THE NON-EXISTENCE OF CERTAIN AFFINE RESOLVABLE BALANCED INCOMPLETE BLOCK DESIGNS

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1. Summary. A method of proving the impossibility of certain Affine Resolvable Balanced Incomplete Block Designs (A.R.B.I.B.D.) has been given by the author elsewhere [**9**]. More complete results in the same direction are obtained here using the ideas of a paper by Connor [**4**].

2. Preliminary results. A Balanced Incomplete Block Design (B.I.B.D.) with parameters v, b, r, k, and λ is said to be *affine resolvable* if the b blocks can be separated into r sets, each forming a complete replication such that any two blocks of different sets have the same number of treatments in common. It has been shown [**1**] that the parameters of such a design can be expressed in terms of two integers n and t ($n \geqslant 2$, $t \geqslant 0$) in the following manner:

2.00 $$v = nk = n^2[(n-1)t+1], \quad b = nr = n(n^2t+n+1), \quad \lambda = nt+1.$$

Further any two blocks of the same set have no treatment in common, whereas those from different sets have exactly

$$\frac{k^2}{v} = (n-1)t+1$$

treatments in common.

Let A be a symmetric matrix of order m with elements in the rational field. Then A is said to be rationally equivalent to B, $A \sim B$ if and only if there exists a non-singular matrix P with elements in the same field such that $B = P'AP$, where P' is the transpose of P. The equivalence of matrices satisfies the requirements of an "equals" relationship.

Consider the Hasse invariant

2.01 $$c_p(A) = (-1, -D_m)_p \prod_{j=1}^{m-1} (D_j, -D_{j+1})_p$$

where p is a prime, D_j is the leading principal minor determinant of order j in A and $(a, b)_p$ is Pall's [**6**] generalization of the Hilbert norm residue symbol. Let i = index of A, and d = the square free part of A. Then we have

THEOREM A. *Let A and B be two non-singular matrices of order m with elements in the rational field. Then $A \sim B$, if and only if A and B have the same values for the invariants i, d, and c_p for every prime p.*

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The following useful properties of the Hilbert norm residue symbol are quoted from [3] for the sake of completeness. They have been used in the following section.

THEOREM B. *If m and m′ are integers not divisible by an odd prime p, then*

$$(m, m')_p = 1,$$

$$(m, p)_p = (p, m)_p = (m/p),$$

where (m/p) *is the Legendre symbol. Moreover, if* $m \equiv m' \not\equiv 0 \pmod{p}$, *then*

$$(m, p)_p = (m', p)_p.$$

THEOREM C. *For arbitrary non-zero integers m, m′, n, n′ and for every prime p,*

$$(-m, m)_p = 1,$$

$$(m, n)_p = (n, m)_p,$$

$$(mm', n)_p = (m, n)_p(m', n)_p.$$

Further for p an odd prime and every positive integer m,

$$(m, m+1)_p = (-1, m+1)_p.$$

The results in the remaining part of this section are due to Connor [4]. Let N be the incidence matrix of v rows and b volumns, i.e., the elements n_{ju} in row j and column u is 1 or 0 according as treatment j does or does not occur in block u. Let the matrix N be augmented to the matrix N_1 of $v + l$ rows and b columns where

2.02
$$N_1 = \left(\begin{array}{c|c} \multicolumn{2}{c}{N} \\ \hline I_l & 0 \end{array}\right)$$

and I_l is the identity matrix of order l and 0 is a matrix with all elements zero. Then

2.03
$$N_1N_1' = \begin{pmatrix} NN' & N_l \\ N_l' & I_l \end{pmatrix},$$

where N_l is the submatrix of the first l columns of N. Obviously

2.04
$$NN' = \begin{pmatrix} r & \lambda & \ldots & \lambda \\ \lambda & r & \ldots & \lambda \\ & & \ldots & \\ \lambda & \lambda & \ldots & r \end{pmatrix},$$

and hence

2.05
$$|NN'| = kr(r-\lambda)^{v-1}.$$

From 2.03 is it easy to show that

2.06 $$|N_1N_1'| = kr^{-l+1}(r - \lambda)^{v-l-1}|C_l|,$$

where

2.07 $$c_{jj} = (r - k)(r - \lambda)$$

and

2.08 $$c_{ju} = \lambda k - rd_{ju},$$

for $j \neq u = 1, 2, \ldots, l$; d_{ju} is the number of treatments common to blocks j and u.

