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EVERYWHERE NONRECURSIVE R.E. SETS IN RECURSIVELY PRESENTED TOPOLOGICAL SPACES

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Abstract

Recursively presented topological spaces are topological spaces with a recursive system of basic neighbourhoods. A recursively enumerable (r.e.) open set is a r.e. union of basic neighbourhoods. A set is *everywhere* r.e. open if its intersection with each basic neighbourhood is r.e. Similarly we define everywhere creative, everywhere simple, everywhere r.e. non-recursive sets and show that there exist sets both with and without these everywhere properties.

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Kalantari and Retzlaff's [7] saw the first introduction of recursion theoretic methods, in particular priority arguments, to the study of effectiveness in topology. Subsequently Kalantari and others have extended this work (see, for example, [4]—[10]).

Topology is just one branch of mathematics to be studied in an effective setting in the spirit of the programme begun by Metakides and Nerode [11]. Earlier work, in particular Fröhlich and Shepherdson [1] and Rabin [12], considered the application of recursion theory to algebra (basically field theory) and did not use priority arguments but was restricted to diagonalization arguments. Of course these arguments are used for negative results; for a long time positive results have been achieved by the explicit construction of algorithms.

In the present paper we treat recursively presented topological spaces which have also been considered by Hingston [2, 3]. They are related to Kalantari *et al.*'s *fully effective topological spaces* but here we only consider countable spaces.

An extension to uncountable ones may be possible (see below). Our main contribution is the introduction of the notion of "everywhere" properties. Thus a set is *everywhere r.e. open* if it is r.e. open in every neighbourhood. (R.e. open is defined formally below, but roughly speaking means an r.e. union of neighbourhoods (basic open sets).) We consider everywhere creative, everywhere simple, everywhere hypersimple, everywhere hyperimmune, everywhere incomparable and everywhere r.e. non-recursive sets. Our methods are standard recursion-theoretic ones put into a topological setting. The principal technique is the finite injury priority method.

1. Recursively presented topological spaces

First we introduce Kalantari and Retzlaff's definition of an effective topological space.

DEFINITION 1.1 (KALANTARI & RETZLAFF [7]). Let X be a topological Hausdorff space and Δ be a countable base for the topology on X. We say $\langle X, \Delta\rangle$ is an *effective topological space* if the following properties hold.

TOPOLOGICAL PROPERTIES.

(1) Δ is closed under finite intersections.

(2) $\varnothing, X \in \Delta$. (The elements of Δ are called *basic open sets*.)

(3) No basic open set can be written as a disjoint union of two or more nonempty basic open sets.

(4) Every nonempty basic open set contains two disjoint nonempty basic open subsets.

We assume a one-to-one Gödel numbering of the Δ. For $\delta \in \Delta$, $\lceil \delta \rceil$ denotes the Gödel number of δ and for $x \in \omega$, $\lfloor x \rfloor$ denotes the basic open set with Gödel number x. We assume that $\lceil\ \rceil$ and $\lfloor\ \rfloor$ are inverses.

RECURSION-THEORETIC PROPERTIES.

(1) There is a partial recursive binary function ψ such that for all $x, y \in \omega$, $\lfloor x \rfloor \cap \lfloor y \rfloor \in \Delta$ implies $\psi(x, y)$ converges and $\lfloor \psi(x, y) \rfloor = \lfloor x \rfloor \cap \lfloor y \rfloor$.

(2) There exists a uniform effective procedure which determines whether or not $\delta \subseteq \varepsilon_1 \cup \cdots \cup \varepsilon_n$, where $\delta, \varepsilon_1, \ldots, \varepsilon_n \in \Delta$.

Now we introduce the definition of a recursively presented topological space.

DEFINITION 1.2. Let $\langle X, \Delta\rangle$ be an effective topological space. We say $\langle X, \Delta\rangle$ is *recursively presented* if $\langle X, \Delta\rangle$ satisfies the following axiom $(*)$:

$$(*) \qquad x \in \delta \text{ is a binary recursive relation in } X \times \Delta.$$

It follows at once that a recursively presented topological space is countable. However, we conjecture that many of our results will also hold for uncountable spaces if instead of using (*) in our constructions, we use the fact (see Lemma 1.3 below) that $\{\langle i, j\rangle: \delta_i \subseteq \delta_j\}$ is recursive and instead of using a point x with $x \in \delta$ use a basic neighbourhood δ_x with $\delta_x \subseteq \delta$ in those constructions. However, we have not had the opportunity to check this.

The following basic lemmas are simple but useful fundamental results on recursively presented topological spaces.

BASIC LEMMA 1.3. *Let (X, Δ) be a recursively presented topological space. Then for all $\delta \in \Delta$, either $\delta = \varnothing$ or δ is an infinite recursive set.*

PROOF. By the axiom (*), $\{x\colon\ x \in \delta\} = \delta$ is recursive. By Definition 1.1, topological property (4), it follows that if $\delta \neq \varnothing$ then δ is infinite.

Note 1.4. It follows from the Basic Lemma 1.3 and the recursion theoretic properties in Definition 1.1 that, without loss of generality, we can assume that $X = \{0, 1, 2, \dots\}$, $\Delta = \{\delta_i\}_{i \in \omega}$ and $\delta_0 = \varnothing$. Moreover, we can assume $i = 0 \leftrightarrow \delta_i = \varnothing$. Throughout the remainder of this article we take $\langle X, \Delta\rangle$ to be a recursively presented topological space, $X = \omega$, $\Delta = \{\delta_i\}_{i \in \omega}$ and we shall write Δ^* for $\Delta - \{\varnothing\}$.

BASIC LEMMA 1.5. *The following sets are recursive*

(1) $\{\langle i, j\rangle\colon\ i \in \delta_j\}$,
(2) $\{i\colon\ \delta_i = \varnothing\}$,
(3) $\{\langle i, j\rangle\colon\ \delta_i = \delta_j\}$,
(4) $\{\langle i, j\rangle\colon\ \delta_i \subseteq \delta_j\}$,
(5) $\{\langle i, j\rangle\colon\ \delta_i \cap \delta_j = \varnothing\}$,
(6) $\{\langle i, j, l\rangle\colon\ \delta_i \cap \delta_j = \delta_l\}$, *and*
(7) $\{\langle i, j, l\rangle\colon\ \delta_i \cup \delta_j = \delta_l\}$.

PROOF. The proofs are immediate from the axiom (*) and the recursion theoretic properties in Definition 1.1.

The next basic lemma shows that there is a uniform effective procedure which, given a non-empty basic neighbourhood, produces an infinite set of disjoint subneighbourhoods of the given neighbourhood.

We use a standard recursive pairing function $\langle\ ,\ \rangle\colon \omega \times \omega \to \omega$ with recursive inverses l, r such that

$$\langle l(z), r(z)\rangle = z.$$

BASIC LEMMA 1.6. *There exists a recursive function $f(i, j)$ such that for all $j \in \omega$, if δ_j is non-empty then*

$$\forall i\left(\delta_{f(i,j)} \neq \varnothing\right) \& \bigcup_{i<\omega} \delta_{f(i,j)} \subseteq \delta_j \& \forall i_i, i_2\left(i_1 \neq i_2 \Rightarrow \delta_{f(i_1,j)} \cap \delta_{f(i_2,j)} = \varnothing\right).$$

PROOF. Let $j \in \omega$.

Case (i). if $\delta_j = \varnothing$, define $f(i, j) = 0$ for all $i \in \omega$.

Case (ii). if $\delta_j \neq \varnothing$, define x_s^j, ε_s^j, and $\varepsilon_s'^j$ as follows.

Stage 0.

$$x_0^j = \mu x\left(\delta_{l(x)} \cap \delta_{r(x)} = \varnothing \& \delta_{l(x)} \cup \delta_{r(x)} \subseteq \delta_j \& \delta_{l(x)} \neq \varnothing \& \delta_{r(x)} \neq \varnothing\right),$$
$$\varepsilon_0^j = \delta_{l(x_0^j)},$$
$$\varepsilon_0'^j = \delta_{r(x_0^j)}.$$

Stage $t = s + 1$.

$$x_t^j = \mu x\left(\delta_{l(x)} \cap \delta_{r(x)} = \varnothing \& \delta_{l(x)} \cup \delta_{r(x)} \subseteq \varepsilon_s'^j \& \delta_{l(x)} \neq \varnothing \& \delta_{r(x)} \neq \varnothing\right),$$
$$\varepsilon_t^j = \delta_{l(x_t^j)},$$
$$\varepsilon_t'^j = \delta_{r(x_t^j)}.$$

By the Basic Lemmas 1.3 and 1.5, the construction is effective, so there exists a recursive function f such that for all i, j

$$f(i, j) = \begin{cases} 0, & \text{if } \delta_j = \varnothing, \\ l\left(x_j^i\right), & \text{if } \delta_j \neq \varnothing. \end{cases}$$

Then, by the construction, f satisfies the condition of Lemma 1.6.

