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A. I. Badulescu and D. Renard

Abstract

In a paper by Badulescu [*Global Jacquet–Langlands correspondence, multiplicity one and classification of automorphic representations*, Invent. Math. **172** (2008), 383–438], results on the global Jacquet–Langlands correspondence, (weak and strong) multiplicity-one theorems and the classification of automorphic representations for inner forms of the general linear group over a number field were established, under the assumption that the local inner forms are split at archimedean places. In this paper, we extend the main local results of that article to archimedean places so that the above condition can be removed. Along the way, we collect several results about the unitary dual of general linear groups over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ which are of independent interest.

Contents

1 Introduction **1116**
1.1 Preliminary notation . . . 1116
1.2 Classification of unitary representations . . . 1117
1.3 Jacquet–Langlands correspondence for unitary representations . . . 1118
1.4 Character identities and ends of complementary series . . . 1119
1.5 Global results . . . 1119
1.6 Some comments . . . 1122
2 Notation **1122**
2.1 Multisets . . . 1122
2.2 Local fields and division algebras . . . 1122
2.3 GL . . . 1122
2.4 Standard parabolic and Levi subgroups . . . 1123
3 The Langlands classification **1124**
4 Jacquet–Langlands correspondence **1125**
5 Support and infinitesimal character **1127**
5.1 Non-archimedean case . . . 1127
5.2 Archimedean case . . . 1127
6 Bruhat G-order **1128**
7 Unitary dual **1129**
7.1 Representations $u(\delta, n)$ and $\pi(\delta, n; \alpha)$ . . . 1129
7.2 Tadić's hypotheses and classification of the unitary dual . . . 1130

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8 Classification of generic irreducible unitary representations 1131
9 Classification of discrete series: archimedean case 1132
9.1 $A = \mathbb{C}$ . . . 1132
9.2 $A = \mathbb{R}$ . . . 1132
9.3 $A = \mathbb{H}$ . . . 1134
10 Hypothesis $U(3)$ for $A = \mathbb{H}$ 1138
11 Hypothesis $U(1)$ for the archimedean case 1140
11.1 $A = \mathbb{C}$ . . . 1141
11.2 $A = \mathbb{R}$ . . . 1141
11.3 $A = \mathbb{H}$ . . . 1142
12 Vogan's classification and $U(0)$ in the archimedean case 1143
13 Jacquet–Langlands correspondence in the archimedean case 1147
13.1 Jacquet–Langlands correspondence and coherent families . . . 1147
13.2 Jacquet–Langlands correspondence and cohomological induction . . . 1148
14 Character formulas and ends of complementary series 1151
14.1 $A = \mathbb{C}$ . . . 1151
14.2 $A = \mathbb{R}$ . . . 1152
14.3 $A = \mathbb{H}$ . . . 1153
15 Compatibility and further comments 1154
16 Notation for the global case 1155
17 Further insight into some local results 1157
17.1 Hypothesis $U(1)$ and the transfer of $u(\delta, k)$ . . . 1157
17.2 Irreducibility and transfer of all unitary representations . . . 1158
18 Global results 1159
19 L-functions, ϵ-factors and transfer 1161
References 1162

1. Introduction

In [Bad08], results on the global Jacquet–Langlands correspondence, (weak and strong) multiplicity-one theorems and the classification of automorphic representations for inner forms of the general linear group over a number field were established, under the condition that the local inner forms are split at archimedean places. The main goal of this paper is to remove this hypothesis. The paper consists of two parts: in the first part, we extend the main local results of [Bad08] to archimedean places; in the second part, we explain how to use these local results to establish the global results in their full generality. Along the way, we collect several results about the unitary dual of general linear groups over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ which are of independent interest. Let us now describe in more detail the content of this paper.

1.1 Preliminary notation

Let A be one of the division algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. If $A = \mathbb{R}$ or $A = \mathbb{C}$ and $n \in \mathbb{N}^\times$, we denote by det the determinant map on $\mathrm{GL}(n, A)$ (taking values in A). If $A = \mathbb{H}$, let RN be the reduced norm map on $\mathrm{GL}(n, \mathbb{H})$ (taking values in $\mathbb{R}_+^\times$).

If $n \in \mathbb{N}$ and $\sum_{i=1}^{s} n_i = n$ is a partition of n, then the group $\mathrm{GL}(n_1, A) \times \mathrm{GL}(n_2, A) \times \cdots \times \mathrm{GL}(n_s, A)$ is identified with the subgroup of $\mathrm{GL}(n, A)$ of block-diagonal matrices of

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sizes $n_1, \dots, n_s$. Let $G_{(n_1,\dots,n_s)}$ denote this subgroup and $P_{(n_1,\dots,n_s)}$ the parabolic subgroup of $\mathrm{GL}(n, A)$ containing $G_{(n_1,\dots,n_s)}$ and the Borel subgroup of invertible upper-triangular matrices. For $1 \leqslant i \leqslant s$, let π_i be an admissible representation of $\mathrm{GL}(n_i, A)$ of finite length. We write $\pi_1 \times \pi_2 \times \cdots \times \pi_s$ for the representation that is parabolically induced from the representation $\pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_s$ of $G_{(n_1,\dots,n_s)}$ with respect to $P_{(n_1,\dots,n_s)}$. We also use this notation for the image of a representation in the Grothendieck group of virtual characters, which makes the above product commutative. Often we shall not distinguish between a representation and its isomorphy class and will write 'equal' for 'isomorphic'.

1.2 Classification of unitary representations

First, we recall Tadić's classification of the unitary dual of the groups $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$, following [Tad09]. The classification is similar to that for non-archimedean local fields [Tad86, Tad90] and is explained in detail in § 7. In the $\mathrm{GL}(n, \mathbb{H})$ case, parts of the arguments have not appeared in the literature, so we give the complete proofs in §§ 10, 11 and 12, using Vogan's classification [Vog86].

Let $X_{\mathbb{C}}$ be the set of unitary characters of $\mathbb{C}^\times$. If $\chi \in X_{\mathbb{C}}$ and $n \in \mathbb{N}^\times$, let χ_n be the character $\chi \circ \det$ of $\mathrm{GL}(n, \mathbb{C})$. Let ν_n be the character of $\mathrm{GL}(n, \mathbb{C})$ given by the square of the module of the determinant. If σ is a representation of $\mathrm{GL}(n, \mathbb{C})$ and $\alpha \in \mathbb{R}$, write $\pi(\sigma, \alpha)$ for the representation $\nu_n^\alpha \sigma \times \nu_n^{-\alpha} \sigma$ of $\mathrm{GL}(2n, \mathbb{C})$. Set

$$\mathcal{U}_{\mathbb{C}} = \{\chi_n, \pi(\chi_n, \alpha) \mid \chi \in X_{\mathbb{C}}, n \in \mathbb{N}^\times, \alpha \in \left]0, \tfrac{1}{2}\right[\,\}.$$

Let $X_{\mathbb{R}}$ be the set of unitary characters of $\mathbb{R}^\times$. Let sgn denote the sign character. If $\chi \in X_{\mathbb{R}}$ and $n \in \mathbb{N}^\times$, let χ_n be the character $\chi \circ \det$ of $\mathrm{GL}(n, \mathbb{R})$ and χ'_n the character $\chi \circ \mathrm{RN}$ of $\mathrm{GL}(n, \mathbb{H})$. For fixed n, the map $\chi \mapsto \chi_n$ is an isomorphism from the group of unitary characters of $\mathbb{R}^\times$ to the group of unitary characters of $\mathrm{GL}(n, \mathbb{R})$, while $\chi \mapsto \chi'_n$ is a surjective map from the group of unitary characters of $\mathbb{R}^\times$ to the group of unitary characters of $\mathrm{GL}(n, \mathbb{H})$, with kernel $\{1, \mathrm{sgn}\}$.

Let ν_n (respectively, ν'_n) be the character of $\mathrm{GL}(n, \mathbb{R})$ (respectively, $\mathrm{GL}(n, \mathbb{H})$) given by the absolute value (respectively, the reduced norm) of the determinant. If σ is a representation of $\mathrm{GL}(n, \mathbb{R})$ (respectively, $\mathrm{GL}(n, \mathbb{H})$) and $\alpha \in \mathbb{R}$, write $\pi(\sigma, \alpha)$ for the representation $\nu_n^\alpha \sigma \times \nu_n^{-\alpha} \sigma$ of $\mathrm{GL}(2n, \mathbb{R})$ (respectively, the representation $\nu'^{\alpha}_n \sigma \times \nu'^{-\alpha}_n \sigma$ of $\mathrm{GL}(2n, \mathbb{H})$).

Let D_2^u be the set of isomorphy classes of square integrable (modulo center) representations of $\mathrm{GL}(2, \mathbb{R})$. For $\delta \in D_2^u$ and $k \in \mathbb{N}^\times$, write $u(\delta, k)$ for the Langlands quotient of the representation

$$\nu_2^{(k-1)/2} \delta \times \nu_2^{(k-3)/2} \delta \times \nu_2^{(k-5)/2} \delta \times \cdots \times \nu_2^{-(k-1)/2} \delta.$$

Then $u(\delta, k)$ is a representation of $\mathrm{GL}(2k, \mathbb{R})$. Set

$$\begin{aligned}\mathcal{U}_{\mathbb{R}} = \{&\chi_n, \pi(\chi_n, \alpha) \mid \chi \in X_{\mathbb{R}},\ n \in \mathbb{N}^\times, \alpha \in \left]0, \tfrac{1}{2}\right[\,\}\\ &\cup \{u(\delta, k), \pi(u(\delta, k), \alpha) \mid \delta \in D_2^u, k \in \mathbb{N}^\times, \alpha \in \left]0, \tfrac{1}{2}\right[\,\}.\end{aligned}$$

Now let D be the set of isomorphism classes of irreducible unitary representations of $\mathbb{H}^\times$ *which are not one-dimensional*. For $\delta \in D$ and $k \in \mathbb{N}^\times$, write $u(\delta, k)$ for the Langlands quotient of the representation

$$\nu'^{(k-1)/2}_1 \delta \times \nu'^{(k-3)/2}_1 \delta \times \nu'^{(k-5)/2}_1 \delta \times \cdots \times \nu'^{-(k-1)/2}_1 \delta.$$

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Then $u(\delta, k)$ is a representation of $\mathrm{GL}(k, \mathbb{H})$. Set

$$\begin{aligned}\mathcal{U}_{\mathbb{H}} = \{\chi'_n, \pi(\chi'_n, \alpha) \mid \chi \in X_{\mathbb{R}}, n \in \mathbb{N}^{\times}, \alpha \in]0, 1[\,\} \\ \cup \{u(\delta, k), \pi(u(\delta, k), \alpha) \mid \delta \in D, k \in \mathbb{N}^{\times} \alpha \in]0, \tfrac{1}{2}[\,\}.\end{aligned}$$

Theorem 1.1. *For $A = \mathbb{C}, \mathbb{R}$ or $\mathbb{H}$, any representation in $\mathcal{U}_A$ is irreducible and unitary, any product of representations in $\mathcal{U}_A$ is irreducible and unitary, and any irreducible unitary representation π of $\mathrm{GL}(n, A)$ can be written as a product of elements in $\mathcal{U}_A$. Moreover, π determines the factors of the product (up to permutation).*

Notice the two different ranges for the possible values of α in the $A = \mathbb{H}$ case.

