ON SUBPLANES OF FREE PLANES

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In 1945, L. I. Kopejkina **(4)**, at the suggestion of A. G. Kurosh, began a programme of studying the properties of free projective planes and the analogies between free planes and free groups. In this paper, this study will be extended by proving a tool theorem and several of its consequences. The theorem deals with the existence of "minimal free generators" for subplanes of free planes.

A set of points and lines and an incidence relation are said to form a projective plane if the following three axioms are satisfied:

I. Any two distinct points are incident with exactly one line.

II. Given any two distinct lines, there is exactly one point incident with both.

III. There exist four points, no three incident with one line.

If, on the other hand, the following two axioms are satisfied, the set is said to constitute a *partial plane*:

1. There exists at most one line through any two distinct points.
2. There is at most one point incident with any two distinct lines.

Any extension of a partial plane, π_0, formed by letting each new point be the intersection of exactly two old lines, and each new line the union of exactly two old points, is called a *free extension* of π_0. A partial plane, π_0, which satisfies conditions 1 and 2 and which, after a finite number of free extensions, satisfies III, is called *non-degenerate*. Otherwise, π_0 is called *degenerate*.

Given any non-degenerate partial plane π_0 which is not a projective plane, we can define new partial planes as follows. For every two points in π_0 not already connected by a line in π_0, we adjoin a new line. Let L_0 be the set of all new lines adjoined in this manner, and set

$$\pi_1 = \pi_0 \cup L_0. \tag{1}$$

Then, for every two lines of π_1 not already intersecting in a point of π_1, adjoin a new point as their point of intersection. Let P_0 be the set of these points, and

$$\pi_2 = \pi_1 \cup P_0. \tag{2}$$

One can define $\pi_3, \pi_4, \ldots$ in an analogous manner, and set

$$\pi = \bigcup_{i=0}^{\infty} \pi_i, \tag{3}$$

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with the obvious incidences. The partial plane π is easily seen to be a projective plane, and is called the *free completion* of π_0. The free plane π^n $(n > 2)$ is defined to be the free completion of the partial plane $\pi_0{}^n$ consisting of n points on a line and two points not on that line.

Let π be any plane, and ρ_0 be a partial plane contained in π. Then if $\rho_1 = \rho_0 \cup$ (those points of π which are intersections of lines of ρ_0), $\rho_2 = \rho_1 \cup$ (those lines of π which are intersections of points of ρ_1), . . . , and if $\rho = \cup \rho_i$, ρ is *the subplane of π generated by* ρ_0. The set of points and lines, ρ_0, generates ρ *freely* if each ρ_i is the free extension of ρ_{i-1}, for $i \geqslant 0$. A notion which can now be introduced is that of the *stage* of a point or line relative to a given set of generators. If $\pi_0 \subset \pi$ is a partial plane which generates π freely, and if x is a point or line of π such that $x \in \pi_n$, $x \notin \pi_{n-1}$, then define

$$s(x) = n, \tag{4}$$

i.e., x is defined at the nth stage in the construction of π by means of the freely generating set π_0. For the sake of definiteness, let x be a point of π. Then if x is contained in a subplane ρ of π, which is freely generated by ρ_0, x has a finite representation in terms of the elements $\{y_i\}$ of ρ_0. For x, in the construction of ρ, was defined by a unique pair of lines L_1 and L_2. Likewise, each L_i, if not already an element of ρ_0, was defined by a unique pair of points. Continuing (the process must be finite for any x), we arrive at a representation of x which involves a finite number of elements of ρ_0.

In **(3)**, Hall introduced the idea of the *rank* of a finite partial plane π_0, by defining

$$r(\pi_0) = 2(p + l) - i, \tag{5}$$

where p is the number of points in π_0, l the number of lines, and i the number of incidences holding in π_0. He further showed (Theorem 4.10) that $r(\pi_0)$ is invariant under free extension, and was able to deduce from this that if $n \neq m$, the free planes π^n and π^m are non-isomorphic.

Finally, we wish to introduce the notion of the *free product* of a set of partial planes $\{\pi_i\}$ (some of which can be projective), which is defined as the free completion of the set-theoretic union of all the π_i, and written

$$\pi = \prod_i{}^* \pi_i. \tag{6}$$

This definition does not assume that all of the π_i are non-degenerate. In fact, in taking free products, the case where some of the π_i are extremely degenerate is quite interesting. In particular, let π_0 be a non-degenerate partial plane and let π_1 consist of a single element (point or line) x not incident with an element of π_0. In this situation the element x is said to have been *freely adjoined* to π_0 in the free product $\pi_0 * \pi_1$.