2. R.e. dense sets

In this section we consider the notion of denseness in the context of a recursively presented topological space.

DEFINITION 2.1. Let $A \subseteq X$ be an open set in the space X. We say A is an *r.e. open set* if there exists a recursive function f such that $A = \bigcup_{i<\omega} \delta_{f(i)}$.

Note 2.2. $\varnothing$, X are r.e. open sets.

PROPOSITION 2.3. *If A and B are r.e. open sets, then $A \cup B$ and $A \cap B$ are r.e. open sets.*

PROOF. Suppose f and g are recursive functions such that $A = \bigcup_{i<\omega}\delta_{f(i)}$, $B = \bigcup_{i<\omega}\delta_{g(i)}$. Define a recursive function h by $h(2i) = f(i)$, $h(2i+1) = g(i)$. Then $A \cup B = \bigcup_{i<\omega}\delta_{h(i)}$.

By recursion-theoretic property (1), $\lfloor\psi(f(i), g(j))\rfloor = \delta_{f(i)} \cap \delta_{g(j)}$. Hence the function k is recursive where $k(z) = \psi(fl(z), gr(z))$, and $A \cap B = \bigcup_{i<\omega}\delta_{k(i)}$ by the infinite distributive law for sets.

DEFINITION 2.4. Let $K \subseteq X$ be an open set and $A \subseteq X$ a r.e. set. A is said to be *r.e. dense in* K if A is dense in K, that is, for all $i \in \omega$,

$$\delta_i \subseteq K \,\&\, \delta_i \neq \varnothing \text{ imply } \delta_i \cap A \neq \varnothing .$$

PROPOSITION 2.5. *Let K be an r.e. open set. Then there exists a recursive set $A \subseteq K$ such that A is dense in K.*

PROOF. *Case* (i). If $K = \varnothing$, then $A = \varnothing$ satisfies the proposition.

Case (ii). If $K \neq \varnothing$, we give a construction for enumerating A in successive stages. Let $A^{(n)}$ be the members of A which have been enumerated by the end of stage n. By Definition 2.1 and Basic Lemma 1.5, there is a one-to-one recursive function f such that $K = \bigcup_{i<\omega}\delta_{f(i)}$.

Stage 0. Set $A^{(0)} = 0$, $E_0 = \varnothing$.

Stage $t = s + 1$.

Case (i). If $\exists i \leqslant s(\delta_i \cap (\bigcup_{j\leqslant t}\delta_{f(j)})) \neq \varnothing$ and $i \notin E_s$, define

$$i_t = \mu i \leqslant s\left(\delta_i \cap \left(\bigcup_{j\leqslant t}\delta_{f(j)}\right) \neq \varnothing \,\&\, i \notin E_s\right),$$

$$E_t = E_s \cup \{i_t\},$$

$$a_t = \mu a\left(a > \operatorname{Max} A^{(s)} \,\&\, a \in \delta_{i_t} \cap \left(\bigcup_{j\leqslant t}\delta_{f(j)}\right)\right),$$

$$A^{(t)} = A^{(s)} \cup \{a_t\}.$$

Case (ii). Otherwise, $A^{(t)} = A^{(s)}$, $E_t = E_s$ (end of construction).

Clearly the construction is recursive, so $A = \bigcup_{j<\omega} A^{(j)}$ is an r.e. set and $A \subseteq K$. It is clear from the fact that the open set $K \neq \varnothing$ and Basic Lemma 1.6 that A is infinite. Observe that for all s_1, s_2

$$s_2 > s_1 \ \& \ a' \in A^{(s_1)} \ \& \ a'' \in A^{(s_2)} - A^{(s_1)} \text{ imply } a'' > a',$$

and hence A is recursive. It remains to show that A is dense in K.

We first show that it is sufficient to prove that if $\delta_i \subseteq K$ and $\delta_i \neq \varnothing$ then there exists t such that $i \in E_t$.

Since $E_0 = \varnothing$ we then have $t = s+1$ for some s. In this case $i = i_t$ and for the corresponding a_t, we have $a_t \in \delta_i \cap \bigcup_{j \leqslant t} \delta_{f(j)}$ and also $a_t \in A$. Therefore $\delta_i \cap A \neq \varnothing$.

Suppose then, for a contradiction, that for some i, $\delta_i \subseteq K$, $\delta_i \neq \varnothing$ and yet $i \notin E_s$ for all s. Let i be minimum with this property. Now $\delta_i \subseteq K$ and $\delta_i \neq \varnothing$ imply $\delta_i \cap K \neq \varnothing$ so for some s and all $t \geqslant s$

$$\delta_i \cap \bigcup_{j \leqslant t} \delta_{f(j)} \neq \varnothing.$$

Now if $j < i$, the assumption that i is minimum implies $j \in E_{u_j}$ for some u_j. Let t be the least number such that $t \geqslant s$, $t > i$ and $t > \operatorname{Max}\{u_j \colon j < i\}$. Then $t = s' + 1$ for some s' and i is the least number such that $i \leqslant s$, $\delta_i \cap \bigcup_{j \leqslant t} \delta_{f(j)} \neq \varnothing$ and $i \notin E_s$. But then, by the construction, $i = i_t$ and $i \in E_t$, which gives the required contradiction.

By Lemma 1.6, given any r.e. open set K we can find two disjoint basic open sets within K. Therefore if we apply the construction in Proposition 2.5 to one of these basic open sets, we obtain the following corollary.

COROLLARY 2.6. *If K is an r.e. open set, then there exists a recursive set A not dense in K.*

Corollary 2.6 shows that Proposition 2.5 is not trivial. We shall use this technique several times to establish non-triviality. We can establish a stronger result as follows.

THEOREM 2.7. *Let $K \neq \varnothing$ be a r.e. open set, and A a recursive set. If A is dense in K, then there exist recursive sets A_1 and A_2 both dense in K such that*

$$A_1 \cup A_2 = A \quad \textit{and} \quad A_1 \cap A_2 = \varnothing.$$

PROOF. *Stage* 0. Set $B_1^{(0)} = \varnothing$, $B_2^{(0)} = \varnothing$, $E_0 = \varnothing$.

Stage $t = s + 1$. Case (i). If $\exists i \leqslant s[(A \cap \delta_i \cap (\bigcup_{j \leqslant t} \delta_{f(j)})) \neq \varnothing$ and $i \notin E_s)]$, define

$$i_t = \mu i \leqslant s\left[A \cap \delta_i \cap \left(\bigcup_{j \leqslant t} \delta_{f(j)}\right) \neq \varnothing \;\&\; i \notin E_s\right],$$

$$a_t = \mu a\left(a > \mathrm{Max}\left(B_1^{(s)} \cup B_2^{(s)}\right) \text{ and } a \in \left(A \cap \delta_{i_t} \cap \left(\bigcup_{j \leqslant t} \delta_{f(j)}\right)\right)\right),$$

$$B_1^{(t)} = B_1^{(s)} \cup \{a_t\},$$

$$b_t = \mu b\left(b > \mathrm{Max}\left(B_1^{(t)} \cup B_2^{(s)}\right) \text{ and } b \in \left(A \cap \delta_{i_t} \cap \left(\bigcup_{j \leqslant t} \delta_{f(j)}\right)\right)\right),$$

$$B_2^{(t)} = B_2^{(s)} \cup \{b_t\},$$

$$E_t = E_s \cup \{i_t\}.$$

Case (ii). Otherwise, define

$$B_1^{(t)} = B_1^{(s)}, \qquad B_2^{(t)} = B_2^{(s)}, \qquad E_t = E_s.$$

Then $A_1 = B_1$ and $A_2 = A - B_1$ satisfy the theorem for the following reasons. Observe that in the construction, if $t < t'$ then $a_t < b_t < a_{t'} < b_{t'}$ and that b_t can be found since if $A \cap \delta_i \cap \bigcup_{j \leqslant t} \delta_{f(j)}$ is non-empty then it is infinite (by Basic Lemma 1.6). The fact that B_1, B_2 are infinite, recursive and dense in K follows, as in the proof of Proposition 2.5. It then follows that $A - B_1$ which contains B_2 is infinite, recursive and dense in K.