1.3 Jacquet–Langlands correspondence for unitary representations

Any element in $\mathrm{GL}(n, \mathbb{H})$ has a characteristic polynomial of degree $2n$ with coefficients in $\mathbb{R}$. We say that two elements $g \in \mathrm{GL}(2n, \mathbb{R})$ and $g' \in \mathrm{GL}(n, \mathbb{H})$ *correspond* (to each other) if they have the same characteristic polynomial and this polynomial has distinct roots in $\mathbb{C}$ (this last condition means that g and g' are regular semisimple). We then write $g \leftrightarrow g'$.

Let $\mathbf{C}$ denote the Jacquet–Langlands correspondence between irreducible square integrable representations of $\mathrm{GL}(2, \mathbb{R})$ and irreducible unitary representations of $\mathbb{H}^{\times}$ (see [JL70]). This correspondence can be extended to a correspondence $|\mathbf{LJ}|$ between *all* irreducible unitary representations of $\mathrm{GL}(2n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{H})$ (it comes from a ring morphism $\mathbf{LJ}$ between the respective Grothendieck groups, defined in § 4, whence the notation). In what follows, it will be understood that whenever we write the relation $|\mathbf{LJ}|(\pi) = \pi'$ for representations π and π' of $\mathrm{GL}(2n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{H})$, respectively, we have that π and π' satisfy the character relation $\Theta_\pi(g) = \varepsilon(\pi)\Theta'_\pi(g')$ for all $g \leftrightarrow g'$, where $\varepsilon(\pi)$ is an explicit sign (π clearly determines π' and ε). The correspondence $|\mathbf{LJ}|$ for unitary representations is given first on elements in $\mathcal{U}_{\mathbb{R}}$, as follows.

(a) $|\mathbf{LJ}|(\chi_{2n}) = \chi'_n$ and $|\mathbf{LJ}|(\pi(\chi_{2n}, \alpha)) = \pi(\chi'_n, \alpha)$ for all $\chi \in X_{\mathbb{R}}$ and $\alpha \in]0, 1/2[$.

(b) If $\delta \in D_2^u$ is such that $\mathbf{C}(\delta)$ is in D (i.e. is not one-dimensional), then $|\mathbf{LJ}|(u(\delta, k)) = |\mathbf{LJ}|(\mathbf{C}(\delta), k)$ and $\mathbf{LJ}(\pi(u(\delta, k), \alpha)) = \pi(u(\mathbf{C}(\delta), k), \alpha)$ for all $\alpha \in]0, 1/2[$.

(c) If $\delta \in D_2^u$ is such that $\mathbf{C}(\delta)$ is a one-dimensional representation χ'_1, then:
   - $|\mathbf{LJ}|(u(\delta, k)) = \pi(\chi'_{k/2}, 1/2)$ and $|\mathbf{LJ}|(\pi(u(\delta, k), \alpha)) = \pi(\pi(\chi'_{k/2}, 1/2), \alpha)$ if k is even and $\alpha \in]0, 1/2[\,$;
   - $|\mathbf{LJ}|(u(\delta, k)) = \chi'_{(k+1)/2} \times \chi'_{(k-1)/2}$ and $|\mathbf{LJ}|(\pi(u(\delta, k), \alpha)) = \pi(\chi'_{(k+1)/2}, \alpha) \times \pi(\chi'_{(k-1)/2}, \alpha)$ if $k \neq 1$ is odd and $\alpha \in]0, 1/2[\,$;
   - $|\mathbf{LJ}|(\delta) = \chi'_1$ and $|\mathbf{LJ}|(\pi(\delta, \alpha)) = \pi(\chi'_1, \alpha)$ for $\alpha \in]0, 1/2[$.

Let π be an irreducible unitary representation of $\mathrm{GL}(2n, \mathbb{R})$. If writing π as a product of elements in $\mathcal{U}_{\mathbb{R}}$ involves a factor not listed in (a), (b) or (c), it is easy to show that π has a character which vanishes on elements that correspond to elements of $\mathrm{GL}(n, \mathbb{H})$, and we set $|\mathbf{LJ}|(\pi) = 0$. If all the factors σ_i of π are in (a), (b) or (c) above, $|\mathbf{LJ}|(\pi)$ is the product of the $|\mathbf{LJ}|(\sigma_i)$ (an irreducible unitary representation of $\mathrm{GL}(n, \mathbb{H})$). Elements of $\mathcal{U}_{\mathbb{R}}$ not listed in (a), (b) or (c) are of type χ or $\pi(\chi, \alpha)$, with χ being a character of some $\mathrm{GL}(k, \mathbb{R})$ and k odd.

Note that some unitary irreducible representations of $\mathrm{GL}(n, \mathbb{H})$ are not in the image of this map (if $n \geqslant 2$). For instance, when $\chi \in X_{\mathbb{R}}$ and $1/2 < \alpha < 1$, then both $\pi(\chi_2, \alpha)$ and $\pi(\chi'_1, \alpha)$ are irreducible and they correspond to each other via the character relation, but $\pi(\chi'_1, \alpha)$ is unitary while $\pi(\chi_2, \alpha)$ is not. Using the classification of unitary representations for $\mathrm{GL}(4, \mathbb{R})$ and

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basic information from the infinitesimal character, it is clear that no (possibly other) unitary representation of $\mathrm{GL}(4,\mathbb{R})$ has character matching with $\pi(\chi_1', \alpha)$.

As a consequence of the above results, we get the following theorem.

Theorem 1.2. *Let u be a unitary irreducible representation of $\mathrm{GL}(2n,\mathbb{R})$. Then either the character Θ_u of u vanishes on the set of elements of $\mathrm{GL}(2n,\mathbb{R})$ which correspond to some element of $\mathrm{GL}(n,\mathbb{H})$ or there exists a unique irreducible unitary (smooth) representation u' of $\mathrm{GL}(n,\mathbb{H})$ such that*

$$\Theta_u(g) = \varepsilon(u)\Theta_{u'}(g')$$

for all $g \leftrightarrow g'$, where $\varepsilon(u) \in \{-1, 1\}$.

The above results are proved in § 13 and are based on the fact that $\mathrm{GL}(2n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{H})$ share Levi subgroups (of θ-stable parabolic subgroups, i.e. the ones used in cohomological induction [KV95]) which are products of $\mathrm{GL}(n_i,\mathbb{C})$. The underlying principle (a nice instance of Langlands' functoriality) is that the Jacquet–Langlands morphism **LJ** commutes with cohomological induction. The same principle, with Kazhdan–Patterson lifting instead of Jacquet–Langlands correspondence, was already used in [AH97].

1.4 Character identities and ends of complementary series

In § 14, we give the composition series of the ends of complementary series in most cases. This is not directly related to the main theme of the paper, the global theory of the second part, but it solves some old conjectures of Tadić which will be important in understanding the topology of the unitary dual of the groups $\mathrm{GL}(n, A)$, for $A = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. The starting point is the Zuckerman formula for the trivial representation of $\mathrm{GL}(n, A)$. Together with cohomological induction, it gives character formulas for unitary representations of the groups $\mathrm{GL}(n, A)$. In the case of $A = \mathbb{C}$, the Zuckerman formula is given by a determinant (see formula (14.2)), and the Lewis Carroll identity of [CR08] allows us to deduce formulas (14.3), (14.5), (14.6), (14.7) and (14.10) for the ends of complementary series.

1.5 Global results

Let $\mathcal{F}$ be a global field of characteristic zero and $\mathcal{D}$ a central division algebra over $\mathcal{F}$ of dimension d^2. Let $n \in \mathbb{N}^*$. Set $A' = M_n(\mathcal{D})$. For each place v of $\mathcal{F}$, let $\mathcal{F}_v$ be the completion of $\mathcal{F}$ at v and set $A'_v = A' \otimes \mathcal{F}_v$. For every place v of $\mathcal{F}$, A'_v is isomorphic to the matrix algebra $M_{r_v}(\mathcal{D}_v)$ for some positive number r_v and some central division algebra $\mathcal{D}_v$ of dimension d_v^2 over $\mathcal{F}_v$ such that $r_v d_v = nd$. We will fix once and for all an isomorphism and identify these two algebras with each other. Let V be the (finite) set of places where $M_n(\mathcal{D})$ is not split (i.e. $d_v \neq 1$).

Let $G'(\mathcal{F})$ be the group $A'^{\times} = \mathrm{GL}(n,\mathcal{D})$. For every place $v \in V$, set $G'_v = A'^{\times}_v = \mathrm{GL}(r_v, \mathcal{D}_v)$ and $G_v = \mathrm{GL}(n, \mathcal{F}_v)$. For a given place v (which will be clear from the context), write $g \leftrightarrow g'$ if $g \in G_v$ and $g' \in G'_v$ are regular semisimple and have equal characteristic polynomial.

If $v \notin V$, the algebras A_v and A'_v are isomorphic, hence we get an identification of G'_v with G_v.

Theorem 1.2 has been proved in the p-adic case as well [Bad08, Tad06]. So, if $v \in V$, then using the same notation and conventions for the p-adic and archimedean cases gives us the following.

THEOREM 1.3. *Let u be a unitary irreducible smooth representation of G_v. Then one and only one of the following two possibilities holds.*

(i) *The character Θ_u of u vanishes on the set of elements of G_v which correspond to elements of G'_v.*

(ii) *There exists a unique unitary smooth irreducible representation u' of G'_v such that*

$$\Theta_u(g) = \varepsilon(u)\Theta_{u'}(g')$$

for any $g \leftrightarrow g'$, where $\varepsilon(u) \in \{-1, 1\}$.

In case (ii) we say that u is *compatible*. We denote the map $u \mapsto u'$ defined on the set of compatible (unitary) representations by $|\mathbf{LJ}_v|$.

Let $\mathbb{A}$ be the ring of adeles of $\mathcal{F}$. The group $G'(\mathcal{F})$ (respectively, $G(\mathcal{F})$) is a discrete subgroup of $G'(\mathbb{A})$ (respectively, $G(\mathbb{A})$). The centers of G' and G consist of scalar non-zero matrices and so can both be identified with the multiplicative group $\mathbb{G}_m$ defined over $\mathcal{F}$; both will be denoted by Z.

We endow these local and global groups with measures as in [AC89]. For every unitary continuous character (also known as a 'grössencharacter') ω of $Z(\mathbb{A})$ that is trivial on $Z(\mathcal{F})$, we let $L^2(G'(\mathcal{F})Z(\mathbb{A})\backslash G'(\mathbb{A}); \omega)$ be the space of functions f defined on $G'(\mathbb{A})$ with values in $\mathbb{C}$ such that:

(i) f is left invariant under $G'(\mathcal{F})$;

(ii) $f(zg) = \omega(z)f(g)$ for all $z \in Z(\mathbb{A})$ and all $g \in G'(\mathbb{A})$;

(iii) $|f|^2$ is integrable over $G'(\mathcal{F})Z(\mathbb{A})\backslash G'(\mathbb{A})$.