Let N_2 be the square matrix of side b:

2.09 $$N_2 = \left(\begin{array}{c|c} \multicolumn{2}{c}{N} \\ \hline I_{b-v} & 0 \end{array}\right).$$

Then

2.10 $$N_2N_2' = \begin{pmatrix} NN' & N_{b-v} \\ N'_{b-v} & I_{b-v} \end{pmatrix},$$

where N_{b-v} is the submatrix of the first b-v columns of N. The principal minor determinants of N_2N_2' of order up to v are the same as those of NN' and those of higher orders can be calculated from 2.06.

Let P be matrix

2.11 $$P = \begin{pmatrix} NN' & 0 \\ 0 & C_{b-v}E_{b-v} \end{pmatrix},$$

where

2.12 $$E_{b-v} = [r(r - \lambda)]^{-1}I_{b-v}.$$

Then it is easily verified that the corresponding principal minor determinants of N_2N_2' and P are equal. Hence

$$c_p(P) = c_p(N_2N_2').$$

But we know from [**5**] that

$$c_p(P) = c_p(NN')c_p(C_{b-v}E_{b-v})(|NN'|, |C_{b-v}E_{b-v}|)_p$$

for every odd prime p. Hence we have for any odd prime p,

2.13 $$c_p(N_2N_2') = c_p(NN')c_p(C_{b-v}E_{b-v})(|NN'|, |C_{b-v}E_{b-v}|)_p\,.$$

The value of $c_p(NN')$ can be calculated as in [**3**] and is given by

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2.14 $c_p(NN') = (-1, rk)_p(-1, r-\lambda)_p^{\frac{1}{2}v(v-1)}(r-\lambda, rk)_p^{v-1}(v, rk)_p(v, r-\lambda)_p.$

3. Impossibility of some A.R.B.I.B. designs.

THEOREM 1. *An* A.R.B.I.B.D. *with parameters* 2.00 *does not exist when n and t are odd and*

(i) $n[(n-1)t+1]$ *is not a perfect square, or*

(ii) $n[(n-1)t+1]$ *is a perfect square and* $nt \equiv 1 \pmod 4$ *and the square-free part of n contains a prime* $\equiv 3 \pmod 4$.

THEOREM 2. *An* A.R.B.I.B.D. *with parameters* 2.00 *does not exist when n is odd and t is even and*

(i) $(n-1)t+1$ *is not a perfect square, or*

(ii) $(n-1)t+1$ *is a perfect square and* $n+t \equiv 1 \pmod 4$ *and the square-free part of n contains a prime* $\equiv 3 \pmod 4$.

THEOREM 3. *An* A.R.B.I.B.D. *with parameters* 2.00 *does not exist for any value of t, if* $n \equiv 2 \pmod 4$ *and the square-free part of n contains a prime* $\equiv 3$ (mod 4).

Proofs. Suppose the A.R.B.I.B.D. actually exists, then there are r sets of n blocks each so that any two blocks of different sets have exactly $(n-1)t+1$ treatments in common. Since $b - v = n^2t + n < n^2t + n + 1 = r$, we can pick out $b - v$ blocks one from each of the $b - v$ sets so that from 2.07 and 2.08 the matrix C_{b-v} for these blocks is given by (c_{ju}) where

3.00 $$c_{jj} = n(nt+1)[(n-1)t+1]$$

and

3.01 $$c_{ju} = -[(n-1)t+1]$$

for $j \neq u = 1, 2, \ldots, n^2t+n$. It is easily verified that

3.02 $$C_{b-v} = [(n-1)t+1]^{n^2t+n}(n^2t+n+1)^{n^2t+n-1}.$$

Let the blocks of the design be permuted so that these are the first n^2t+n blocks of the design. Taking N_2 as in 2.09 we get from 2.06 that

3.03 $$|N_2N_2'| = n^{n^2t-2n^2t+n^2-n}[(n-1)t+1]^{n^2[(n-1)t+1]}.$$

But $|N_2N_2'| = |N_2|^2$. Hence the right-hand side of 3.03 must be a perfect square. Hence we get the following results.

A necessary condition for the existence of the design is that

(a) $n[(n-1)t+1]$ should be a perfect square if both n and t are odd, and

(b) $(n-1)t+1$ should be a perfect square if n is odd and t is even.