As in Corollary 2.6 above, part (b) of the following theorem shows part (a) is non-trivial. As usual W_e is the eth r.e. set in an acceptable enumeration.

THEOREM 2.8. (a) *There exists a non-recursive r.e. set A which is dense.*

(b) *There exists a non-recursive r.e. set A which is not dense.*

(c) *If A is a non-recursive r.e. dense set, then there exist non-recursive r.e. sets A_1 and A_2 such that both A_1 and A_2 are dense and*

$$A_1 \cup A_2 = A \;\&\; A_1 \cap A_2 = \varnothing.$$

PROOF. Take a simple set A. Then A is a non-recursive r.e. set and A intersects every infinite r.e. set, and hence A is dense and therefore A satisfies (a).

To show (b), let $A = \{f(i)\colon i \in W_i\}$, where f is a one-to-one recursive function such that

$$\exists i, j\colon \delta_i \neq \varnothing \;\&\; \delta_j \neq \varnothing \;\&\; \delta_i \cap \delta_j = \varnothing \;\&\; \delta_i = \{f(i)\colon i < \omega\}.$$

Then A satisfies (b).

PROOF OF (c). Using the S_n^m-theorem define a recursive function f such that $W_{f(i)} = A \cap \delta_i$. Since A is dense, by Lemma 1.6, for all i,

$$\delta_i \neq \varnothing \text{ implies } W_{f(i)} \text{ is an infinite set.}$$

We construct an r.e. set B as follows, where $W_{i,t}$ denotes the finite subset of the ith r.e. set enumerated at stage t.

Stage 0. $B^{(0)} = \varnothing$, $E_0 = \varnothing$.

Stage $t = s + 1$.

Case (i). If $\exists\, i \leqslant s(i \notin E_s \;\&\; W_{f(i),t} \neq \varnothing)$ define

$$\begin{aligned} i_t &= \mu i \leqslant s\big(i \notin E_s \;\&\; W_{f(i),t} \neq \varnothing\big), \\ b_t &= \mu b\big(b \in W_{f(i)} \;\&\; b > \operatorname{Max} B^{(s)}\big), \\ B^{(t)} &= B^{(s)} \cup \{b_t\}, \\ E_t &= E_s \cup \{i_t\}. \end{aligned}$$

Case (ii). Otherwise, define $B^{(t)} = B^{(s)}$, $E_t = E_s$. Then $B = \bigcup_{s<\omega} B^{(s)} \subseteq A$ and B is r.e.. Since, for all i, $\delta_i \neq \varnothing$ implies $W_{f(i)}$ is infinite, the same technique as in the proof of Proposition 2.5 shows that B is infinite, recursive and dense. Hence $A - B$ is a non-recursive r.e. set. By Friedberg's theorem [3] there exist sets B_1' and B_2' such that B_1' and B_2' are r.e. but not recursive and

$$B_1' \cup B_2' = A - B \;\&\; B_1' \cap B_2' = \varnothing.$$

Then define $A_1 = B_1 \cup B_1'$, $A_2 = B_2 \cup B_2'$, where $B_1 \cup B_2 = B$ and $B_1 \cap B_2 = \varnothing$ and B_1, B_2 are each dense by Theorem 2.7, so A_1, A_2 satisfy Theorem 2.8(c).

3. Everywhere properties

In this section we introduce the notion of everywhere properties. Intuitively a property is an everywhere property if it holds in every basic neighbourhood. The following definition covers many cases.

DEFINITION 3.1. Let $\mathscr{P}$ be a property of sets. A set $A \subseteq X$ is said to be *everywhere* $\mathscr{P}$ if, for all i,

$$\delta_i \neq \varnothing \text{ implies } A \cap \delta_i \text{ has property } \mathscr{P}.$$

Thus $A \subseteq X$ is everywhere r.e. non-recursive if, for all i, $A \cap \delta_i$ is a r.e. non-recursive set.

PROPOSITION 3.2. *A is everywhere non-recursive implies*

(a) *A is dense,*

(b) *$X - A$ is dense.*

The next theorem shows that some r.e. non-recursive sets are everywhere r.e. non-recursive and some are not. However, we shall defer the proof of part (b) as it is a corollary of a later result.

THEOREM 3.3. (a) *There exists an r.e. non-recursive set A which is not everywhere r.e. non-recursive.*

(b) *There exists an r.e. non-recursive set A which is everywhere r.e. non-recursive.*

PROOF. Part (Ia) follows from Theorem 2.8(b) and Proposition 3.2(a). Part (b) follows from Proposition 4.2(d) and Theorem 4.4(a) below.

In order to introduce an appropriate definition of everywhere creative, we have to depart slightly from Definition 3.1.

DEFINITION 3.4. $A \subseteq X$ is said to be *everywhere creative* if (i) A is r.e. and (ii) for all $i \in \omega$, $\delta_i \neq \varnothing$ implies $\delta_i - A$ is productive.

Note that A is r.e. implies that A is everywhere r.e. and similarly for recursive.

The next theorem shows the existence of sets which are and are not everywhere creative. For part (a) we use a simple technique related to that for establishing Corollary 2.6.

THEOREM 3.5. (a) *If A is everywhere creative then A is creative.*

(b) *There exists a creative set A which is not everywhere creative.*

PROOF. (a) Trivial.

(b) Let i, j be such that $\delta_i \neq \varnothing \neq \delta_j$ and $\delta_i \cap \delta_j = \varnothing$. Let f be a one-to-one recursive function such that $\delta_i = \{f(n)\colon n < \omega\}$. Let $A = \{f(n)\colon n \in W_n\}$. Then, for all $n \in \omega$, $n \in W_n \leftrightarrow f(n) \in A$. Hence $\{n\colon n \in W_n\} \leqslant_1 A$ and A is therefore creative. But $A \cap \delta_j = \varnothing$, and therefore A is a creative set which is not everywhere creative.

Finally we introduce a definition and two lemmas due to Hingston [2].

LEMMA 3.6. *There is an algorithm which, given a basic open set δ and distinct elements $x, y_1, \ldots, y_m$ in δ, produces basic open sets $\varepsilon_x, \varepsilon_1, \ldots, \varepsilon_m$ such that $\varepsilon_x \subseteq \delta$, $\varepsilon_i \subseteq \delta$, $x \in \varepsilon_x$, $y_i \in \varepsilon_i$ and $\varepsilon_x \cap \delta_i = \varnothing$ for $i = 1, \ldots, m$.*

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PROOF. First observe that since X is Hausdorff, $\delta_m \cap \delta_n \neq \varnothing$ and $x \in \delta_n$ are recursive, then given two distinct points x and y we can find numbers k_x and k_y such that $x \in \delta_{k_x}$, $y \in \delta_{k_y}$ and $\delta_{k_x} \cap \delta_{k_y} = \varnothing$.

It follows that for $i = 1, \ldots, m$, we can compute k_i, l_i such that $x \in \delta_{k_i}$ and $y_i \in \delta_{l_i} = \varnothing$. Let $\varepsilon_x = \delta \cap \bigcap\{\delta_{k_i}: i = 1, \ldots, m\}$ and $\varepsilon_i = \delta \cap \delta_{l_i}$. Note that the Gödel numbers of ε_x and the ε_i can be computed using the ψ function of recursion-theoretic property (1) of Definition 1.1.

In the obvious way we call a collection $\Gamma = \{\gamma_i: i \in \omega\}$, of basic open sets, an r.e. collection if the set $\{\lceil \gamma_i \rceil: i \in \omega\}$ is r.e.

DEFINITION 3.7. An r.e. collection of basic open sets is said to be a *partition* of a set $A \subseteq X$ if

(i) $\gamma_i \neq \varnothing$ for all $i \in \omega$,
(ii) $i \neq j$ implies $\gamma_i \cap \gamma_j = \varnothing$,
(iii) $\bigcup\{\gamma_i: i \in \omega\} \subseteq A$, and
(iv) $\bigcup\{\gamma_i: i \in \omega\}$ is dense in A.

LEMMA 3.8 (HINGSTON). *Let X be a recursively presented topological space and $A \subseteq X$ an r.e. open set. Then X contains a partition for A.*

PROOF. We shall construct a recursive function f in stages and define $\gamma_i = \delta_{f(i)}$; f^s will denote the part of f constructed up to stage s. Since A is r.e. open, $A = \bigcup\{\delta_n: n \in W_e\}$ for some e and we let $A^{(s)} = \bigcup\{\delta_n: n \in W_{e,s}\}$ where $W_{e,s}$ is the set of elements of W_i enumerated by stage s.