Let us denote by R'_ω the representation of $G'(\mathbb{A})$ on $L^2(G'(\mathcal{F})Z(\mathbb{A})\backslash G'(\mathbb{A}); \omega)$ by right translations. A *discrete series of* $G'(\mathbb{A})$ is the equivalence class of an irreducible subrepresentation of R'_ω for some smooth unitary character ω of $Z(\mathbb{A})$ that is trivial on $Z(\mathcal{F})$. Then ω is the central character of π. Let $R'_{\omega,\mathrm{disc}}$ be the subrepresentation of R'_ω generated by irreducible subrepresentations. It is known that a discrete series representation of $G'(\mathbb{A})$ appears with finite multiplicity in $R'_{\omega,\mathrm{disc}}$; see [GGP90].

Similar definitions and statements can be made with G instead of G', with obvious adjustments to the notation. Every discrete series π of $G'(\mathbb{A})$ (respectively, $G(\mathbb{A})$) is 'isomorphic' to a restricted Hilbert tensor product of irreducible unitary smooth representations π_v of the groups G'_v (respectively, G_v); see [Fla79] for a precise statement and proof. The local components π_v are determined by π.

Let DS (respectively, DS′) denote the set of discrete series of $G(\mathbb{A})$ (respectively, of $G'(\mathbb{A})$). We say that a discrete series π of $G(\mathbb{A})$ is *$\mathcal{D}$-compatible* if π_v is compatible for all places $v \in V$.

THEOREM 1.4.

(a) *There exists a unique map $\mathbf{G}: \mathrm{DS}' \to \mathrm{DS}$ such that for every $\pi' \in \mathrm{DS}'$, if $\pi = \mathbf{G}(\pi')$, then one has that:*

- *π is $\mathcal{D}$-compatible;*
- *if $v \notin V$, then $\pi_v = \pi'_v$;*
- *if $v \in V$, then $|\mathbf{LJ}_v|(\pi_v) = \pi'_v$.*

The map $\mathbf{G}$ is injective. The image of $\mathbf{G}$ is the set of all $\mathcal{D}$-compatible discrete series of $G(\mathbb{A})$.

(b) *If $\pi' \in \mathrm{DS}'$, then the multiplicity of π' in the discrete spectrum is one (multiplicity-one theorem).*

(c) *If $\pi', \pi'' \in \mathrm{DS}'$ and $\pi'_v \simeq \pi''_v$ for almost all v, then $\pi' = \pi''$ (strong multiplicity-one theorem).*

With $\mathcal{D}$ fixed, we now need to consider all possible $n \in \mathbb{N}^\times$ at the same time. We add a subscript to the notation and write, for example, $A_n = M_n(\mathcal{F})$, $A'_n = M_n(\mathcal{D})$, G_n, G'_n, DS_n, DS'_n and so on. We recall the Moeglin–Waldspurger classification of the residual spectrum for the groups $G_n(\mathbb{A})$, $n \in \mathbb{N}^*$. Let ν be the character of $G_n(\mathbb{A})$ or $G'_n(\mathbb{A})$ given by the restricted product of characters $\nu_v = |\det|_v$, where $|\cdot|_v$ is the v-adic norm and det is the reduced norm at the place v. Let $m \in \mathbb{N}^*$ and $\rho \in \mathrm{DS}_m$ be a cuspidal representation. If $k \in \mathbb{N}^*$, then the induced representation to $G_{mk}(\mathbb{A})$ from $\bigotimes_{i=0}^{k-1}(\nu^{(k-1)/2-i}\rho)$ has a unique constituent π (in the sense of [Lan79]) which is a discrete series (i.e. $\pi \in \mathrm{DS}_{mk}$). We then set $\pi = \mathrm{MW}(\rho, k)$. Discrete series π of groups $G_n(\mathbb{A})$, $n \in \mathbb{N}^*$, are all of this type, and k and ρ are determined by π. The discrete series π is cuspidal if $k = 1$ and residual if $k > 1$. These results are proved in [MW89].

The proofs of the following propositions and corollary are the same as those in [Bad08], once the local and global transfer are established without the condition on archimedean places. First, concerning cuspidal representations of $G'(\mathbb{A})$, we have the following result.

Proposition 1.5. *Let $m \in \mathbb{N}^*$ and let $\rho \in \mathrm{DS}_m$ be a cuspidal representation. Then the following hold.*

(a) *There exists $s_{\rho,\mathcal{D}} \in \mathbb{N}^*$ such that for $k \in \mathbb{N}^*$, $\mathrm{MW}(\rho, k)$ is $\mathcal{D}$-compatible if and only if $s_{\rho,\mathcal{D}}|k$; we have $s_{\rho,\mathcal{D}}|d$.*

(b) *$\mathbf{G}^{-1}(\mathrm{MW}(\rho, s_{\rho,\mathcal{D}})) = \rho' \in \mathrm{DS}'_{ms_{\rho,\mathcal{D}}/d}$ is cuspidal; the map $\mathbf{G}^{-1}$ sends cuspidal $\mathcal{D}$-compatible representations to cuspidal representations.*

(c) *Every cuspidal representation in $\mathrm{DS}'_{ms_{\rho,\mathcal{D}}/d}$ is obtained as in (b).*

Let us call the twist of a cuspidal representation by a real power of ν an *essentially cuspidal* representation. If $n_1, n_2, \ldots, n_k$ are positive integers such that $\sum_{i=1}^k n_i = n$, then the subgroup L of $G'_n(\mathbb{A})$ of diagonal matrices by blocks of sizes $n_1, n_2, \ldots, n_k$ will be called the *standard Levi subgroup* of $G'_n(\mathbb{A})$. We identify L with $\times_{i=1}^k G'_{n_i}(\mathbb{A})$. All the definitions extend in an obvious way to L. The two statements in the following proposition generalize, respectively, [MW89] and [JS81, Theorem 4.4].

Proposition 1.6.

(a) *Let $\rho' \in \mathrm{DS}'_m$ be a cuspidal representation and let $k \in \mathbb{N}^*$. The induced representation from $\bigotimes_{i=0}^{k-1}(\nu_{\rho'}^{(k-1)/2-i}\rho')$ has a unique irreducible quotient π' (also characterized among irreducible subquotients by being in the discrete series), denoted by $\pi' = \mathrm{MW}'(\rho', k)$. Every discrete series π' of a group $G'_n(\mathbb{A})$ with $n \in \mathbb{N}^*$ is of this type, and k and ρ' are determined by π'. The representation π' is cuspidal if $k = 1$ and residual if $k > 1$. If $\pi' = \mathrm{MW}'(\rho', k)$, then $\mathbf{G}(\rho') = \mathrm{MW}(\rho, s_{\rho,\mathcal{D}})$ if and only if $\mathbf{G}(\pi') = \mathrm{MW}(\rho, ks_{\rho,\mathcal{D}})$.*

(b) *Let (L_i, ρ'_i), $i = 1$ or 2, be such that L_i is a standard Levi subgroup of $G'_n(\mathbb{A})$ and ρ'_i is an essentially cuspidal representation of L_i for $i = 1, 2$. Fix any finite set of places V' containing the infinite places and all the finite places v where $\rho'_{1,v}$ or $\rho'_{2,v}$ is ramified (i.e. has no non-zero vector fixed under K_v). If for all places $v \notin V'$ the unramified subquotients of the representation of $G'_n(\mathbb{A})$ induced from the $\rho'_{i,v}$ are equal, then (L_1, ρ'_1) and (L_2, ρ'_2) are conjugate.*

We know from [Lan79] that if π' is an automorphic representation of G'_n, then there exists (L, ρ') where L is a standard Levi subgroup of G'_n and ρ' is an essentially cuspidal representation of L such that π' is a constituent of the representation of G'_n induced from ρ'. A corollary of assertion (b) of the proposition is the following.

Corollary 1.7. *(L, ρ') is unique up to conjugation.*

1.6 Some comments

The length of this paper can be explained by our desire to give complete proofs and/or references for all of the statements. For instance, the proof in § 10 of $U(3)$ for $\mathrm{GL}(n, \mathbb{H})$ is already quite long in itself but moreover requires material about the Bruhat G-order introduced in the preceding section, which is not needed elsewhere. We could have saved four or five pages by referring to [Tad09], which gives the proof of $U(3)$ for $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$; however, [Tad09] was still unpublished at the time of writing this paper, and our arguments by means of the Bruhat G-order could be used to simplify the proofs in [Tad09]. Our paper is also aimed at the reader who might be interested in comparing the archimedean and non-archimedean theory, so we have tried to make the expositions as parallel as possible. Our discussion of Vogan's classification in § 12 is also longer than is strictly necessary, but we feel that it is important for the relation between Vogan's and Tadić's classifications to be explained in detail somewhere.

We thank D. Vogan for answering many questions concerning his work.

2. Notation

2.1 Multisets

Let X be a set. We denote by $M(X)$ the set of functions from X to $\mathbb{N}$ with finite support, and we consider an element $m \in M(X)$ as a 'set with multiplicities'. Such an element $m \in M(X)$ will typically be written as

$$m = (x_1, x_2, \ldots, x_r).$$

It is a (non-ordered) list of elements x_i in X.

The multiset $M(X)$ is endowed with the structure of a monoid induced from the one on $\mathbb{N}$: if $m = (x_1, \ldots, x_r)$ and $n = (y_1, \ldots, y_s)$ are in $M(X)$, we get

$$m + n = (x_1, \ldots, x_r, y_1, \ldots, y_s).$$

2.2 Local fields and division algebras

We will use the following notation: F is a local field, $|\cdot|_F$ is the normalized absolute value on F, and A is a central division algebra over F with $\dim_F(A) = d^2$.

If F is archimedean, then either $F = \mathbb{R}$ and $A = \mathbb{R}$ or $A = \mathbb{H}$, the algebra of quaternions, or $F = A = \mathbb{C}$.

2.3 GL

For $n \in \mathbb{N}^\times$, we set $G_n = \mathrm{GL}(n, A)$ and $G_0 = \{1\}$. We denote the reduced norm on G_n by

$$\mathrm{RN} : G_n \to F^\times.$$

We set

$$\nu_n : G_n \to |\mathrm{RN}(g)|_F.$$

When the value of n is not relevant to the discussion, we will simply write G for G_n and ν for ν_n.

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Remark 2.1. If $A=F$, the reduced norm is just the determinant.

When F is non-archimedean, the character ν of G is unramified and, in fact, the group of unramified characters of G is

$$\mathcal{X}(G)=\{\nu^s : s\in\mathbb{C}\}.$$

The notation for the group of complex powers of ν will also be used in the archimedean case.

If G is one of the groups G_n or, more generally, the group of rational points of any reductive algebraic connected group defined over F, we denote by $\mathcal{M}(G)$ the category of smooth representations of G (in the non-archimedean case) or the category of Harish-Chandra modules (in the archimedean case) with respect to a fixed maximal compact subgroup K of G. For $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{GL}(n,\mathbb{H})$, these maximal compact subgroups are chosen to be $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$, respectively, embedded in the standard way. Then $\mathcal{R}(G)$ denotes the Grothendieck group of the category of finite-length representations in $\mathcal{M}(G)$. This is the free $\mathbb{Z}$-module with basis $\mathrm{Irr}(G)$, the set of equivalence classes of irreducible representations in $\mathcal{M}(G)$. If $\pi\in\mathcal{M}(G)$, of finite length, we will again denote by π its image in $\mathcal{R}(G)$. When confusion may occur, we will state precisely if we consider π as a representation or as an element in $\mathcal{R}(G)$.