Kopejkina shows **(4)** that the rank of the free product of a finite number of planes of finite rank is the sum of their ranks. Thus, if one has a non-degenerate partial plane π_0 of finite rank, and adjoins freely a point or line x, then

$$r(\pi_0 * x) = r(\pi_0) + r(x) = r(\pi_0) + 2,$$

since $r(x) = 2$. Kopejkina's theorem also holds in the very special situation when π_1 consists merely of a single point (line) contained in a single line (point) of π_0, with no further incidence holding. Clearly, $r(\pi_0 \cup \pi_1) = r(\pi_0) + 1$, since one element and one incident have been added to π_0. This process is described as the *free extension of a partial plane by a plane of rank* 1.

Let π be a free plane generated freely by the partial plane π_0 which is assumed to consist of a single line containing all but two of the points of π_0, and two points off π_0 (with no further incidences holding). The partial plane π_k consists of all those points and lines of π which have stage $\leqslant k$ relative to the generation of π by π_0. Now let τ be any non-degenerate subplane of π, and denote by τ_k the subplane generated by $\{\tau \cap \pi_k\}$. Further define

$$M_k = \tau \cap \pi_k - \tau_{k-1}. \tag{7}$$

Then we have the following theorem.

THEOREM 1. *τ is a free plane. In fact, τ is freely generated by*

$$M = \bigcup_{i=0}^{\infty} M_i.$$

Proof. That every subplane of a free plane is free was first proved by Kopejkina **(4)** and appears in English in Skornyakov's paper **(6**, Theorem 3**)**. The second statement of the theorem is a trivial consequence of Skornyakov's proof in which he shows that $\tau_k = \tau_{k-1} * M_k$.

As above, let π be a free plane generated freely by π_0, and let τ be a subplane of π. If $x \in \tau$ is any element of τ, then $s(x) = n$ for some integer n. By definition of the sets M_k, either $x \in M_n$, or x is generated by elements contained in

$$\bigcup_{i=0}^{n-1} M_n.$$

This proves the following theorem.

THEOREM 2. *Let π be a free plane and let π_0 be a set of free generators for π consisting of a collection of independent points on a line, the line L, and two points off L. Finally if $x \in \pi$ is any element of π, let $s(x)$ denote the stage in the generation of π by π_0 in which x first appears. Then if τ is any subplane of π, and M is defined as in Theorem* 1 *above, given any $x \in \tau$, there exists a finite subset of M, $\{m_i\}$, such that x is generated by the $\{m_i\}$ and $s(x) > s(m_i)$, for each m_i.*

The results of Theorem 2 can be used to prove some interesting theorems, in a manner analogous to the existence of "level sets" **(2)** in the study of free groups.

THEOREM 3. *The set S_m (ordered by inclusion) of all subplanes of rank $\leqslant m$ of a free plane π^k of finite rank satisfies the ascending chain condition.*

Proof. We shall prove a proposition which immediately implies the correctness of Theorem 3. Let

$$\pi_1{}^n \subset \pi_2{}^n \subset \ldots \subset \pi^k \tag{8}$$

be an ascending chain of subplanes of type π^n (and rank $n+6$) of π^k; then this chain must be finite.

This proposition is proved by induction. If $n = 2$, let $\{y_i\}$ be the set M of Theorems 1 and 2 corresponding to $\pi_1{}^2$. Then any $\pi_j{}^2$ must have a set $\{x_i\}$ of generators constructed in the same way. Now, $\pi_j{}^2$ is finitely generated, and hence $\{x_i\}$ will be finite. Let $x_0 \in \{x_i\}$ be an element of maximal stage in $\{x_i\}$. Then if x_0 is a point, x_0 cannot lie on two lines of $\{x_i\}$. For let $x_0 \in L_1, L_2$. Then either $s(L_1) < s(x_0)$ and $s(L_2) < s(x_0)$, or $s(L_i) > s(x_0)$ for $i = 1$ or 2. In the former case $x_0 \notin M$, and in the latter, x_0 is not of maximal stage in $\{x_i\}$. Likewise, if x_0 is a line, x_0 cannot contain more than one point of $\{x_i\}$. Thus the rank, $r(\rho)$, of the partial plane ρ consisting of $\{x_i\} - x_0$ is equal to 6 or 7, since $8 = r\{x_i\} = r(\rho) + 1$ or $r(\rho) + 2$. Now there must exist at least one element y of the set $\{y_i\}$ such that x_0 is necessary in the generation of y from $\{x_i\}$. For otherwise the plane $\pi_1{}^2$ would be contained in the plane generated by ρ, which is impossible since $r(\rho) < 8$, and ρ generates a degenerate plane. By Theorem 2, then, $s(y) \geqslant s(x_0)$ for some $y \in \{y_i\}$. In particular, then