Construction. Stage 0. $f^0 = \varnothing$.

Stage $s = t + 1$. Effectively find the least n such that $\delta_n \neq \varnothing$, $\delta_n \subseteq A^{(s)}$ and δ_n is disjoint from $\bigcup\{\gamma_i: i \leqslant t\}$. Such an n exists since the construction will ensure $\bigcup\{\gamma_i: i \leqslant t\}$ is not dense in A. Since $\delta_n \neq \varnothing$, δ_n is infinite and since $\delta_n \subseteq A^{(s)}$ we can compute two elements $x, y \in \delta_n$.

By Lemma 3.6 we can find basic open sets $\delta_k, \delta_l \subseteq \delta_n$ such that $\delta_k \cap \delta_l = \varnothing$, $x \in \delta_k$ and $y \in \delta_l$. Set $f(s) = k$. Note that $\bigcup\{\gamma_i: i \subseteq S\}$ is not dense in A since it does not meet δ_l.

Finally put $f = \bigcup f^s$, $\Gamma = \{\gamma_i: i \in \omega\}$ (end of construction).

By construction, Γ is an r.e. open set. Γ is dense in A since if δ_n did not meet $\bigcup\{\gamma_i: i \in \omega\}$ there would be a stage when δ_n did not meet $\bigcup\{\gamma_i: i \leqslant t\}$ and n was least. At that stage a subneighbourhood of δ_n would be put into Γ.

4. Everywhere simple sets

DEFINITION 4.1. (a) We say A is *simple in* δ_i if

(i) $A \cap \delta_i$ is a r.e. set,

(ii) $\delta_i - A$ is infinite, and

(iii) for all j, $W_j \subseteq \delta_i$ & W_j is infinite imply $W_j \cap A \neq \varnothing$.

(b) We say that A is *an everywhere simple set* if, for all i,

$$\delta_i \neq \varnothing \text{ implies } A \text{ is simple in } \delta_i.$$

PROPOSITION 4.2. (a) *A is simple $\Leftrightarrow$ A is simple in X.*

(b) *A is everywhere simple $\Rightarrow$ A is simple.*

(c) *A is everywhere simple $\Rightarrow$ $\forall i(\delta_i \neq \varnothing \Rightarrow \delta_i - A$ is immume).*

(d) *A is everywhere simple $\Rightarrow$ A is an everywhere non-recursive r.e. set.*

PROPOSITION 4.3. *Let A be an r.e. set. Then A is everywhere simple if the following hold for all $i \in \omega$:*

(a) *W_i is infinite $\Rightarrow$ $W_i \cap A \neq \varnothing$ and*

(b) *$\delta_i \neq \varnothing \Rightarrow \delta_i - A \neq \varnothing$.*

PROOF. A is r.e. implies $A \cap \delta_i$ is r.e., so (i) of Definition 4.1 holds for all i.

Condition (a) trivially implies condition (iii) of Definition 4.1 for all i.

We now show (b) implies condition (ii) of Definition 4.1 for all i. Condition (b) says that *every* non-empty basic neighbourhood meets the complement of A (in X). Now by Lemma 1.6, every basic neighbourhood contains two disjoint basic subneighbourhoods. These in turn contain at least two distinct points not in A. By induction it follows that every basic neighbourhood contains an infinite number of points not in A. Hence, for all i, $\delta_i \neq \varnothing$ implies $\delta_i - A$ is infinite. That is, condition (ii) of Definition 4.1 holds for all i.

THEOREM 4.4. (a) *There exists an everywhere simple set.*

(b) *There exists a simple set which is not everywhere simple.*

PROOF. (a) By Proposition 4.3, it suffices to construct an r.e. set A meeting the requirements

$$P_e\colon W_e \text{ is infinite implies } W_e \cap A \neq \varnothing,$$

$$N_e\colon \delta_e \neq \varnothing \text{ implies } \delta_e - A \neq \varnothing.$$

The priority ranking of the requirements is $N_0, P_0, N_1, P_1, \ldots$. $A^{(s)}$ consists of the elements enumerated in A by the end of stage s. To aid in meeting N_e, given $A^{(s)}$, define, for all e, the *restraint function*

$$r(e,s) = \operatorname{Min}\left(\delta_e - A^{(s)}\right).$$

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$r(e, s)$ is a recursive function because $\{A^{(s)}\}_{s \in \omega}$ is a recursive sequence of finite sets.

Recall that W_e is the eth r.e. set and $W_{e,s}$ is the finite subset of W_e enumerated in s steps.

Construction of A. *Stage* 0. Let $A^{(0)} = \varnothing$.

Stage $s + 1$.

Case (i). $\exists i \leqslant s[W_{i,s} \cap A^{(s)} = \varnothing \;\&\; (\exists x)(x \in W_{i,s} \;\&\; (\forall e \leqslant i)(r(e,s) < x))]$. Define

$$i_{s+1} = \mu i \leqslant s\left[W_{i,s} \cap A^{(s)} = \varnothing \;\&\; (\exists x)(x \in W_{i,s} \;\&\; (\forall e \leqslant i)(r(e,s) < x))\right],$$
$$x_{s+1} = \mu x\left[x \in W_{i_{s+1},s} \;\&\; (\forall e \leqslant i_{i+1})(r(e,s) < x)\right],$$
$$A^{(s+1)} = A^{(s)} \cup \{x_{s+1}\},$$

and say $P_{i_{s+1}}$ receives attention.

Case (ii). Otherwise, define $A^{(s+1)} = A^{(s)}$ (end of construction).

We say that x *injures* N_e at stage $s + 1$ if $x \in A^{(s+1)} - A^{(s)}$ and $x \leqslant r(e, s)$. Define the injury set I_e for N_e as follows: $I_e = \{x\colon (\exists s)[x \in A^{(s+1)} - A^{(s)} \;\&\; x \leqslant r(e,s)]\}$.

LEMMA 1. $(\forall e)[I_e$ *is finite*$]$.

PROOF. By construction, each positive requirement P_i contributes at most one element to A, and N_e can be injured by P_i only if $i < e$. Hence, I_e is a finite set.

LEMMA 2. *For every* e, *requirement* N_e *is met and* $r(e) = \lim_{s \to \infty} r(e, s)$ *exists.*

PROOF. Fix e. By Lemma 1, choose s_e such that N_e is not injured at any stage $s \geqslant s_e$. Then for any $s \geqslant s_e$, $r(e, s) = r(e, s_e)$ and $r(e) = r(e, s_e) \in \delta_e - A$.

LEMMA 3. *For every* i, *requirement* P_i *is met.*

PROOF. Fix i such that W_i is infinite. By Lemma 2, choose s such that

$$(\forall t \geqslant s)(\forall e \leqslant i)[r(e,t) = r(e)].$$

Choose $s' \geqslant s$ such that no P_j with $j < i$ receives attention after stage s'. Choose $t > s'$ such that

$$(\exists x)[x \in W_{i,t} \;\&\; (\forall e \leqslant i)[r(e) < x]].$$

Then either $W_{i,t} \cap A^{(t)} \neq \varnothing$ or else P_i receives attention at stage $s + 1$. In either case $W_{i,t} \cap A^{(t+1)} \neq \varnothing$, so P_i is met by the end of stage $t + 1$.

(b) Let A be everywhere simple. Take δ_i and δ_j such that $\delta_i \neq \varnothing$, $\delta_j \neq \varnothing$, and $\delta_i \cap \delta_j = \varnothing$. Then $A \cup \delta_i$ is a simple set which is not an everywhere simple set.

We can now complete the proof of Theorem 3.3(c). By Theorem 4.4(a) there is an everywhere simple set. By Proposition 4.2(d) this set is everywhere r.e. non-recursive.

THEOREM 4.5. *There is an everywhere simple set A which is low (that is, $A' \equiv_T \varnothing'$ where $\equiv_T$ means "Turing equivalent to").*

PROOF. It suffices to construct an r.e. set A to meet, for all e, the requirements

$$P_e\colon W_e \text{ is infinite implies } W_e \cap A \neq \varnothing,$$
$$N_e\colon \delta_e \neq \varnothing \text{ implies } \delta_e - A \neq \varnothing \ \&\ (\exists^\infty s)\left[\{e\}_s^{A^{(s)}}(e)_\downarrow \Rightarrow e^A(e)_\downarrow\right],$$

where $f(x)_\downarrow$ means $f(x)$ is defined and $f(x)^\uparrow$ means $f(x)$ is undefined. The requirements $(\forall e)(\exists^\infty s)[\{e\}_s^{A^{(s)}}(e)_\downarrow \Rightarrow \{e\}^A(e)_\downarrow]$ guarantee $A' \leqslant_T \varnothing'$, where $(\exists^\infty s)$ denotes "there exist infinitely many s such that".