Set

$$\mathrm{Irr}_n=\mathrm{Irr}(G_n),\quad \mathrm{Irr}=\coprod_{n\in\mathbb{N}}\mathrm{Irr}_n,\quad \mathcal{R}=\bigoplus_{n\in\mathbb{N}}\mathcal{R}(G_n).$$

If $\tau\in\mathcal{M}(G_n)$ or $\mathcal{R}(G_n)$, we set $\deg\tau=n$.

2.4 Standard parabolic and Levi subgroups

Let $n\in\mathbb{N}$ and let $\sum_{i=1}^s n_i=n$ be a partition of n. The group

$$\prod_{i=1}^s G_{n_i}$$

is identified with the subgroup of G_n of block-diagonal matrices of sizes $n_1,\dots,n_s$. Let $G_{(n_1,\dots,n_s)}$ denote this subgroup, and let $P_{(n_1,\dots,n_s)}$ (respectively, $\bar{P}_{(n_1,\dots,n_s)}$) denote the parabolic subgroup of G_n generated by $G_{(n_1,\dots,n_s)}$ and the Borel subgroup of invertible upper-triangular (respectively, lower-triangular) matrices. The subgroup $G_{(n_1,\dots,n_s)}$ is a Levi factor of the standard parabolic subgroup $P_{(n_1,\dots,n_s)}$.

In this setting, we denote by $i_{(n_1,\dots,n_s)}$ (respectively, $\underline{i}_{(n_1,\dots,n_s)}$) the functor of normalized parabolic induction from $\mathcal{M}(G_{(n_1,\dots,n_s)})$ to $\mathcal{M}(G_n)$ with respect to the parabolic subgroup $P_{(n_1,\dots,n_s)}$ (respectively, $\bar{P}_{(n_1,\dots,n_s)}$).

Definition 2.2. Let $\pi_1\in\mathcal{M}(G_{n_1})$ and $\pi_2\in\mathcal{M}(G_{n_2})$ both be of finite length. We can then form the induced representation

$$\pi_1\times\pi_2:=i_{(n_1,n_2)}(\pi_1\otimes\pi_2).$$

The image of $i_{n_1,n_2}(\pi_1\otimes\pi_2)$ in the Grothendieck group $\mathcal{R}_{n_1+n_2}$ will still be denoted by $\pi_1\times\pi_2$. This extends linearly to a product

$$\times:\mathcal{R}\times\mathcal{R}\to\mathcal{R}.$$

Remark 2.3. We warn the reader again that it is important to know when we are considering $\pi_1\times\pi_2$ as a representation and when we are considering it as an element in $\mathcal{R}$. For instance, $\pi_1\times\pi_2=\pi_2\times\pi_1$ in $\mathcal{R}$ (see below), but $i_{(n_1,n_2)}(\pi_1\otimes\pi_2)$ is not isomorphic to $i_{(n_1,n_2)}(\pi_2\otimes\pi_1)$ in general.

Proposition 2.4. *The ring* $(\mathcal{R}, \times)$ *is graded commutative. Its identity is the unique element in* Irr_0.

3. The Langlands classification

We recall how to combine the Langlands classification of Irr in terms of irreducible essentially tempered representations with the fact that, for the groups G_n, tempered representations are induced fully from irreducible square integrable modulo center representations to obtain a classification of Irr in terms of irreducible essentially square integrable modulo center representations.

Let

$$D_n^u \subset \mathrm{Irr}_n \quad \text{and} \quad D_n \subset \mathrm{Irr}_n$$

denote, respectively, the sets of equivalence classes of irreducible square integrable modulo center and irreducible essentially square integrable modulo center representations of G_n, and set

$$D^u = \coprod_{n \in \mathbb{N}^\times} D_n^u \quad \text{and} \quad D = \coprod_{n \in \mathbb{N}^\times} D_n.$$

Similarly,

$$T_n^u \subset \mathrm{Irr}_n \quad \text{and} \quad T_n \subset \mathrm{Irr}_n$$

denote, respectively, the sets of equivalence classes of irreducible tempered and irreducible essentially tempered representations of G_n. Set

$$T^u = \coprod_{n \in \mathbb{N}^\times} T_n^u \quad \text{and} \quad T = \coprod_{n \in \mathbb{N}^\times} T_n.$$

For all $\tau \in T$, there exist a unique $e(\tau) \in \mathbb{R}$ and a unique $\tau^u \in T^u$ such that

$$\tau = \nu^{e(\tau)} \tau^u.$$

Theorem 3.1. *Let* $d = (\delta_1, \ldots, \delta_l) \in M(D^u)$. *Then*

$$\delta_1 \times \delta_2 \times \cdots \times \delta_l$$

is irreducible and therefore in T^u. *This defines a one-to-one correspondence between* $M(D^u)$ *and* T^u.

This result is due to Jacquet and Zelevinsky in the $A = F$ non-archimedean case (see [Jac77] or [Zel80]). For a non-archimedean division algebra, it is established in [DKV84]. In the archimedean case, reducibility of induced-from-square-integrable representations are well understood in terms of R-groups (see [KZ82]), and for the groups G_n, the R-groups are trivial.

Definition 3.2. Let $t = (\tau_1, \ldots, \tau_l) \in M(T)$. We say that t is written in a standard order if

$$e(\tau_1) \geqslant \cdots \geqslant e(\tau_l).$$

Theorem 3.3. *Let* $d = (d_1, \ldots, d_l) \in M(D)$ *be written in a standard order, i.e.*

$$e(d_1) \geqslant e(d_2) > \cdots \geqslant e(d_l).$$

Then the following hold.

(i) *The representation*

$$\lambda(d) = d_1 \times \cdots \times d_l$$

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has a unique irreducible quotient $\mathrm{Lg}(d)$ *appearing with multiplicity one in a Jordan–Hölder sequence of* $\lambda(d)$*. It is also the unique subrepresentation of*

$$d_l \times d_{l-1} \times \cdots \times d_2 \times d_1.$$

(ii) *Up to a multiplicative scalar, there is an unique intertwining operator*

$$J : d_1 \times \cdots \times d_l \longrightarrow d_l \times \cdots \times d_1.$$

We then have $\mathrm{Lg}(d) \simeq \lambda(d)/\ker J \simeq \mathrm{Im}\, J$*.*

(iii) *The map*

$$d \mapsto \mathrm{Lg}(d)$$

is a bijection between $M(D)$ *and* Irr*.*

For a proof in the non-archimedean case, the reader may consult [Ren10].

Representations of the form $\lambda(d) = d_1 \times \cdots \times d_l$ with $d = (d_1, \ldots, d_l) \in M(D)$ written in a standard order are called *standard representations*.

Remark 3.4. If d is a multiset of representations in Irr, we denote by $\deg d$ the sum of the degrees of representations in d. Let $M(D)_n$ be the subset of $M(D)$ consisting of multisets of degree n. Then the theorem gives a one-to-one correspondence between $M(D)_n$ and Irr_n.

PROPOSITION 3.5. *The ring* R *is isomorphic to* $\mathbb{Z}[D]$*, the ring of polynomials in* X_d $(d \in D)$ *with coefficients in* $\mathbb{Z}$*; that is,* $\{[\lambda(d)]\}_{d \in D}$ *is a* $\mathbb{Z}$*-basis of* $\mathcal{R}$*.*

See [Zel80, Proposition 8.5] for a proof.

We give some easy consequences of the above proposition.

COROLLARY 3.6.

(i) *The ring* R *is a factorial domain.*

(ii) *If* $\delta \in D$*, then* $[\delta]$ *is prime* $\mathcal{R}$*.*

(iii) *If* $\pi \in \mathcal{R}$ *is homogeneous and* $\pi = \sigma_1 \times \sigma_2$ *in* $\mathcal{R}$*, then* σ_1 *and* σ_2 *are homogeneous.*

(iv) *The group of invertible elements in* $\mathcal{R}$ *is* $\{\pm \mathrm{Irr}_0\}$*.*

4. Jacquet–Langlands correspondence

In this section, we fix a central division algebra A of dimension d^2 over the local field F. We recall the Jacquet–Langlands correspondence between $\mathrm{GL}(n, A)$ and $\mathrm{GL}(nd, F)$. Since we need F and A simultaneously in the notation, we write G_n^A and G_n^F for $\mathrm{GL}(n, A)$ and $\mathrm{GL}(n, F)$, respectively, and similarly with other notation, e.g. $\mathcal{R}(G_n^A)$ or $\mathcal{R}(G_n^F)$, D_n^A or D_n^F etc.

There is a standard way of defining the determinant and the characteristic polynomial for elements of G_n^A, despite A being non-commutative (see, for example, [Pie82, § 16]), and the reduced norm RN introduced earlier is just given by the constant term of the characteristic polynomial. If $g \in G_n^A$, then the characteristic polynomial of g has coefficients in F; it is monic and has degree nd. If $g \in G_n^A$ for some n, we say that g is *regular semisimple* if the characteristic polynomial of g has distinct roots in an algebraic closure of F.

If $\pi \in \mathcal{R}(G_n)$, then we denote by Θ_π the function character of π as a locally constant map, stable under conjugation, defined on the set of regular semisimple elements of G_n.

We say that $g' \in G_n^A$ *corresponds* to $g \in G_{nd}^F$ if g and g' are regular semisimple and have the same characteristic polynomial, and we write $g' \leftrightarrow g$. Notice that if $g' \leftrightarrow g$ and if g_1' and g_1 are conjugate to g' and g, respectively, then $g_1' \leftrightarrow g_1$. In other words, $\leftrightarrow$ is really a correspondence between conjugacy classes.

Theorem 4.1. *There is a unique bijection $\mathbf{C} : D_{nd}^F \to D_n^A$ such that for all $\pi \in D_{nd}^F$ we have*

$$\Theta_\pi(g) = (-1)^{nd-n}\Theta_{\mathbf{C}(\pi)}(g')$$

for all $g \in G_{nd}^F$ and $g' \in G_n^A$ such that $g' \leftrightarrow g$.

For the proof, see [DKV84] for the case where the characteristic of the base field F is zero and [Bad02] for the non-zero characteristic case. For the archimedean case, see §§ 9.2–9.3; also see Remark 9.6 for more details about this correspondence [DKV84, JL70].

We identify the centers of G_{nd}^F and G_n^A via the canonical isomorphisms with $F^\times$. Then the correspondence $\mathbf{C}$ preserves central characters so that, in particular, σ is unitary if and only if $\mathbf{C}(\sigma)$ is.

The correspondence $\mathbf{C}$ can be extended in a natural way to a correspondence $\mathbf{LJ}$ between Grothendieck groups.