$$s_0 = s(y_0) \geqslant s(x_i), \tag{9}$$

for all i, where $y_0 \in \{y_i\}$ has maximal stage in $\{y_i\}$. Then if $\pi_j{}^2$ is any plane of rank 8 containing $\pi_1{}^2$, we have shown that $\pi_j{}^2$ must have a set of generators all of which have stage $\leqslant s_0$. But, by assumption, π^k was finitely generated, which implies that $\pi_n{}^k$ {the set of points and lines of π^k of stage $\leqslant n$} is finite. Thus we have shown that there can be only a finite number of planes of type π^2 containing $\pi_1{}^2$, and this is even stronger than the proposition we were trying to prove for $n = 2$. For $n > 2$, assume the proposition to be true for all integers $<n$, and consider the chain (8). For each pair of integers $(i, i+1)$, either $\pi_i{}^n$ is contained in a subplane η of π^n_{i+1} which is a free factor of π^n_{i+1} (i.e., $\pi^u_{i+1} = \eta * \eta'$, $\pi^n_i \subset \eta$) or not. By the induction hypothesis, the former possibility can occur only a finite number of times, for an ascending chain of subplanes of type π^{n-1} or less is defined by these occurrences, and would in that case be infinite, contrary to assumption. Thus there is an integer N such that if $i \geqslant N$, $\pi_i{}^n$ is not contained in a free factor of π^n_{i+1}. Let $\{y_i\}$ be the set of generators M for $\pi_N{}^n$ as discussed in Theorems 1 and 2. Let $\{x_i\}$ be the set M for π^n_{N+1}. As in the case $n = 2$, if x_0 is an element of $\{x_i\}$ of maximal stage, then x_0 cannot be incident with more than one other element of $\{x_i\}$. Again it must be the case that for some $y \in \{y_i\}$, x_0 is one of the generators for y considered as an element of π^u_{N+1}. For, if not, $\pi_N{}^n$ is contained in the subplane ρ generated by $\{x_i\} - x_0$. But since x_0 is not incident with more than

one element of $\{x_i\}$. $\pi_{N+1}^n = \rho * x_0$, contradicting the assumption that π_N^n is contained in no free factor of π_{N+1}^n. Thus $s(y) \geqslant s(x_0)$. Again, if y_0 is an element of $\{y_i\}$ of maximal stage, we have $s_0 = s(y_0) \geqslant s(x_i)$. Now if $\{z_i\}$ is the set M of generators for π_{N+2}^n, one easily sees that $s_0 \geqslant s(x_0) \geqslant s(z_i)$ for all i. Continuing, we see that every one of the π_{N+j}^n has a set of generators each element of which has stage $\leqslant s_0$. As in the case $n = 2$, only a finite number of such sets of generators can exist, and hence the chain must be finite as asserted.

THEOREM 4. *If π^k is a free plane of finite rank, and if ρ is a subplane of π^k which is also of finite rank, then there are only finitely many planes, ϵ, $\rho \subset \epsilon \subset \pi^k$, such that ρ is contained in no free factor of ϵ.*

Proof. Let π_0 generate π^k freely, and define $s(x)$ relative to π_0. Then ρ has a set M of free generators $\{x_i\}$ as described in Theorems 1 and 2. Now if $\epsilon \subset \rho$ is a subplane of π^k, then ϵ also has such a system $\{z_i\}$. If ρ is contained in no free factor of ϵ, then, as above, if z_0 is an element of $\{z_i\}$ of maximal stage, z_0 must appear in at least one of the expressions of the elements $x \in \{x_i\}$ in terms of $\{z_i\}$. By Theorem 2,

$$s(z_i) \leqslant \max s(z_i) \leqslant \max s(x_i),$$

and, as before, since π_0 is finite there are only a finite number of points and lines of stage $\leqslant n$ for any finite number n, and a fortiori there can be only a finite number of such planes ϵ.