The priority ordering of requirements will be $N_0, P_0, N_1, P_1, \ldots$. First, we define the *use function*

$$u(A^{(s)}; e, x, s) = \begin{cases} m+1, & \text{where } m = \text{the maximum element used in the} \\ & \text{computation of } \{e\}_s^{A^{(s)}}(x), \text{ if } \{e\}_s^{A^{(s)}}(x)_\downarrow, \\ \uparrow, & \text{if } \{e\}_s^{A^{(s)}}(x)^\uparrow, \end{cases}$$

and the *restraint function* $r(e, s) = \mathrm{Max}\{\mathrm{Min}(\delta_e - A^{(s)}),\ u(A^{(s)}; e, e, s)\}$. Then $r(e, s)$ will be a recursive function because $\{A^{(s)}\}_{s \in \omega}$ is a recursive sequence.

Construction of A. Stage 0. Let $A^{(0)} = \varnothing$, $E_0 = \varnothing$.

Stage $s + 1$. *Case* (i): $\exists i \leqslant s[W_{i,s} \cap A^{(s)} = \varnothing \ \&\ (\exists x)(x \in W_{i,s} \ \&\ (\forall e \leqslant i)(r(e, s) < x))$. Define

$$i_{s+1} = \mu i \leqslant s\left[W_{i,s} \cap A^{(s)} = \varnothing \ \&\ (\exists x)\big(x \in W_{i,s} \ \&\ (\forall e \leqslant i)(r(e,s) < x)\big)\right],$$
$$x_{s+1} = \mu x\left[x \in W_{i_{s+1},s} \ \&\ (\forall e \leqslant i_{s+1})(r(e,s) < x)\right],$$
$$A^{(s+1)} = A^{(s)} \cup \{x_{s+1}\},$$
$$E_{s+1} = E_s \cup \{i_{s+1}\},$$

and say that $P_{i_{s+1}}$ *receives attention.*

Case (ii). Otherwise, define $A^{(s+1)} = A^{(s)}$, $E_{s+1} = E_s$ (end of construction).

We say that x injures N_e at stage $s+1$ if $x \in A^{(s+1)} - A^{(s)}$ and $x \leqslant r(e,s)$. Define the injury set, I_e, for N_e as follows

$$I_e = \{x\colon x \text{ injures } N_e \text{ at some stage } s\}.$$

LEMMA 1. $(\forall e)[I_e$ *is finite*].

PROOF. By construction, each pointwise requirement P_i contributes at most one element to S, and N_e can be injured by P_i only if $i < e$. Hence I_e is a finite set.

LEMMA 2. *For every* e, *requirement* N_e *is met and* $r(e) = \lim_{s\to\infty} r(e,s)$ *exists.*

PROOF. Fix e. By Lemma 1, choose a state s_e such that N_e is not injured at any stage $s \geqslant s_e$. Then for any $s \geqslant s_e$, $r(e,s) = r(e,s_e)$. If $\{e\}_t^{A^{(t)}}(e) = \{e\}_s^{A^{(s)}}(e)$ for all $t \geqslant s$, then $\{e\}^A(e)_\downarrow$ by the Use Principle (see [14, p. 18]). It is clear that $\operatorname{Min}(\delta_e - A^{(s_e)}) \in \delta_e - A$.

LEMMA 3. *for every* i, *requirement* P_i *is met.*

PROOF. Fix i so that W_i is finite. By Lemma 2, choose s such that

$$(\forall t \geqslant s)(\forall e \leqslant i)(r(e,t) = r(e)).$$

Choose $s' > s$ such that no P_j, $j < i$, receives attention after stage s'. Choose $t > s'$ such that

$$(\exists x)(x \in W_{i,t} \;\&\; (\forall e \leqslant i)(r(e) < x)).$$

Then either $W_{i,t} \cap A_t \neq \varnothing$ or else P_i receives attention at stage $t+1$. In either case $W_{i,t} \cap A_{t+1} \neq \varnothing$ so P_i is met by the end of stage $t+1$.

THEOREM 4.6. *There exists an everywhere simple set* A *which is everywhere low* (*that is*, $\forall i\ (\delta_i \neq \varnothing \Rightarrow (\delta_i \cap A)' \equiv_T \varnothing')$).

PROOF. It suffices to construct an r.e. set A to meet, for all e, the requirements

P_e: W_e is infinite implies $W_e \cap A \neq \varnothing$,

N_e: $\delta_e \neq \varnothing$ implies $\delta_e - A \neq \varnothing \;\&\; (\exists^\infty s)\left[\{e\}_s^{(A\cap\delta_e)^{(s)}}(e)_\downarrow \Rightarrow \{e\}^{A\cap\delta_e}(e)_\downarrow\right]$

where $(A \cap \delta_e)^{(s)} = A^{(s)} \cap \delta_{e,s}$, $W_{f(e)} = \delta_e$, and $\delta_{e,s} = W_{f(e),s}$.

Now by the proof of Proposition 4.3, $\forall i(\delta_e \neq \varnothing$ implies $\delta_e - A \neq \varnothing)$ implies

$$\forall i(\delta_e \neq \varnothing \text{ implies } \delta_e - A \text{ is infinite}).$$

Modify the construction in the proof of Theorem 4.5 by using $(A \cap \delta_e)^{(s)}$ in place of $A^{(s)}$ in the definition of the use function. It is then clear that the A constructed in this way satisfies Theorem 4.6.

DEFINITION 4.7. A is said to be *everywhere hyperimmune* if A is infinite and for all i and every recursive function f, $\delta_i \neq \varnothing \Rightarrow f$ does not majorize $\delta_i \cap A$.

PROPOSITION 4.8. (a) *A is hyperimmune does not imply A is everywhere hyperimmune.*

(b) *A is everywhere hyperimmune implies A is hyperimmune.*

PROOF. (a) Take $\delta_i \neq \varnothing$, $\delta_j \neq \varnothing$ such that $\delta_i \cap \delta_j = \varnothing$. Define

$$g(0) = \mu x(x \in \delta_i \;\&\; x > f_0(0) + 1),$$
$$g(n+1) = \mu x(x \in \delta_i \;\&\; x > g(n) \;\&\; x > f_{n+1}(n+1)),$$

where $f_0, f_1, \ldots, f_n, \ldots$ is a sequence of total functions which includes all the recursive functions. Then $A = \operatorname{range} g$ is hyperimmune but is not everywhere hyperimmune.

(b) This is immediate.

PROPOSITION 4.9. *Let A be a hyperimmune set. Then A is everywhere hyperimmune if and only if $\forall i(\delta_i \neq \varnothing \Rightarrow \delta_i \cap A \neq \varnothing)$.*

PROOF. The "only if" statement is immediate. To prove the if statement, let $a_0, a_1, \ldots, a_n, \ldots$ be the members of A in strictly increasing order. By the proof of Proposition 4.3, $\forall i(\delta_i \neq \varnothing \Rightarrow \delta_i \cap A \neq \varnothing)$ if, and only if, $\forall i(\delta_i \neq \varnothing \Rightarrow \delta_i \cap A$ is infinite). Fix $\delta_i \neq \varnothing$. We can assume $b_0, b_1, \ldots, b_n, \ldots$ are the members of $A \cap \delta_i$ in strictly increasing order. Since $\{b_n : n < \omega\} \subseteq \{a_n : n < \omega\}$,

$$(*) \qquad \forall n,\ a_n \leqslant b_n.$$

By hypothesis, A is a hyperimmune set; it follows from $(*)$ that A is everywhere hyperimmune.

THEOREM 4.10. *There exists an everywhere hyperimmune set.*

PROOF. Let $f_0, f_1, \ldots, f_n, \ldots$ be a sequence of functions which includes all the recursive functions. By Lemma 1.5(2) there is a recursive function g such that for every i, $\delta_{g(i)} \neq \varnothing$, and every non-empty δ_j is $\delta_{g(i)}$ for some i. Define h by

$$h(0) = \mu x(x \in \delta_{g(0)} \;\&\; x \geqslant f_0(0) + 1),$$
$$h(n+1) = \mu x(x \in \delta_{g(n+1)} \;\&\; x > h(n) \;\&\; x \geqslant f_{n+1}(n+1)).$$

Then $A = \operatorname{range} h$ satisfies Theorem 4.10.