- If $\sigma \in D_{nd}^F$, viewed as an element in $\mathcal{R}(G_{nd}^F)$, then we set
$$\mathbf{LJ}(\sigma) = (-1)^{nd-n}\mathbf{C}(\sigma),$$
viewed as an element in $\mathcal{R}(G_n^A)$.
- If $\sigma \in D_r^F$, where r is not divisible by d, then we set $\mathbf{LJ}(\sigma) = 0$.
- Since $\mathcal{R}^F$ is a polynomial algebra in the variables $d \in D^F$, one can extend $\mathbf{LJ}$ in a unique way to an algebra morphism between $\mathcal{R}^F$ and $\mathcal{R}^A$. It is clear that $\mathbf{LJ}$ is surjective.

The fact that $\mathbf{LJ}$ is a ring morphism means that it 'commutes with parabolic induction'. Let us describe how to (theoretically) compute $\mathbf{LJ}(\pi)$, for $\pi \in \mathcal{R}^F$. Since $\{\lambda(a)\}_{a \in M(D^F)}$ is a basis of $\mathcal{R}^F$, we first write π in this basis as

$$\pi = \sum_{a \in M(D^F)} M(a,\pi)\lambda(a)$$

with $M(a,\pi) \in \mathbb{Z}$ (see § 6). Since $\mathbf{LJ}$ is linear, we have

$$\mathbf{LJ}(\pi) = \sum_{a \in M(D^F)} M(a,\pi)\,\mathbf{LJ}(\lambda(a)),$$

so it remains to describe $\mathbf{LJ}(\lambda(a))$. If $a = (d_1, \dots d_k)$, then

$$\lambda(a) = d_1 \times \cdots \times d_k$$

(because we consider $\lambda(a)$ as an element in $\mathcal{R}^F$, the order of the d_j is not important). Since $\mathbf{LJ}$ is an algebra morphism, we have

$$\mathbf{LJ}(\lambda(a)) = \mathbf{LJ}(d_1) \times \cdots \times \mathbf{LJ}(d_k).$$

If d does not divide one of the $\deg d_i$, this is 0; if d divides all the $\deg d_i$, then upon setting $\sum_i \deg d_i = n$ we get

$$\mathbf{LJ}(\lambda(a)) = \prod_{i=1}^{k}(-1)^{d\deg d_i - \deg d_i}\mathbf{C}(d_i) = (-1)^{nd-n}\mathbf{C}(d_1) \times \cdots \times \mathbf{C}(d_k).$$

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5. Support and infinitesimal character

The goal of this section is, again, to introduce the necessary notation and recall some well-known results, but we wish to adopt a uniform terminology for the archimedean and non-archimedean cases. In the non-archimedean case, some authors, by analogy with the archimedean case, refer to as 'infinitesimal character' the cuspidal support of a representation (a multiset of irreducible supercuspidal representations). We take the opposite view of considering infinitesimal characters in the archimedean case as multisets of complex numbers.

5.1 Non-archimedean case

We start with the case where F is non-archimedean. We denote by C (respectively, C^u) the subset of Irr consisting of supercuspidal representations (respectively, unitary supercuspidal representations, i.e. those with $e(\rho)=0$).

For all $\pi \in \mathrm{Irr}$, there exist $\rho_1, \ldots, \rho_n \in C$ such that π is a subquotient of $\rho_1 \times \rho_2 \times \cdots \times \rho_n$. The multiset $(\rho_1, \ldots, \rho_n) \in M(C)$ is uniquely determined by π, and we denote it by $\mathrm{Supp}(\pi)$, called the cuspidal support of π. When π is a finite-length representation whose irreducible subquotients have the same cuspidal support, we denote this cuspidal support by $\mathrm{Supp}(\pi)$. If $\tau = \pi_1 \times \pi_2$ with $\pi_1, \pi_2 \in \mathrm{Irr}$, we have

$$\mathrm{Supp}(\tau) = \mathrm{Supp}(\pi_1) + \mathrm{Supp}(\pi_2). \tag{5.1}$$

For all $\omega \in M(C)$, denote by Irr_ω the set of $\pi \in \mathrm{Irr}$ whose cuspidal support is ω. We obtain a decomposition

$$\mathrm{Irr} = \coprod_{\omega \in M(C)} \mathrm{Irr}_\omega. \tag{5.2}$$

Set

$$\mathcal{R}_\omega = \bigoplus_{\pi \in \mathrm{Irr}_\omega} \mathbb{Z}\pi.$$

Then

$$\mathcal{R} = \bigoplus_{\omega \in M(C)} \mathcal{R}_\omega \tag{5.3}$$

is a graduation of $\mathcal{R}$ by $M(C)$.

We recall the following well-known result.

Proposition 5.1. *Let $\omega \in M(C)$. Then Irr_ω is finite.*

5.2 Archimedean case

Let $\mathfrak{g}_n$ denote the complexification of the Lie algebra of G_n, $\mathfrak{U}_n = \mathfrak{U}(\mathfrak{g}_n)$ its enveloping algebra, and $\mathfrak{Z}_n$ the center of the latter. Let $\mathfrak{h}_n$ be a Cartan subalgebra of $\mathfrak{g}_n$ and $W_n = W(\mathfrak{g}_n, \mathfrak{h}_n)$ its Weyl group. Harish-Chandra defined an algebra isomorphism from $\mathfrak{Z}_n$ to the Weyl group invariants in the symmetric algebra over $\mathfrak{h}_n$:

$$\mathrm{HC}_n : \mathfrak{Z}_n \longrightarrow S(\mathfrak{h}_n)^{W_n}.$$

Using this isomorphism, every character of $\mathfrak{Z}_n$ (i.e. a morphism of algebra with unit $\mathfrak{Z}_n \to \mathbb{C}$) is identified with a character of $S(\mathfrak{h}_n)^{W_n}$. Such characters are given by orbits of W_n in $\mathfrak{h}_n^*$, by evaluation at a point of the orbit.

A representation (recall that this means a Harish-Chandra module in the archimedean case) admits an infinitesimal character if the center of the enveloping algebra acts on it by scalars.

Irreducible representations admit infinitesimal character. For all $\lambda \in \mathfrak{h}_n^*$, denote by Irr_λ the set of $\pi \in \mathrm{Irr}$ whose infinitesimal character is given by λ.

We are now going to identify infinitesimal characters with multisets of complex numbers.

$A = \mathbb{R}$. In this case, $\mathfrak{g}_n = M_n(\mathbb{C})$ and we can choose $\mathfrak{h}_n$ to be the space of diagonal matrices, identified with $\mathbb{C}^n$. Its dual space is also identified with $\mathbb{C}^n$ via the canonical duality

$$\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}, \quad ((x_1, \ldots, x_n), (y_1, \ldots y_n)) \mapsto \sum_{i=1}^n x_i y_i.$$

The Weyl group W_n is then identified with the symmetric group $\mathfrak{S}_n$, acting on $\mathbb{C}^n$ by permuting coordinates. Thus, an infinitesimal character for G_n is given by a multiset of n complex numbers.

$A = \mathbb{C}$. In this case, $\mathfrak{g}_n = M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ and we can choose $\mathfrak{h}_n$ to be the space of pairs of diagonal matrices, identified with $\mathbb{C}^n \times \mathbb{C}^n$. Its dual space is also identified with $\mathbb{C}^n \times \mathbb{C}^n$ as above. The Weyl group is then identified with $\mathfrak{S}_n \times \mathfrak{S}_n$, acting on $\mathfrak{h}_n^* \simeq \mathbb{C}^n \times \mathbb{C}^n$ by permuting coordinates. Thus, an infinitesimal character for G_n is given by a pair of multisets of n complex numbers.

$A = \mathbb{H}$. The group G_n is a real form of $\mathrm{GL}(2n, \mathbb{C})$, so $\mathfrak{g}_n = M_{2n}(\mathbb{C})$. The discussion is then the same as for $F = \mathbb{R}$, with $2n$ replacing n.

By analogy with the non-archimedean case, we denote by $M(C)$ the set of multisets (or pairs of multisets if $A = \mathbb{C}$) described above.

Definition 5.2. Let $\omega \in M(C)$ be a multiset (or a pair of multisets of the same cardinality if $A = \mathbb{C}$) of complex numbers. If $\pi \in \mathrm{Irr}_n$, we set

$$\mathrm{Supp}(\pi) = \omega$$

where $\omega \in M(C)$ is the multiset (or pair of multisets of the same cardinality if $F = \mathbb{C}$) defined by the infinitesimal character of π. We say that ω is the support of π. When π is a finite-length representation whose subquotients all have same support, we denote the support by $\mathrm{Supp}(\pi)$. If $\pi \in \mathrm{Irr}$ and $\pi = \mathrm{Lg}(a)$ for $a \in M(D)$, we set

$$\mathrm{Supp}(a) := \mathrm{Supp}(\pi).$$

We denote by $M(D)_\omega$ the set of $a \in M(D)$ with support ω.

Proposition 5.3. *The results of § 5.1 are valid in the archimedean case.*

By 'results of § 5.1' we mean (5.1)–(5.3) and Proposition 5.1 above.

6. Bruhat G-order

We continue using the notation of the previous section. In what follows, we will use a partial order $\leqslant$ on $M(D)$, called the Bruhat G-order, which is obtained from partial orders on each $M(D)_\omega$, $\omega \in M(C)$.

Proposition 6.1. *Let $a \in M(D)$. Then the decomposition of $\lambda(a)$ in the basis $\{\mathrm{Lg}(b)\}_{b \in M(D)}$ of $\mathcal{R}$ is of the form*

$$\lambda(a) = \sum_{b \leqslant a} m(b, a)\, \mathrm{Lg}(b),$$

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where the $m(a,b)$ are non-negative integers. The decomposition of $\mathrm{Lg}(a)$ *in the basis* $\{\lambda(b)\}_{b\in M(D)}$ *of* $\mathcal{R}$ *is of the form*

$$\mathrm{Lg}(a) = \sum_{b\leqslant a} M(b,a)\,\lambda(b),$$

where the $M(b,a)$ are integers. In particular, all the factors $\mathrm{Lg}(b)$ *(respectively,* $\lambda(b)$*) appearing in the decomposition of* $\lambda(a)$ *(respectively, of* $\mathrm{Lg}(a)$*) have the same support. Furthermore,* $m(a,a) = M(a,a) = 1$.

In the non-archimedean case, Bruhat G-order was described by Zelevinsky [Zel80] (in the case where $A = F$) and Tadić [Tad90] in terms of linked segments. On arbitrary real reductive groups, Bruhat G-order was defined by Vogan, on a different set of parameters, in terms of integral roots (see [Vog82, Definition 12.12]). In all cases, Bruhat G-order is constructed by first defining *elementary operations*, starting from an element $a \in M(D)$ and obtaining another element $a' \in M(D)$; this is written as

$$a' \prec a.$$

Bruhat G-order is then generated by $\prec$. (In the case of $A = \mathbb{R}$, the situation is a little more complicated.) Another important property of Bruhat G-order is the following. One can define on all $M(D)_\omega$ a length function

$$l : M(D)_\omega \to \mathbb{N}$$

such that: if $b \leqslant a$, then $l(b) \leqslant l(a)$; if $b \leqslant a$ and $l(b) = l(a)$, then $b = a$; and, finally, if $b \leqslant a$ and $l(b) = l(a) - 1$, then $b \prec a$. In particular, if $b \prec a$, there is no $c \in M(D)_\omega$ such that $b \leqslant c < a$ but $b = c$.

We then have the following proposition.