In the rest of the paper p stands for an odd prime and will be suppressed in the symbol $(a, b)_p$ whenever no confusion is likely to arise.

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Using Theorems B and C it is easily verified that

$$3.04 \quad c_p(NN') = (-1, n^2t + n + 1)(-1, n)^{\frac{1}{2}v(v+1)}(-1, (n-1)t + 1)^{\frac{1}{2}v(v+1)} \cdot (n, n^2t + n + 1)^{v-1}((n-1)t + 1, n^2t + n + 1)^v.$$

From 2.12, 3.00, and 3.01 it is seen that for the matrix $C_{b-v} E_{b-v}$, the diagonal elements are $(nt + 1)/(n^2t + n + 1)$ whereas the non-diagonal elements are $-1/n(n^2t + n + 1)$. Obviously $C_{b-v} E_{b-v} \sim Q$, where $Q = (q_{ju})$ with

$$q_{jj} = n^2(n^2t + n + 1)(nt + 1)$$

and

$$q_{ju} = -n(n^2t + n + 1).$$

It is easily proved that:

$$3.05 \quad |Q| = n^{n^2t+n}(n^2t + n + 1)^{2n^2t+2n-1}, \quad c_p(C_{b-v}E_{b-v}) = c_p(Q),$$

and

$$(|NN'|, |C_{b-v}E_{b-v}|) = (|NN'|, |Q|),$$

where, from 2.05,

$$3.06 \quad |NN'| = (n^2t + n + 1)[n(n-1)t + 1]^{n^2[(n-1)t+1]}.$$

Hence

$$3.07 \quad c_p(N_2N_2') = c_p(NN')c_p(Q)(|NN'|, |Q|).$$

The value of $c_p(Q)$ can be calculated in exactly the same way as $c_p(NN')$ and is given by

$$3.08 \quad c_p(Q) = (-1, n)^{\frac{1}{2}(n^2t+n)(n^2t+n+1)}(n, n^2t + n + 1)^{n^2t+n-1}.$$

With these general results we now proceed to consider the various cases. First take the case where both n and t are odd. If $n[(n-1)t + 1]$ is not a perfect square the design is impossible. Hence we consider only those values for which $n\,[(n-1)t + 1]$ is a perfect square. From 3.04 to 3.08,

$$c_p(NN') = (-1, n^2t + n + 1)((n-1)t + 1, n^2t + n + 1),$$

$$c_p(Q) = (-1, n)^{\frac{1}{2}(n^2t+n)(n^2t+n+1)}(n, n^2t + n + 1),$$

$$(|NN'|, |Q|) = (-1, n^2t + n + 1).$$

Hence

$$c_p(N_2N_2') = (-1, n)^{\frac{1}{2}(n^2t+n)(n^2t+n+1)} = (-1, n)^{\frac{1}{2}(nt+1)}.$$

Hence $c_p(N_2N_2') = 1$ for those values of n and t for which $nt + 1 \equiv 0 \pmod 4$. If however $nt \equiv 1 \pmod 4$ then

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$$c_p(N_2N_2') = (-1, n)_p = (-1, p)_p = (-1/p),$$

if p is a factor of the square-free part of n. Hence if $p \equiv 3 \pmod 4$,

$$c_p(N_2N_2') = -1.$$

But $N_2N_2' \sim I_b$ and hence $c_p(N_2N_2') = c_p(I_b) = 1$, which is a contradiction. Hence the design 2.00 is impossible. This proves Theorem 1.

Now consider the case when n is odd and t is even. We only consider those values of n and t for which $(n-1)t+1$ is a perfect square. For these values of n and t, from 3.04 to 3.08 we have

$$c_p(NN') = (-1, n^2t+n+1)(-1, n),$$

$$c_p(Q) = (-1, n)^{\frac{1}{2}(n^2t+n+1)},$$

$$(|NN'|, |Q|) = (-1, n)(-1, n^2t+n+1).$$

Hence

$$c_p(N_2N_2') = (-1, n)^{\frac{1}{2}(n^2t+n+1)} = (-1, n)^{\frac{1}{2}(n+t+1)}.$$

Obviously the right-hand side is always 1 except possibly when $n+t \equiv 1 \pmod 4$, in which case

$$c_p(N_2N_2') = (-1, n).$$

Hence, as before, if the square-free part of n contains a prime $\equiv 3 \pmod 4$ the design is impossible. This proves Theorem 2.