DEFINITION 4.11. A is *everywhere hypersimple* if A is recursively enumerable and for any $\delta_i \neq \varnothing$, $\delta_i - A$ is hyperimmune.

THEOREM 4.12. (a) *There exists a hypersimple set which is not everywhere hypersimple.*

(b) *There exists an everywhere hypersimple set.*

PROOF. (a) Choose $\delta_i \neq \varnothing$, $\delta_j \neq \varnothing$ with $\delta_i \cap \delta_j = \varnothing$. Let A be a given recursively enumerable, non-recursive set. Let h be a one-to-one function with range $h = \delta_i$. Let f be a one-to-one recursive function with range $f = A$. Define

$$B = \{h(x)\colon (\exists y)(x < y \;\&\, f(y) \leqslant f(x))\}.$$

Then B is recursively enumerable. $X - B$ is hyperimmune, for if not, let g majorize $X - B$, then $x \in A$ if and only if $x \in \{f(0), f(1), \ldots, f(g(x))\}$, so A would be recursive. Since $B \subseteq \delta_i$, it follows that B is hypersimple but not everywhere hypersimple.

(b) Let A be a given everywhere non-recursive r.e. set. Let f be a one-to-one recursive function such that $A = \operatorname{range} f$. Define

$$B = \{x\colon (\exists y)(x < y \;\&\, f(y) \leqslant f(x))\}.$$

Then B is an everywhere hypersimple set. For, if not, let g majorize $\delta_i - B$. Let f be an increasing recursive function such that $\delta_i = \operatorname{range} h$. Then

$$h(x) \in A \cap \delta_i \Leftrightarrow h(x) \in \{f(0), f(1), \ldots, f(g(h(x)))\}.$$

So $A \cap \delta_i$ would be a recursive set.

5. Everywhere incomparable r.e. sets

DEFINITION 5.1. We say r.e. sets A and B are *everywhere incomparable* if for every $\delta_i \neq \varnothing$, $A \cap \delta_i \not\leqslant_T B \cap \delta_i$ and $B \cap \delta_i \not\leqslant_T A \cap \delta_i$.

THEOREM 5.2. *There exist r.e. sets A and B such that A and B are everywhere incomparable.*

PROOF. By Lemmas 1.3 and 1.5, we can assume that, for all i, $\delta_i \neq \varnothing$ and take a one-to-one recursive function f such that $\forall n(\{f(n, m)\colon m < \omega\} \subseteq \delta_n)$. To establish Theorem 5.2, it suffices to recursively enumerate A and B to meet the requirements

$$R_{2e}\colon A \cap \delta_{r(e)} \neq \{l(e)\}^{B \cap \delta_{r(e)}},$$
$$R_{2e+1}\colon B \cap \delta_{r(e)} \neq \{l(e)\}^{A \cap \delta_{r(e)}}.$$

The priority ordering of the requirements will be $R_0, R_1, R_2, \ldots$.

Construction of A and B. Stage 0. $A^{(0)} = B^{(0)} = \varnothing$, $x_{2e}^0 = f(r(2e),0)$, $x_{2e+1}^0 = f(r(2e+1),0)$, and $\bar{r}(e,0) = -1$ for all e.

Stage $s+1$. If $\{l(e)\}^{B^{(s)} \cap \delta_{r(e)}}(x_{2e}^s) = 0$ & $\bar{r}(2e,s) = -1$, then we say the requirement R_{2e} *requires attention*; if

$$\{l(e)\}_s^{A^{(s)} \cap \delta_{r(e)}}(x_{2e+1}^s) = 0 \;\&\; \bar{r}(2e+1,s) = -1,$$

then we say the requirement R_{2e+1} *requires attention*.

Case (i). $\exists\, i \leqslant s(R_i$ requires attention). We define

$$i_{s+1} = \mu i \leqslant s \quad (R_i \text{ requires attention}),$$

and say $R_{i_{s+1}}$ *receives attention*.

Subcase (i). $i_{s+1} = 2e$. Define $A^{(s+1)} = A^{(s)} \cup \{x_{2e}^s\}$, $B^{(s+1)} = B^{(s)}$,

$$\bar{r}(j,s+1) = \begin{cases} \bar{r}(j,s), & \text{if } j < 2e, \\ u\big(B^{(s)} \cap \delta_{r(e)};\, l(e), x_{2e}^s, s\big), & \text{if } j = 2e, \\ -1, & \text{if } j > 2e, \end{cases}$$

where u is the use function from Theorem 4.5,

$$x_j^{s+1} = \begin{cases} x_j^s & \text{if } j \leqslant 2e, \\ \mu x \in \{f(j,2m)\colon m < \omega\} \text{ such that} & \text{if } j > 2e \text{ and} \\ x \notin A^{(s+1)} \cup B^{(s+1)} \;\&\; x > x_j^s & \quad j \text{ is even,} \\ \&\; x > \max\{\bar{r}(k,s+1)\colon k \leqslant 2e\}, & \\ \mu x \in \{f(j,2m+1)\colon m < \omega\} \text{ such that} & \text{if } j > 2e \text{ and} \\ x \notin A^{(s+1)} \cup B^{(s+1)} \;\&\; x > x_j^2 & \quad j \text{ is odd.} \\ \&\; x > \max\{\bar{r}(k,s+1)\colon k \leqslant 2e\} & \end{cases}$$

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Subcase (ii). $i_{s+1} = 2e + 1$. Do as for subcase (i) with A and B interchanged.

Case (ii). Otherwise, no R_i requires attention for $i \leqslant s$. Define

$$A^{(s+1)} = A^{(s)}, \qquad B^{(s+1)} = B^{(s)},$$
$$\bar{r}(j, s+1) = \bar{r}(j, s) \quad \text{for all } j,$$
$$x_j^{s+1} = x_j^s \quad \text{for all } j \text{ (end of construction).}$$

LEMMA. *For every $e < \omega$, requirement R_e receives attention at most finitely often and is eventually satisfied.*

PROOF. The proof is by induction on e.

Fix i and assume by induction that the Lemma holds for all $j < i$. Then we can choose $\bar{s}$ such that

$$\bar{s} = \mu s\left[(\forall j < i)(\forall t > s)\left(R_j \text{ does not recieve attention at stage } t\right)\right].$$

Then

$$\forall t \geqslant \bar{s}\colon x_i^t = x_i^{\bar{s}} = x_i \;\&\; x_i \notin \left(A^{(\bar{s})} \cup B^{(\bar{s})}\right) \cap \delta_{r(i)}$$

where $x_i = \lim_t x_i^t$.

Case (i). $i = 2e$.

Subcase (i). R_{2e} never receives attention after $\bar{s}$. Then $\{l(e)\}^{B \cap \delta_{r(e)}}(x_{2e}) \neq 0$ since $\bar{r}(2e, s) \neq -1$ for sufficiently large s. Hence $x_{2e} \in \{l(e)\}^{B \cap \delta_{r(e)}}$. But x_{2e}^s is never put into $A^{(s)}$ for $s > \bar{s}$, and therefore $x_{2e} \notin A \cap \delta_{r(e)}$. Hence

$$A \cap \delta_{r(e)} \neq \{l(e)\}^{B \cap \delta_{r(e)}}.$$

Subcase (ii). R_{2e} receives attention at some stage $t + 1 > s$. Then $\{l(e)\}^{B^{(t)} \cap \delta_{r(e)}}(x_{2e}) = 0$, and

$$B^{(t)} \cap \delta_{r(e)} \cap \left\{x\colon x \leqslant u\left(B^{(t)} \cap \delta_{r(e)}; l(e), x_{2e}, t\right)\right\}$$
$$= B \cap \delta_{r(e)} \cap \left\{x\colon x \leqslant u\left(B^{(t)} \cap \delta_{r(e)}; l(e), x_{2e}, t\right)\right\},$$

so

$$\{l(e)\}^{B \cap \delta_{r(e)}}(x_{2e}) = 0.$$

But

$$x_{2e} \in \left(A^{(t+1)} - A^{(t)}\right) \cap \delta_{r(e)} \subseteq A \cap \delta_{r(e)},$$

that is,

$$A \cap \delta_{r(e)} \neq \{l(e)\}^{B \cap \delta_{r(e)}}.$$

Case (ii). $i = 2e + 1$. This is similar to Case (i).