Proposition 6.2. *Let $a, b \in M(D)_\omega$ such that $b \prec a$. Then $m(b,a) \neq 0$ and $M(b,a) \neq 0$.*

Proof. The first assertion follows from the recursion formulas for Kazhdan–Lusztig–Vogan polynomials in the archimedean case [Vog83]; we even have $m(b,a) = 1$ in this case. In the non-archimedean case, the first assertion was established by Zelevinsky [Zel80] and Tadić [Tad90]. The second assertion follows from Proposition 6.1. □

7. Unitary dual

7.1 Representations $u(\delta, n)$ and $\pi(\delta, n; \alpha)$

Let $\delta \in D$; then $\delta \times \delta$ is irreducible. Indeed, if $\delta \in D^u$, this is Theorem 3.1, and the general case follows from tensoring with an unramified character. Consider $\delta \times \nu^\alpha \delta$ with $\alpha > 0$. There exists a smallest $\alpha_0 > 0$ such that $\delta \times \nu^{\alpha_0}\delta$ is reducible.

Definition 7.1. Let $\delta \in D$. Set $\nu_\delta = \nu^{\alpha_0}$, where $\alpha_0 > 0$ is the smallest real number $\alpha > 0$ such that $\delta \times \nu^\alpha \delta$ is reducible.

For all $\delta \in D$ and $n \in \mathbb{N}^\times$, set

$$a(\delta,n) = (\nu_\delta^{(n-1)/2}\delta, \nu_\delta^{((n-1)/2)-1}\delta, \dots, \nu_\delta^{-(n-1)/2}\delta) \in M(D), \tag{7.1}$$

$$u(\delta,n) = \mathrm{Lg}(a(\delta,n)). \tag{7.2}$$

For all $\delta \in D$, $n \in \mathbb{N}^\times$ and $\alpha \in \mathbb{R}$, set

$$\pi(\delta, n; \alpha) = \nu_\delta^\alpha u(\delta, n) \times \nu_\delta^{-\alpha} u(\delta, n). \tag{7.3}$$

7.2 Tadić's hypotheses and classification of the unitary dual

We recall Tadić's classification of the unitary dual of the groups G_n. For a fixed division algebra A, consider the following hypotheses.

$U(0)$: if $\sigma, \tau \in \mathrm{Irr}^u$, then $\sigma \times \tau \in \mathrm{Irr}^u$.

$U(1)$: if $\delta \in D^u$ and $n \in \mathbb{N}^\times$, then $u(\delta, n) \in \mathrm{Irr}^u$.

$U(2)$: if $\delta \in D^u$, $n \in \mathbb{N}^\times$ and $\alpha \in]0, 1/2[$, then $\pi(\delta, n; \alpha) \in \mathrm{Irr}^u$.

$U(3)$: if $\delta \in D$, then $u(\delta, n)$ is prime in $\mathcal{R}$.

$U(4)$: if $a, b \in M(D)$, then $L(a) \times L(b)$ contains $L(a+b)$ as a subquotient.

Suppose Tadić's hypotheses are satisfied for A. Then we have the following theorem.

Theorem 7.2. *The set Irr^u is endowed with the structure of a free commutative monoid with product $(\sigma, \tau) \mapsto \sigma \times \tau$ and basis*

$$\mathcal{B} = \{u(\delta, n), \pi(\delta, n; \alpha) \mid \delta \in D^u, n \in \mathbb{N}^\times, \alpha \in]0, 1/2[\,\}.$$

More explicitly, if $\pi_1, \ldots, \pi_k \in \mathcal{B}$, then $\pi_1 \times \cdots \times \pi_k \in \mathrm{Irr}^u$, and if $\pi \in \mathrm{Irr}^u$, there exist $\pi_1, \ldots, \pi_k \in \mathcal{B}$, unique up to permutation, such that $\pi = \pi_1 \times \cdots \times \pi_k$.

This result is established in [Tad95, Proposition 2.1]. The proof is formal.

First, observe that $U(4)$ is a fairly simple consequence of the Langlands classification, established by Tadić for all A in [Tad06] (the proof works also for archimedean A; see [Tad09]). It is also easy to see that $U(2)$ can be deduced from $U(0)$ and $U(1)$ using the following simple principle: if $(\pi_t)_{t \in I}$ (where I is an open interval containing 0) is a family of hermitian representations in $\mathcal{M}(G)$, continuous in a sense that we shall not make precise here, and if π_0 is unitary and irreducible, then π_t is unitary on the largest interval $J \subset I$ containing 0 where π_t is irreducible (the signature of the hermitian form can change only when crossing reducibility points). The representations $\pi(\delta, n; \alpha)$, $\alpha \in \mathbb{R}$, are hermitian, $\pi(\delta, n; 0) = u(\delta, n) \times u(\delta, n)$ is unitary and irreducible (by $U(0)$ and $U(1)$), and $\pi(\delta, n; \alpha)$ is irreducible for $\alpha \in]-\frac{1}{2}, \frac{1}{2}[$. For details, see [Tad09] and the references therein.

For the remaining hypotheses $U(0)$, $U(1)$ and $U(3)$, the situation is more complicated.

- $U(3)$ was proved by Tadić in the non-archimedean case in [Tad86] and for $A = \mathbb{R}$ or $\mathbb{C}$ in [Tad09]. Below we give the proof for $A = \mathbb{H}$ following Tadić's ideas.
- $U(1)$ was proved by Tadić in the non-archimedean case in [Tad86] for the field case $A = F$. The generalization to all division algebras over F was given by the authors in [BR04], using unitarity of some distinguished representations closely related to the $u(\delta, n)$ established by the first author in [Bad07] using global methods. For $F = \mathbb{C}$, $u(\delta, n)$ is a unitary character, so the statement is obvious. For $F = \mathbb{R}$, $U(1)$ was first proved by Speh in [Spe83] using global methods. It can also be proved using Vogan's results on cohomological induction (see details below). Finally, for $A = \mathbb{H}$, $U(1)$ can be established by using again the general results on cohomological induction and the argument in [BR04]. A more detailed discussion of the archimedean case can be found in § 11.
- $U(0)$ is by far the most delicate. For $A = F$ non-archimedean, $U(0)$ was established by Bernstein in [Ber84], using reduction to the mirabolic subgroup. For $A = \mathbb{R}$ or $\mathbb{C}$,

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although the same approach can be used, some serious technical difficulties remained unsolved until the paper of Baruch [Bar03]. For A being a general non-archimedean division algebra, $U(0)$ was established by Sécherre [Sec09] using his deep results on Bushnell and Kutzko's type theory for the groups $\mathrm{GL}(n, A)$; these give Hecke algebra isomorphisms and allow one to reduce the problem to the field case (the proof also uses in a crucial way Barbash and Moy's results on unitarity for Hecke algebra representations [BM89]). In the $A = \mathbb{H}$ case there are, to our knowledge, no written references, but it is well known to some experts that $U(0)$ can be deduced from Vogan's classification of the unitary dual of G_n in the archimedean case [Vog86]. Vogan's classification is conceptually very different from Tadić's classification. It has its own merits, but the final result is quite difficult to state and to understand, since it uses sophisticated concepts and techniques from the theory of real reductive groups. So for people who are mainly interested in applications, for instance to automorphic forms, Tadić's classification is much more convenient. Before Baruch's paper was published, in the literature one could often find the statement of Tadić's classification along with a reference to Vogan's paper [Vog86] for the proof. It may not be totally obvious to non-experts how to derive Tadić's classification from Vogan's, so in this paper we take the opportunity to explain (see § 12) some aspects of Vogan's classification and how it is related to Tadić's classification, as well as how to deduce $U(0)$ from it. Of course, an independent proof of $U(0)$ would be highly desirable in this case. It would be even better to have a uniform proof of $U(0)$ for all cases; however, for this, new ideas are clearly needed.

- All these results are true if the characteristic of F is positive (as explained in [BHLS]).

8. Classification of generic irreducible unitary representations

From the classification of the unitary dual of $\mathrm{GL}(n, \mathbb{R})$ given above and the classification of irreducible generic representations of real reductive groups [Kos78, Vog78], we deduce the classification of generic irreducible unitary representations of $\mathrm{GL}(n, \mathbb{R})$. Let us first recall that Vogan gave a classification of 'large' irreducible representations of a quasi-split real reductive group (i.e. one having maximal Gelfand–Kirillov dimension), that Kostant showed that such a group admits generic representations if and only if the group is quasi-split, and that 'generic' is equivalent to 'large'. Therefore, Vogan's result can be stated as follows.

THEOREM 8.1. *Any generic irreducible representation of any quasi-split real reductive group is irreducibly induced from a generic limit of discrete series; and, conversely, a representation which is irreducibly induced from a generic limit of discrete series is generic.*

Note that in the above theorem, one can replace 'limit of discrete series' by 'essentially tempered', because according to [KZ82] any tempered representation is fully induced from a limit of discrete series. In the case of $\mathrm{GL}(n, \mathbb{R})$, all discrete series are generic; so by Theorem 3.1, all essentially tempered representations are generic.

Let us denote by $\mathrm{Irr}^u_{\mathrm{gen}}$ the subset of Irr^u consisting of generic representations. We then have the following specialization of Theorem 7.2.

THEOREM 8.2. *The set $\mathrm{Irr}^u_{\mathrm{gen}}$ is endowed with the structure of a free commutative monoid with product $(\sigma, \tau) \mapsto \sigma \times \tau$ and basis*

$$\mathcal{B}_{\mathrm{gen}} = \{u(\delta, 1), \pi(\delta, 1; \alpha) \mid \delta \in D^u, \alpha \in]0, 1/2[\,\}.$$

More explicitly, if $\pi_1, \ldots, \pi_k \in \mathcal{B}_{\mathrm{gen}}$, then $\pi_1 \times \cdots \times \pi_k \in \mathrm{Irr}^u_{\mathrm{gen}}$, and if $\pi \in \mathrm{Irr}^u_{\mathrm{gen}}$, then there exist $\pi_1, \ldots, \pi_k \in \mathcal{B}_{\mathrm{gen}}$, unique up to permutation, such that $\pi = \pi_1 \times \cdots \times \pi_k$.

9. Classification of discrete series: archimedean case

In this section, we describe explicitly square integrable modulo center irreducible representations of G_n in the archimedean case. For $A=\mathbb{H}$ we also give details about supports, Bruhat G-order etc. Since the Bruhat G-order is defined by Vogan on a set of parameters for irreducible representations consisting of (conjugacy classes of) characters of Cartan subgroups, we also describe the bijections between the various sets of parameters.

9.1 $A=\mathbb{C}$

There are square integrable modulo center irreducible representations of $\mathrm{GL}(n,\mathbb{C})$ only when $n=1$. Thus

$$D=D_1=\mathrm{Irr}_1.$$

An element $\delta\in D$ is then a character

$$\delta:\mathrm{GL}(1,\mathbb{C})\simeq\mathbb{C}^\times\to\mathbb{C}^\times.$$

Let $\delta\in D$. Then there exist a unique $n\in\mathbb{Z}$ and a unique $\beta\in\mathbb{C}$ such that

$$\delta(z)=|z|^{2\beta}\left(\frac{z}{|z|}\right)^n=|z|_{\mathbb{C}}^{\beta}\left(\frac{z}{|z|}\right)^n.$$

Take $x,y\in\mathbb{C}$ satisfying

$$\begin{cases}x+y=2\beta,\\ x-y=n.\end{cases}$$

With the above notation (and abusing it by writing a complex power of a complex number), we set

$$\delta(z)=\gamma(x,y)=z^x\bar{z}^y.$$

The following result is well known.

