$
$$\langle V_1 \oplus V_2, W \rangle = \langle V_1, W \rangle + \langle V_2, W \rangle.$$

Let V_1, V_2, V_3 be finite-dimensional F-vector spaces. Recall that there exist canonical vector space isomorphisms

$$V_2^* \otimes V_3 \to \mathrm{Hom}_F(V_2, V_3)$$

and

$$\mathrm{Hom}_F(V_1 \otimes V_2, V_3) \to \mathrm{Hom}_F(V_1, \mathrm{Hom}_F(V_2, V_3))$$

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and consequently a canonical isomorphism

$$\text{(1)} \qquad \mathrm{Hom}_F(V_1 \otimes V_2, V_3) \to \mathrm{Hom}_F(V_1, V_2^* \otimes V_3).$$

If V_1, V_2, V_3 are all G-modules then these canonical maps are also G-homomorphisms. Hence, it follows that the isomorphism (1) induces a vector space isomorphism

$$\mathrm{Hom}_G(V_1 \otimes V_2, V_3) \to \mathrm{Hom}_G(V_1, V_2^* \otimes V_3).$$

Therefore we have

$$\text{(2)} \qquad \langle V_1 \otimes V_2, V_3 \rangle = \langle V_1, V_2^* \otimes V_3 \rangle.$$

Taking $V_1 = F$ and writing $V_2 = V$, $V_3 = W$, we obtain

$$\text{(3)} \qquad \langle V, W \rangle = \dim_F (V^* \otimes W)^G.$$

If V is a simple G-module and W a semi-simple G-module then $\langle V, W \rangle$ is the multiplicity of V in W. This follows from the fact that $\mathrm{End}_G(V) = F$, F being algebraically closed. If both V and W are semi-simple G-modules then clearly $\langle V, W \rangle = \langle W, V \rangle$.

We recall that the tensor product of semi-simple G-modules is also semi-simple because char $F = 0$, see [1, p. 85].

If V is a G-module and n an integer ≥ 0 then nV denotes direct sum of n copies of V.

THEOREM 1. *Let U, V, W be simple G-modules and m, n, k their dimensions, respectively. Assume that $m \leq n \leq k$. Then*

$$\dim_F (U \otimes V \otimes W)^G \leq m$$

and equality holds if and only if $n = k$ and $U \otimes V \cong mW^*$

Proof. Recall that the dual of a simple G-module is also simple. Using (2) and (3) we obtain

$$\begin{aligned} \dim_F (U \otimes V \otimes W)^G &= \langle F, U \otimes V \otimes W \rangle \\ &= \langle W^*, U \otimes V \rangle \\ &\leq \frac{mn}{k} \leq m. \end{aligned}$$

The assertion about the equality sign is now obvious.

THEOREM 2. *Let U, V, W be simple G-modules and m, n, k their dimensions, respectively. Then the multiplicity of U in $V \otimes W$ is less than or equal* $\min(m, n, k)$.

Proof. Using (2) we obtain

$$\begin{aligned} \langle U, V \otimes W \rangle &= \langle F, U^* \otimes V \otimes W \rangle \\ &= \dim_F (U^* \otimes V \otimes W)^G \end{aligned}$$

and we can apply Theorem 1 to get the assertion.

THEOREM 3. *Let V and W be simple G-modules and $n = \dim V$. Then*

$$\dim_F \mathrm{End}_G(V \otimes W) \leq n^2.$$

In particular, the length of the G-module $V \otimes W$ is $\leq n^2$.

Proof. Since $V \otimes V^*$ is semi-simple [1, p. 85] we have a direct decomposition $V \otimes V^* = V_1 \oplus \cdots \oplus V_k$ where V_i are simple G-modules. Using (2), (3), and Theorem 2, we obtain

$$\begin{aligned}\dim_F \mathrm{End}_G(V \otimes W) &= \dim_F \mathrm{Hom}_G(F, V \otimes V^* \otimes W \otimes W^*) \\ &= \sum_{i=1}^{k} \dim_F (V_i \otimes W \otimes W^*)^G \\ &\leq \sum_{i=1}^{k} \dim_F (V_i) = n^2.\end{aligned}$$

The second assertion follows from the first.

The bound n^2 is best possible as shown by an example in [2].

REFERENCES

1. G. Hochschild: *Introduction to affine algebraic groups*, Holden-Day, San Francisco 1971.
2. D. Poguntke: *Decomposition of tensor products of irreducible unitary representations*, Proc. Amer. Math-Soc. **52** (1975), 427–432.

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