6. An everywhere splitting theorem

THEOREM 6.1. *For every r.e. nonrecursive set C there is an everywhere simple set A such that $\forall i(\delta_i \neq \varnothing \Rightarrow C \not\leqslant_T A \cap \delta_i)$.*

PROOF. It suffices to construct an A to satisfy, for all e, the requirements

$$N_e'\colon C \neq \{l(e)\}^{A \cap \delta_{r(e)}},$$
$$N_e''\colon \delta_e - A \neq \varnothing,$$
$$P_e\colon W_e \text{ is infinite} \Rightarrow W_e \cap A \neq \varnothing,$$

where, by Lemma 1.5(2) we may assume $(\forall i)(\delta_i \neq \varnothing)$. The order of priorities is $N_0', N_0'', P_0, N_1', N_1'', P_1, N_2', \ldots$. Let $\{C_s\}_{s \in \omega}$ be a recursive enumeration of C.

Construction of A. Stage 0. $A^{(0)} = \varnothing$.

Stage $s + 1$. Given $A^{(s)}$, define, for all e, recursive functions $\bar{l}$, $\bar{r}_1$ and $\bar{r}_2$ as follows:

$$\bar{l}(e,s) = \max\Big\{x\colon \Big(\forall y < x\Big[C_s(y) = \{l(e)\}_s^{A^{(s)} \cap \delta_{r(e)}}(y)\Big]\Big)\Big\},$$
$$\bar{r}_1(e,s) = \max\Big\{u\Big(A^{(s)} \cap \delta_{r(e)}; l(e), x, s\Big)\colon x \leqslant \bar{l}(e,s)\Big\},$$
$$\bar{r}_2(e,s) = \min\big(\delta_e - A^{(s)}\big),$$

where $\{e\}_s^A(x) = y$ means $x, y, e < s$ & $\{e\}^A(x)$ is defined and equals y in strictly less than s steps. For each $i < s$ if $W_{i,s} \cap A^{(s)} = \varnothing$ and

$$(\exists x)\big[x \in W_{i,s} \mathbin{\&} (\forall e \leqslant i)\big[\bar{r}_1(e,s) + \bar{r}_2(e,s) < x\big]\big],$$

then put the least such x into A, otherwise do nothing (end of construction).

LEMMA 1. *For every e, the injury set*

$$I_e = \Big\{x\colon (\exists s)\big[x \in A^{(s+1)} - A^{(s)} \mathbin{\&} x \leqslant \bar{r}_1(e,s) + \bar{r}_2(e,s)\big]\Big\}$$

is a finite set.

PROOF. Each positive requirement P_i contributes at most one element to A A by the construction. But N_e' or N_e'' can be injured by P_i only if $i < e$.

LEMMA 2. $(\forall e)[\delta_e - A \neq \varnothing]$.

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PROOF. Assume for a contradiction that $\delta_e - A = \varnothing$. By Lemma 1, choose s'' such that N_e'' is never injured after stage s''. Then $\forall s \geqslant s''(\bar{r}_2(e,s) = \bar{r}_2(e,s''))$, so $\bar{r}_2(e,s'') = \min(\delta_e - A)$, contrary to hypothesis.

LEMMA 3. $(\forall e)[C \neq \{l(e)\}^{A \cap \delta_{r(e)}}]$.

PROOF. Assume for a contradiction that $C = \{l(e)\}^{A \cap \delta}r(e)$. Then

$$\lim_s \bar{l}(e,s) = \infty. \tag{6.1}$$

By Lemma 1, choose s' such that N_e' is never injured after stage s'. For any $n \in \omega$, by (6.1) find $s > s'$ such that $\bar{l}(e,s) > n$. It follows by induction on $t \leqslant s$ that

(6.2)

$$(\forall t \geqslant s)\left[\bar{l}(e,t) > n \;\&\; \bar{r}_1(e,t) \geqslant \max\left\{u\left(A^{(s)} \cap \delta_{r(e)};\, l(e), x, s\right) : x \leqslant n\right\}\right],$$

and hence that

$$\{l(e)\}^{A^{(s)} \cap \delta_{r(e)}}(n) = \{l(e)\}^{A^{(s)} \cap \delta_{r(e)}}(n) = \{l(e)\}^{A \cap \delta_{r(e)}}(n) = C(n),$$

that is, C is a recursive set, contrary to hypothesis.

Now we show (6.2) by induction on $t \leqslant s$. Since $\bar{l}(e,s) > n$, (6.2) holds for $t < s$. Assume it holds for t. Then $\bar{r}_1(e,t)$ and $s > s'$ ensure that

$$\forall x \leqslant n, \forall z < u\left(A^{(t)} \cap \delta_{r(e)}; l(e), x, t\right)\left(A^{(t+1)} \cap \delta_{r(e)} \cap \{x : x \leqslant z\} \right.$$
$$\left. = A^{(t)} \cap \delta_{r(e)} \cap \{x : x \leqslant z\}\right).$$

Thus $\{l(e)\}_t^{A^{(t)} \cap \delta_{r(e)}}(x)_\downarrow \Rightarrow \{l(e)\}^{A^{(t+1)} \cap \delta_{r(e)}}(x)_\downarrow$ converges and equals $\{l(e)\}^{A^{(t)} \cap \delta_{r(e)}}(x)$ for $x \leqslant n$. So $\bar{l}(e,t+1) > n$ unless $(\exists x < \bar{l}(e,t))[C_{t+1}(x) \neq C_t(x)]$. Assume there is such an x. Define x', t' as follows:

$$x' = \mu x\left\{x \leqslant n : (\exists t \geqslant s)\left[x < \bar{l}(e,t) \;\&\; C_{t+1}(x) \neq C_t(x)\right]\right\},$$
$$t' = \mu t\left\{t \geqslant s : C_{t+1}(x') \neq C_t(x')\right\}.$$

By the definition of $r(e,s)$,

$$C_{t'}(x') \neq \{l(e)\}^{A^{(t)} \cap \delta_{r(e)}}(x')$$

contrary to $C = \{(e)\}^{A \cap \delta_{r(e)}}$. It remains to show

$$\bar{r}_1(e,t+1) \geqslant \max\left\{u\left(A^{(s)} \cap \delta_{r(e)}; l(e), x, s\right) : x \leqslant n\right\}. \tag{6.3}$$

In fact, $\bar{r}_1(e,t+1) = \max\{u(A^{(t+1)} \cap \delta_{r(e)}; l(e), x, t+1) : x \leqslant \bar{l}(e,t+1)\}$, and $\bar{l}(e,t+1) > n$, and $t \geqslant s > s'$, so

$$\max\left\{u\left(A^{(s)} \cap \delta_{r(e)}; l(e), x, s\right) : x \leqslant n\right\}$$
$$\leqslant \max\left\{u\left(A^{(t+1)} \cap \delta_{r(e)}; l(e), x, t+1\right) : x \leqslant \bar{l}(e,t+1)\right\}$$

and hence (6.3) holds.

LEMMA 4. $(\forall e)[\lim_{s\to\infty} \bar{r}_1(e,s)$ *exists and is finite*].

PROOF. By Lemma 3, choose $n = (\mu x)[C(x) \neq \{l(e)\}^{A\cap\delta_{r(e)}}(x)]$. Choose s' sufficiently large such that, for all $s \geqslant s'$,

$$(\forall x < n)\left[\{l(e)\}_s^{A^{(s)}\cap\delta_{r(e)}}(x)_{\downarrow} = \{l(e)\}^{A\cap\delta_{r(e)}}(x)\right], \tag{6.4}$$

$$(\forall x \leqslant n)\left[C_s(x) = C(x)\right], \text{ and} \tag{6.5}$$

$$N_e' \text{ is not injured at stage } s. \tag{6.6}$$

Case (i). $(\forall s \geqslant s')[\{l(e)\}_s^{A^{(s)}\cap\delta_{r(e)}}(n)\uparrow]$. Then $C_s(n) \neq \{l(e)\}^{A^{(s)}\cap\delta_{r(e)}}(n)$ for all $s \geqslant s'$. Hence $(\forall s \geqslant s')[\bar{r}_1(e,s) = \bar{r}_1(e,s')]$.

Case (ii). $(\exists t \geqslant s')[\{l(e)\}_t^{A^{(t)}\cap\delta_{r(e)}}(n)_{\downarrow}]$. Take such a t. By (6.4) and the definition of n, $(\forall s \geqslant t)[\bar{l}(e,s) = n]$, so

$$(\forall s \geqslant t)\left[\bar{r}_1(e,s) \geqslant u\left(A^{(s)} \cap \delta_{r(e)}; l(e), n, s\right)\right].$$

Hence, by induction on $s \geqslant t$ from (6.6) we have

$$(\forall s \geqslant t)\left[\{l(e)\}_s^{A^{(s)}\cap\delta_{r(e)}}(n) = \{l(e)\}_t^{A^{(t)}\cap\delta_{r(e)}}(n)\right].$$

Thus,

$$(\forall s \geqslant t)\left[\{l(e)\}^{A\cap\delta_{r(e)}}(n) = \{l(e)\}^{A^{(s)}\cap\delta_{r(e)}}(n)\right].$$

But $C(n) \neq \{l(e)\}^{A\cap\delta_{r(e)}}(n)$. Therefore $(\forall s \geqslant t)[\bar{r}_1(e,s) = \bar{r}_1(e,t)]$. Hence $r_1(e,t) = \lim_s \bar{r}_1(e,s)$.