Proposition 9.1. *Take $\delta=\gamma(x,y)\in D$ as above. Then $\delta\times\nu^\alpha\delta$ is reducible for $\alpha=1$ and irreducible for $0\leqslant\alpha<1$. Thus $\nu_\delta=\nu$ (cf. Definition 7.1). In the case of reducibility, where $\alpha=1$, we have that in $\mathcal{R}$,*

$$\gamma(x,y)\times\gamma(x+1,y+1)=\mathrm{Lg}((\gamma(x,y),\gamma(x+1,y+1)))+\gamma(x,y+1)\times\gamma(x+1,y).$$

9.2 $A=\mathbb{R}$

There are square integrable modulo center irreducible representations of $\mathrm{GL}(n,\mathbb{R})$ only when $n=1$ or 2:

$$D=D_1\coprod D_2=\mathrm{Irr}_1\coprod D_2.$$

Let us start with the parametrization of D_1. An element $\delta\in D_1$ is a character

$$\delta:\mathrm{GL}(1,\mathbb{R})\simeq\mathbb{R}^\times\to\mathbb{C}^\times.$$

Let $\delta\in D_1$. Then there exist a unique $\epsilon\in\{0,1\}$ and a unique $\alpha\in\mathbb{C}$ such that

$$\delta(x)=|x|^\alpha\mathrm{sgn}(x)^\epsilon\quad\text{for } x\in\mathbb{R}^\times.$$

We set

$$\delta=\delta(\alpha,\epsilon).$$

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Let us now give a parametrization of D_2. Let $\delta_1, \delta_2 \in D_1$. Then $\delta_1 \times \delta_2$ is reducible if and only if there exists $p \in \mathbb{Z}\backslash\{0\}$ such that

$$\delta_1\delta_2^{-1}(x) = x^p \mathrm{sgn}(x) \quad \text{for } x \in \mathbb{R}^\times.$$

If $\delta_i = \delta(\alpha_i, \epsilon_i)$, we rewrite these conditions as

$$\alpha_1 - \alpha_2 = p, \quad \epsilon_1 - \epsilon_2 = p + 1 \mod 2. \tag{9.1}$$

If $\delta_1 \times \delta_2$ is reducible, we have that in $\mathcal{R}$,

$$\delta_1 \times \delta_2 = \mathrm{Lg}((\delta_1, \delta_2)) + \eta(\delta_1, \delta_2) \tag{9.2}$$

where $\eta(\delta_1, \delta_2) \in D_2$ and $\mathrm{Lg}((\delta_1, \delta_2))$ is an irreducible finite-dimensional representation (of dimension $|p|$ in the notation above).

Definition 9.2. If $\alpha_1, \alpha_2 \in \mathbb{C}$ satisfy $\alpha_1 - \alpha_2 \in \mathbb{Z}\backslash\{0\}$, we set

$$\eta(\alpha_1, \alpha_2) = \eta(\delta_1, \delta_2) \tag{9.3}$$

where $\delta_1(x) = |x|^{\alpha_1}$ and $\delta_2(x) = |x|^{\alpha_2}\mathrm{sgn}(x)^{\alpha_1-\alpha_2+1}$. This defines a surjective map from

$$\{(\alpha_1, \alpha_2) \in \mathbb{C}^2 \mid \alpha_1 - \alpha_2 \in \mathbb{Z}\backslash\{0\}\}$$

to D_2, and we have

$$\eta(\alpha_1, \alpha_2) = \eta(\alpha_1', \alpha_2') \Leftrightarrow \{\alpha_1, \alpha_2\} = \{\alpha_1', \alpha_2'\}.$$

This gives a parametrization of D_2 by pairs of complex numbers α_1, α_2 satisfying $\alpha_1 - \alpha_2 \in \mathbb{N}^\times$.

Remark 9.3. The representation $\eta(x, y) \in D_2$ for $x, y \in \mathbb{C}$ with $x - y \in \mathbb{Z}\backslash\{0\}$ is obtained from the character $\gamma(x, y)$ of $\mathbb{C}^\times$ by some appropriate functor of cohomological induction. But even when $x = y$, the functor of cohomological induction maps $\gamma(x, x)$ to an irreducible essentially tempered representation of $\mathrm{GL}(2, \mathbb{R})$, namely the *limit of discrete series* $\delta(x, 0) \times \delta(x, 1)$, which is an irreducible principal series.

For this reason we set, for $x \in \mathbb{C}$,

$$\eta(x, x) := \delta(x, 0) \times \delta(x, 1) \in \mathrm{Irr}_2. \tag{9.4}$$

Proposition 9.4. *Let $\delta \in D$. Then $\delta \times \nu^\alpha\delta$ is reducible for $\alpha = 1$ and irreducible for $0 \leqslant \alpha < 1$. Thus $\nu_\delta = \nu$ (cf. Definition 7.1).*

This result is also well known. Let us be more precise by giving the composition series for $\delta \times \nu\delta$. We start with the case where $\delta = \delta(\alpha, \epsilon) \in D_1$. Then we get from (9.2) that in $\mathcal{R}$,

$$\delta(\alpha, \epsilon) \times \delta(\alpha + 1, \epsilon) = \mathrm{Lg}(\delta(\alpha, \epsilon), \delta(\alpha + 1, \epsilon)) + \eta(\alpha, \alpha + 1). \tag{9.5}$$

In the case where $\delta = \eta(x, y) \in D_2$ with $x - y = r \in \mathbb{N}^\times$, we get that if $r \neq 1$,

$$\eta(x, y) \times \eta(x + 1, y + 1) = \mathrm{Lg}(\eta(x, y), \eta(x + 1, y + 1)) + \eta(x, y + 1) \times \eta(x + 1, y). \tag{9.6}$$

If $r = 1$, the situation degenerates, but the following formulas remain valid by coherent continuation (see § 13.1):

$$\eta(x, y) \times \eta(x + 1, y + 1) = \mathrm{Lg}(\eta(x, y), \eta(x + 1, y + 1)) + \eta(x, y + 1) \times \eta(x + 1, y).$$

Recall that our convention is that

$$\eta(y + 1, y + 1) = \delta(y + 1, 0) \times \delta(y + 1, 1)$$

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is a limit of discrete series; thus

$$\eta(y+1,y)\times\eta(y+2,y+1)=\mathrm{Lg}(\eta(y+1,y),\eta(y+2,y+1)) \\ +\delta(y+1,0)\times\delta(y+1,1)\times\eta(y+2,y). \tag{9.7}$$

9.3 $A=\mathbb{H}$

Let us identify quaternions with 2×2 matrices of the form

$$\begin{pmatrix}\alpha & \beta\\ -\bar\beta & \bar\alpha\end{pmatrix},\quad \alpha,\beta\in\mathbb{C}.$$

The reduced norm is given by

$$\mathrm{RN}\begin{pmatrix}\alpha & \beta\\ -\bar\beta & \bar\alpha\end{pmatrix}=|\alpha|^2+|\beta|^2.$$

The group of invertible elements $\mathbb{H}^\times$ contains $\mathrm{SU}(2)$, the kernel of the reduced norm. Thus we have an exact sequence

$$1\to\mathrm{SU}(2)\hookrightarrow\mathbb{H}^\times\xrightarrow{\mathrm{RN}}\mathbb{R}_+^\times\to 1,$$

and we can identify $\mathbb{H}^\times$ with the direct product $\mathrm{SU}(2)\times\mathbb{R}_+^\times$.

The group $\mathrm{GL}(n,\mathbb{H})$ is a real form of $\mathrm{GL}(2n,\mathbb{C})$; its elements are $2n\times 2n$ matrices composed of 2×2 quaternionic matrices as described above. Complex conjugacy on $\mathrm{GL}(2n,\mathbb{C})$ for this real form is given on the 2×2 blocks by

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\mapsto\begin{pmatrix}\bar\delta & -\bar\gamma\\ -\bar\beta & \bar\alpha\end{pmatrix}.$$

A maximal compact subgroup of $\mathrm{GL}(n,\mathbb{H})$ is then

$$\mathrm{Sp}(n)\simeq\mathrm{U}(2n)\cap\mathrm{GL}(n,\mathbb{H}).$$

Its rank is n, the rank of $\mathrm{GL}(n,\mathbb{H})$ is $2n$, and the split rank of the center is one. Thus there are square integrable modulo center representations only when $n=1$.

For $n=1$ and $D_1=\mathrm{Irr}_1$, all irreducible representations of $\mathbb{H}^\times$ are essentially square integrable modulo center, since $\mathbb{H}^\times$ is compact modulo center. Harish-Chandra's parametrization in this case is as follows: irreducible representations of $\mathbb{H}^\times$ are parametrized by some characters of a fundamental Cartan subgroup; here we choose

$$\mathbb{C}^\times\hookrightarrow\mathbb{H}^\times,\quad \alpha\mapsto\begin{pmatrix}\alpha & 0\\ 0 & \bar\alpha\end{pmatrix},$$

which is connected. Characters of $\mathbb{C}^\times$ were described in the $F=\mathbb{C}$ section; they are of the form $\gamma(x,y)$ where $x-y\in\mathbb{Z}$. An irreducible representation of $\mathbb{H}^\times$ is then parametrized by a pair of complex numbers (x,y) such that $x-y\in\mathbb{Z}$. The pairs (x,y) and (x',y') parametrize the same representation if and only if the characters $\gamma(x,y)$ and $\gamma(x',y')$ are conjugate under the Weyl group, i.e. the multisets (x,y) and (x',y') are equal. Furthermore, $\gamma(x,y)$ corresponds to an irreducible representation if and only if $x\neq y$. Let us denote by $\eta'(x,y)$ the representation parametrized by the multiset (x,y) with $x-y\in\mathbb{Z}\backslash\{0\}$. It is obtained from the character $\gamma(x,y)$ of the Cartan subgroup $\mathbb{C}^\times$ by cohomological induction.

Remark 9.5. In contrast to the $A=\mathbb{R}$ case, when we induced cohomologically the character $\gamma(x,x)$ of the Cartan subgroup $\mathbb{C}^\times$ to $\mathbb{H}^\times$, we get 0: there is no limit of discrete series. Thus we set $\eta'(x,x)=0$.

Remark 9.6. The Jacquet–Langlands correspondence (see § 4) between representations of $\mathrm{GL}(1,\mathbb{H})=\mathbb{H}^\times$ and essentially square integrable modulo center irreducible representations $\mathrm{GL}(2,\mathbb{R})$ is given by

$$\mathbf{C}(\eta(x,y))=\eta'(x,y) \quad \text{for } x,y\in\mathbb{C} \text{ with } x-y\in\mathbb{Z}\backslash\{0\}.$$

The representations $\eta(x,y)$ and $\eta'(x,y)$ are obtained by cohomological induction from the same character $\gamma(x,y)$ of the Cartan subgroup $\mathbb{C}^\times$ of $\mathrm{GL}(2,\mathbb{R})$ and $\mathbb{H}^\times$. In the case where $x=y$, the construction still respects the Jacquet–Langlands character relation since both sides are equal to zero.