COROLLARY. *Let π^k be a free plane of finite rank. If π^k contains a subplane ρ of finite rank n which is contained in no subplane of rank $<n$, then every descending chain*

$$\epsilon_1 \supset \epsilon_2 \supset \ldots \epsilon_n \supset \ldots \supset \rho, \tag{10}$$

where $r(\epsilon_i) = n$, must be finite.

Proof. If ρ is contained in no subplane of rank $<n$, then ρ cannot be contained in a free factor of any of the ϵ_i, and so there can only be a finite number of the ϵ_i by Theorem 4.

One shortcoming of Theorems 3 and 4 is that π^k was always taken to be finitely generated—as seemed necessary if Theorem 2 was to be applicable. In **(1)**, however, Dembowski has shown the following theorem.

THEOREM 5. *π^2 contains a subplane isomorphic to π^∞.*

Thus, as a corollary to Theorem 5, we can state the following theorem.

THEOREM 6. *Theorems 3 and 4 and the Corollary to Theorem 4 hold even if π^k is an infinitely generated free plane.*

Recall that in the proof of Theorem 3, it was shown that a subplane of rank 8 can be contained in at most a finite number of subplanes of rank 8. In general, the results of Theorem 3 might be more or less trivial if this stronger result were true for subplanes of arbitrary rank. To show that this is not at all the case, the following example is of interest. Let π_0 be the partial plane consisting of

$$L_1 : A_1A_2A_3, \qquad L_2 : B_1B_2, \tag{11}$$

where $$L_1 \cap L_2 \neq A_i, \qquad i = 1, 2, 3.$$

Then the free completion of π_0 is π^3, the free plane of rank 9. Let π^2 be the subplane generated by A_1, A_2, B_1, B_2. Then π^2 is a free plane of rank 8. Now consider the free extension of π_0 given by:

$$\begin{aligned} &L_1 : A_1A_2A_3A_4, \\ &L_2 : B_1B_2, \\ &L_3 : A_2B_1P_1, \\ &L_4 : A_3B_2P_1, \\ &L_5 : A_2B_2P_2, \\ &L_6 : A_3B_1P_2, \\ &L_7 : P_1P_2A_4. \end{aligned} \tag{12}$$

Now the partial plane $\pi_0{}^1$ given by:

$$\begin{aligned} &L_1 : A_1A_2A_4, \\ &L_2 : B_1B_2, \\ &L_3 : A_2B_1, \\ &L_5 : A_2B_2 \end{aligned} \tag{13}$$

is complete in π_0, i.e., if two points (lines) of $\pi_0{}^1$ are on a line (point) of π_0, that line (point)is already in $\pi_0{}^1$. By a theorem of Hall (**3**, Theorem 4.3) the free extension of $\pi_0{}^1$, $\pi_1{}^3$, is contained in the free extension of π_0, and the two free extensions are not equal. But, by free contraction, $\pi_0{}^1$ is reducible to a configuration generating a free plane of rank 9, so $\pi_1{}^3 \approx \pi^3$. This construction can be repeated indefinitely and an *infinite* decreasing chain

$$\pi^3 \supsetneqq \pi_1{}^3 \supsetneqq \pi_2{}^3 \supsetneqq \ldots \supsetneqq \pi^2. \tag{14}$$

obtained, where the $\pi_i{}^3$ are all of rank 9. But by another result of Hall (**3**, Theorem 4.5) π^2 contains a free subplane ρ of rank 9. To get an arbitrarily long ascending chain, take

$$\rho \subset \pi_n{}^3 \subset \pi_{n-1}^3 \subset \ldots \subset \pi_1{}^3 \leqslant \pi^3, \tag{15}$$

when n can be taken arbitrarily large. Thus a subplane ρ of finite rank can be determined such that for any integer n there exists an ascending chain of subplanes whose rank is $r(\rho)$ and whose first element is ρ. This example shows that, except for rank 8, Theorem 3 is the best theorem obtainable.

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