LEMMA 5. $(\forall e)[W_e$ *infinite* $\Rightarrow W_e \cap A \neq \varnothing]$.

PROOF. By Lemma 4, let $\bar{r}_1(e) = \lim_{s\to\infty} \bar{r}_1(e,s)$ and $R(e) = \max\{\bar{r}_1(i): i \leqslant e\}$. Now if W_e is infinite then $(\exists x)[x \in W_e \;\&\; x > R(e)]$. But then $W_e \cap A \neq \varnothing$.

Finally, note that $\bar{A} \cap \delta_i$ is infinite since $(\forall e)[\delta_e - A \neq \varnothing]$, and hence A is everywhere simple. This completes the proof of Theorem 6.1.

THEOREM 6.2. *For every everywhere nonrecursive r.e. set C there is an everywhere simple set A such that* $\forall i(\delta_i \neq \varnothing \Rightarrow C \cap \delta_i \not\leqslant_T A \cap \delta_i)$.

PROOF. It suffices to construct A to satisfy, for all e, the requirements

$$N_e': C \cap \delta_{r(e)} \neq \{l(e)\}^{A \cap \delta_{r(e)}},$$
$$N_e'': \delta_e - A \neq \varnothing,$$
$$P_e: W_e \text{ is infinite implies } W_e - A \neq \varnothing.$$

The order of priorities is $N_0', N_0'', P_0, N_1', N_1'', P_1, N_2', \ldots$. Without loss of generality we may assume all $\delta_i \neq \varnothing$.

Construction of A. Stage 0. $A^{(0)} = \varnothing$.

Stage $s+1$. Given $A^{(s)}$, define, for all e, recursive functions

$$\bar{l}(e,s) = \max\Big\{x\colon (\forall y < x)\Big[\min\{C_s(y), \delta_{r(e)}(y)\} = \{l(e)\}_s^{A^{(s)} \cap \delta_{r(e)}}(y)\Big]\Big\},$$
$$\bar{r}_1(e,s) = \max\big\{u\big(A^{(s)} \cap \delta_{r(e)}; l(e), x, s\big)\colon x \leqslant \bar{l}(e,s)\big\},$$
$$\bar{r}_2(e,s) = \min\big(\delta_e - A^{(s)}\big).$$

For each $i < s$ if $W_{i,s} \cap A^{(s)} = \varnothing$ and

$$(\exists x)\big[x \in W_{i,s} \ \&\ (\forall e \leqslant i)[\bar{r}_1(e,s) + \bar{r}_2(e,s) < x]\big]$$

then enumerate the least such x into A; otherwise do nothing (end of construction).

The proof is similar to the proof of Theorem 6.1.

THEOREM 6.3. *Let B and C be r.e. sets such that C is non-recursive. Then there exist r.e. sets A_0 and A_1 such that*

(i) $A_0 \cup A_1 = B$ *and* $A_0 \cap A_1 = \varnothing$ *and*

(ii) $(\forall i)\delta_i \neq \varnothing \Rightarrow [C \not\leqslant_T A_0 \cap \delta_i \ \&\ C \not\leqslant_T A_1 \cap \delta_i]$.

PROOF. If B is a finite set, then Theorem 6.3 holds. So we can assume B is an infinite set.

Let $\{B_s\}_{s \in \omega}$, $\{C_s\}_{s \in \omega}$ be recursive enumerations of B and C such that B_0 is empty and B_{s+1} contains exactly one more element than B_s. It suffices to construct r.e. sets A_0 and A_1 to satisfy the single positive requirement

$$P\colon x \in B_{s+1} - B_s \Rightarrow \big[x \in A_0^{(s+1)} \text{ or } x \in A_1^{(s+1)}\big],$$

and the negative requirements for all e

$$N_e'\colon C \neq \{l(e)\}^{A_0 \cap \delta_{r(e)}},$$
$$N_e''\colon C \neq \{l(e)\}^{A_1 \cap \delta_{r(e)}}.$$

The ordering of priorities is $N_0', N_0'', P_0, N_1', N_1'', P_1, N_2', \ldots$.

Construction of A_0 *and* A_1. *Stage* 0. $A_0^{(0)} = \varnothing$, $A_1^{(0)} = \varnothing$.

Stage $s+1$. Given $A(s)_0$ and $A_1^{(s)}$, define recursive functions

$$\bar{l}^0(e,s) = \max\left\{x \colon (\forall y < x)\left[C_s(y) = \{l(e)\}_s^{A_0^{(s)} \cap \delta_{r(e)}}(y)\right]\right\},$$

$$\bar{l}^1(e,s) = \max\left\{x \colon (\forall y < x)\left[C_s(y) = \{l(e)\}_s^{A_1^{(s)} \cap \delta_{r(e)}}(y)\right]\right\},$$

$$\bar{r}^0(e,s) = \max\left\{u\left(A_0^{(s)} \cap \delta_{r(e)}; l(e), x, s\right) \colon x \leqslant \bar{l}^0(e,s)\right\},$$

$$\bar{r}^1(e,s) = \max\left\{u\left(A_1^{(s)} \cap \phi d_{r(e)}; l(e), x, s\right) \colon x \leqslant \bar{l}^1(e,s)\right\},$$

where u is the use function. Let x be the unique element in $B_{s+1} - B_s$.

Case (i). $\exists e \leqslant s[x \leqslant \bar{r}^0(e,s)$ or $x \leqslant \bar{r}^1(e,s)]$. Set $e_{s+1} = \mu e[x \leqslant \bar{r}^0(e,s)$ or $x \leqslant \bar{r}^1(e,s)]$.

Subcase (i). $x \leqslant \bar{r}^0(e_{s+1}, s)$. Define $A_0^{(s+1)} = A_0^{(s)}$, $A_1^{(s+1)} = A_1^{(s)} \cup \{x\}$.

Subcase (ii). $x > \bar{r}^0(e_{s+1}, s)$, $x \leqslant \bar{r}^1(e_{s+1}, s)$. Define $A_0^{(s+1)} = A_0^{(s)} \cup \{x\}$, $A_1^{(s+1)} = A_1^{(s)}$.

Case (ii). $\forall e \leqslant s[x > \bar{r}^0(e,s)$ & $x > \bar{r}^1(e,s)]$. Define $A_0^{(s+1)} = A_0^{(s)} \cup \{x\}$, $A_1^{(s+1)} = A_1^{(s)}$. (end of construction).

Now we can define the injury sets

$$I_e^0 = \left\{x \colon (\exists x)\left[x \in A_0^{(s+1)} - A_0^{(s)} \;\&\; x \leqslant \bar{r}^0(e,s)\right]\right\},$$

$$I_e^1 = \left\{x \colon (\exists x)\left[x \in A_1^{(s+1)} - A_1^{(s)} \;\&\; x \leqslant \bar{r}^1(e,s)\right]\right\}.$$

It follows by induction on e that

(1) I_e^0, I_e^1 are finite,

(2) $C \neq \{l(e)\}^{A_0 \cap \delta_{r(e)}}$, $C \neq \{l(e)\}^{A_1 \cap \delta_{r(e)}}$, and

(e) $\bar{r}^0(e) = \lim_s \bar{r}^0(e,s)$, $\bar{r}^1(e) = \lim_s \bar{r}^1(e,s)$ exist and are finite.

The proof is similar to ther proof of Theorem 6.1.

We can strengthen Theorem 6.3 as follows.

THEOREM 6.4. *Let* B, C *be r.e. sets such that* C *is everywhere nonrecursive. Then there exist r.e. sets* A_0, A_1 *such that*

(i) $A_0 \cup A_1 = B$, $A_0 \cap A_1 = \varnothing$, *and*

(ii) $(\forall i)\ (\delta_i \neq \varnothing \Rightarrow C \cap \delta_i \not\leqslant_T A_0 \cap \delta_i \;\&\; C \cap \delta_i \not\leqslant_T A_1 \cap \delta_i)$.

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