Lastly, consider the case when n is even. Then

$$c_p(NN') = (-1, n^2t+n+1)(n, n^2t+n+1),$$

$$c_p(Q) = (-1, n)^{\frac{1}{2}n}(n, n^2t+n+1),$$

$$(|NN'|, |Q|) = (-1, n^2t+n+1).$$

Hence

$$c_p(N_2N_2') = (-1, n)^{\frac{1}{2}n}.$$

The value of the right-hand side is always 1 except possible when $n \equiv 2 \pmod 4$ in which case

$$c_p(N_2N_2') = (-1, n).$$

Hence, as before, if the square-free part of n contains a prime $\equiv 3 \pmod 4$ then the design is impossible. This completes the proof of Theorem 3.

It is obvious that the above results are the best possible using this particular method.

COROLLARY. *Putting $t = 0$ in Theorems 2 and 3 above we get that the* A.R.B.I.B.C. *with parameters*

$$v = n^2,\ b = n^2 + n,\ r = n + 1,\ k = n,\ \lambda = 1$$

is impossible when $n \equiv 1$ *or* 2 (mod 4) *and the square-free part of* n *contains a prime* $\equiv 3$ (mod 4). This is equivalent to the result given by Bruck and Ryser [3].

4. Improvement of an inequality for orthogonal arrays of strength 2. Consider a matrix $A = (a_{ij})$ with m rows and N columns where each element a_{ij} represents one of the integers $0, 1, 2, \ldots, n - 1$. Consider all the d-rowed submatrices that can be formed ($d \leqslant m$). Each column of any d-rowed submatrix gives an ordered d-plet. There are n^d possible d-plets. If in the N d-plets obtained from every submatrix each of the n^d possible d-plets occurs exactly μ times ($N = \mu n^d$), then the matrix is called an orthogonal array (N, m, n, d) of size N, m constraints, n levels, and strength d. The idea of orthogonal arrays which is very useful in certain combinatorial problems is due to Rao [8]. The multi-factorial designs considered by Plackett and Burman [7] are orthogonal arrays of strength 2. Give the values of n, d, and N $(= \mu n^d)$, let $f(N, n, d)$ represent the maximum number of constraints possible. Then it is known [7] that

$$4.1 \qquad f(\mu n^2, n, 2) \leqslant I\left(\frac{\mu n^2 - 1}{n - 1}\right)$$

where $I(x)$ is the integral part of x. In some cases this inequality can be improved. When $\mu - 1$ is not divisible by $n - 1$, Bose [2] has given the following

THEOREM D. *If* $\mu - 1 = a(n - 1) + b$, $0 < b < n - 1$ *and* l *is the largest non-negative integer consistent with*

$$n(b - 2l) \geqslant (b - l)(b - l + 1),$$

then

$$f(\mu n^2, n, 2) \leqslant I\left(\frac{\mu n^2 - 1}{n - 1}\right) - l - 1.$$

The results of the previous section can be used to improve the inequality 4.1 in some cases when $\mu - 1$ is actually divisible by $n - 1$.

If $(\mu n^2 - 1)/(n - 1)$ is an integer (which implies that $(\mu n - 1)/(n - 1)$ is also an integer) and further if an orthogonal array exists with the maximum possible number of constraints which is $(\mu n^2 - 1)/(n - 1)$, then such an array is said to be complete. It has been shown [7] that the existence of a complete orthogonal array

$$\left(\mu n^2, \frac{\mu n^2 - 1}{n - 1}, n, 2\right)$$

implies the existence of an A.R.B.I.B.D. with parameters

$$v = nk = \mu n^2,\quad b = nr = n\left(\frac{\mu n^2 - 1}{n - 1}\right),\quad \lambda = \frac{\mu n - 1}{n - 1}$$

and conversely. Hence, in particular, a complete orthogonal array $(n^2[(n-1)t+1], n^2t+n+1, n, 2)$ and an A.R.B.I.B.D. with parameters 2.00 are co-existent. The theorems of the previous section can, therefore, be expressed in terms of the non-existence of the corresponding complete orthogonal arrays. Hence

$$f(n^2[(n-1)t+1], n, 2) \leqslant n^2 t + n,$$

for the values of n and t given in the theorems of the last section.

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