More generally, let us now give the parametrization of irreducible representations of $\mathrm{GL}(n,\mathbb{H})$ by conjugacy classes of characters of Cartan subgroups. The group $\mathrm{GL}(n,\mathbb{H})$ has only one conjugacy class of Cartan subgroups, a representative being T_n, which consist of 2×2 block-diagonal matrices of the form $\left(\begin{smallmatrix}\alpha & 0\\ 0 & \bar{\alpha}\end{smallmatrix}\right)$. Thus $T_n\simeq(\mathbb{C}^\times)^n$ is connected, $\mathfrak{t}_n=\mathrm{Lie}(T)\simeq\mathbb{C}^n$ and $(\mathfrak{t}_n)_\mathbb{C}\simeq(\mathbb{C}\oplus\mathbb{C})^n$.

Let Λ be a character of T_n. Its differential

$$\lambda=d\Lambda:\mathfrak{t}_n\to\mathrm{Lie}(\mathbb{C}^\times)\simeq\mathbb{C},$$

is a $\mathbb{R}$-linear map, with complexification being the $\mathbb{C}$-linear map

$$\lambda=d\Lambda:\mathfrak{t}_\mathbb{C}\simeq(\mathbb{C}\oplus\mathbb{C})^n\to\mathrm{Lie}(\mathbb{C}^\times)\simeq\mathbb{C}.$$

Such a linear form is given by a n-tuple of pairs (λ_i,μ_i) such that $\lambda_i-\mu_i\in\mathbb{Z}$.

Since T_n is connected, a character Λ of T_n is determined by its differential. We write

$$\Lambda=\Lambda(\lambda_1,\mu_1,\dots,\lambda_n,\mu_n)=\Lambda((\lambda_i,\mu_i)_{1\leqslant i\leqslant n})$$

if its differential is given by the n-tuple of pairs (λ_i,μ_i) such that $\lambda_i-\mu_i\in\mathbb{Z}$.

Let $\mathcal{P}$ be the set of characters $\Lambda=\Lambda((\lambda_i,\mu_i)_{1\leqslant i\leqslant n})$ of the Cartan subgroup T_n such that $\lambda_i-\mu_i\in\mathbb{Z}\backslash\{0\}$.

Irreducible representations of $\mathrm{GL}(n,\mathbb{H})$ are parametrized by $\mathcal{P}$, with two characters Λ_1 and Λ_2 giving the same irreducible representation if and only if they are conjugate under $W(\mathrm{GL}(2n,\mathbb{C}),T_n)$. This group is isomorphic to $\{\pm 1\}^n\times\mathfrak{S}_n$. Its action on $\mathfrak{t}_\mathbb{C}\simeq(\mathbb{C}\oplus\mathbb{C})^n$ is as follows: each factor $\{\pm1\}$ acts inside the corresponding factor $\mathbb{C}\oplus\mathbb{C}$ by permutation, and $\mathfrak{S}_n$ acts by permuting the n factors $\mathbb{C}\oplus\mathbb{C}$. Thus we see that irreducible representations of $\mathrm{GL}(n,\mathbb{H})$ are parametrized by multisets of cardinality n consisting of pairs of complex numbers (λ_i,μ_i) such that $\lambda_i-\mu_i\in\mathbb{Z}\backslash\{0\}$. Since such a pair (λ_i,μ_i) parametrizes the representation $\eta'(\lambda_i,\mu_i)$, we recover the Langlands parametrization of Irr by $M(D)$. Let us denote by $\sim$ the equivalence relation on $\mathcal{P}$ given by the Weyl group action $W(\mathrm{GL}(2n,\mathbb{C}),T)$. We have described one-to-one correspondences

$$\mathcal{P}/\!\sim\ \simeq\mathrm{Irr}_n\simeq M(D)_n.$$

Recall that a support for $\mathrm{GL}(n,\mathbb{H})$ is a multiset of $2n$ complex numbers, i.e. an element of the quotient of $\mathfrak{t}^*_\mathbb{C}\simeq(\mathbb{C}\oplus\mathbb{C})^n\simeq\mathbb{C}^{2n}$ by the action of the Weyl group $W_\mathbb{C}\simeq\mathfrak{S}_{2n}$.

Definition 9.7. The support of a character $\Lambda=\Lambda((\lambda_i,\mu_i)_{1\leqslant i\leqslant n})\in\mathcal{P}$ is the multiset

$$(\lambda_1,\mu_1,\dots,\lambda_n,\mu_n).$$

It does not depend on the equivalence class of Λ with respect to $\sim$. If $\Lambda\in\mathcal{P}$ parametrizes the irreducible representation π, we write $\mathrm{Supp}(\Lambda)=\mathrm{Supp}(\pi)$.

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This describes explicitly the map

$$\mathcal{P} \to M(C), \quad \Lambda \mapsto \mathrm{Supp}(\Lambda)$$

and its fibers: two parameters

$$\Lambda_1((\lambda_i^1, \mu_i^1)) \quad \text{and} \quad \Lambda_2((\lambda_i^2, \mu_i^2))$$

have the same support if and only if the multisets

$$(\lambda_1^1, \dots, \lambda_n^1, \mu_1^1, \dots, \mu_n^1) \quad \text{and} \quad (\lambda_1^2, \dots, \lambda_n^2, \mu_1^2, \dots, \mu_n^2)$$

are equal. We denote by $\mathcal{P}(\omega)$ the fiber at ω.

We now give the description of the Bruhat G-order in terms of integral roots. We have the following decomposition of the Lie algebra:

$$\mathrm{Lie}(\mathrm{GL}(2n, \mathbb{C})) = (\mathfrak{g}_{2n})_{\mathbb{C}} = (\mathfrak{t}_n)_{\mathbb{C}} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_{\mathbb{C}}^{\alpha} \right)$$

where $R = \{\pm(e_i - e_j) : 1 \leqslant i < j \leqslant 2n\}$ is the usual root system of type A_{2n-1}. Let us denote by σ the non-trivial element of the Galois group of $\mathbb{C}/\mathbb{R}$.

The roots $\pm(e_{2j-1} - e_{2j})$, $j = 1, \dots, n$, are imaginary compact; thus

$$\sigma \cdot e_{2j-1} = e_{2j} \quad \text{for } j = 1, \dots, n.$$

Other roots are complex: for all j, l such that $1 \leqslant j \neq l \leqslant l$,

$$\sigma \cdot (e_{2j-1} - e_{2l-1}) = e_{2j} - e_{2l}, \quad \sigma \cdot (e_{2j-1} - e_{2l}) = e_{2j} - e_{2l-1}.$$

Let us fix a support ω and take Λ to be a character of T_n such that $\mathrm{Supp}(\Lambda) = \omega$, say $\Lambda = \Lambda((\lambda_i, \mu_i)_{i=1,\dots,n})$ where $\lambda_i - \mu_i \in \mathbb{Z}\backslash\{0\}$. Notice that $W_{\mathbb{C}} \simeq \mathfrak{S}_{2n}$ does not act on $\mathcal{P}(\omega)$, since the condition

$$\lambda_i - \mu_i \in \mathbb{Z}$$

may not hold anymore after some permutation of the λ_i.

Denote by W_Λ the subgroup of $W_{\mathbb{C}}$ consisting of elements w such that

$$w \cdot (\lambda_i, \mu_i)_i - (\lambda_i, \mu_i)_i \in (\mathbb{Z} \times \mathbb{Z})^n.$$

Then W_Λ is the Weyl group of the root system R_Λ of *integral roots* for Λ. A root $\alpha = e_k - e_l$ in R is integral for Λ if, upon writing

$$\lambda_1, \mu_1, \lambda_2, \mu_2, \dots, \lambda_n, \mu_n = \nu_1, \dots, \nu_{2n},$$

we have $\nu_k - \nu_l \in \mathbb{Z}$.

Suppose that the support ω is regular, i.e. that all the ν_i, $1 \leqslant i \leqslant 2n$, are distinct. We choose as a positive root system $R_\Lambda^+ \subset R_\Lambda$ the roots $e_k - e_l$ such that $\nu_k - \nu_l > 0$. This defines simple roots.

Let us state, first, a necessary and sufficient condition for reducibility of standard modules (for regular support).

Proposition 9.8. *Let $a = (\eta'(\lambda_i, \mu_i)_{i=1,\dots,n}) \in M(D)_\omega$ be parametrized by the character $\Lambda = \Lambda((\lambda_i, \mu_i)_{i=1,\dots,n})$ of T_n. Suppose that the support*

$$\omega = (\lambda_1, \mu_1, \dots, \lambda_n, \mu_n)$$

is regular. Then $\lambda(a)$ is reducible if and only if there exists a simple root $e_k - e_l$ in R_Λ^+ which is complex and such that if $e_k - e_l = e_{2i-1} - e_{2j-1}$ or $e_k - e_l = e_{2i} - e_{2j}$ for $i \neq j$, then

$$\lambda_i - \lambda_j > 0 \quad \text{and} \quad \mu_i - \mu_j > 0,$$

and if $e_k - e_l = e_{2i-1} - e_{2j}$ or $e_k - e_l = e_{2i} - e_{2j-1}$ for $i \neq j$, then

$$\lambda_i - \mu_j > 0 \quad \text{and} \quad \mu_i - \lambda_j > 0.$$

When ω is not regular, we still have a necessary condition for reducibility: if $\lambda(a)$ is reducible, then there exists a root $e_k - e_l$ in R_Λ^+ which is not necessarily simple but still satisfies the condition above.

See [Vog82].

DEFINITION 9.9. We still assume $\omega \in M(C)$ to be regular and suppose that $\Lambda \in \mathcal{P}(\omega)$ satisfies the reducibility criterion above for the simple integral complex root $e_k - e_l$. Write

$$\Lambda = \Lambda((\lambda_1, \mu_1), \ldots, (\lambda_n, \mu_n)) = \Lambda((\nu_1, \nu_2), \ldots, (\nu_{2n-1}, \nu_{2n})).$$

Let $\Lambda' \in \mathcal{P}(\omega)$ be obtained from Λ by exchanging ν_k and ν_l, and let $a' \in M(D)_\omega$ correspond to Λ'. We say that a' is obtained from a by an *elementary operation*, and we write $a' \prec a$. The Bruhat G-order on $M(D)_\omega$ is the partial order generated by $\prec$.

Let us now deduce from the above reducibility criterion the invariant ν_δ attached (cf. Definition 7.1) to an essentially square integrable modulo center irreducible representation $\delta = \eta'(x, y)$, where $x, y \in \mathbb{C}$ with $x - y \in \mathbb{Z}\backslash\{0\}$. We may suppose that $x - y = r > 0$, since $\eta'(x, y) = \eta'(y, x)$.

PROPOSITION 9.10. *With the previous notation, $\nu_\delta = \nu$ if $r > 1$ and $\nu_\delta = \nu^2$ if $r = 1$. Since r is the dimension of δ, we see that $\nu_\delta = \nu$ except when δ is a one-dimensional representation of* $\mathrm{GL}(1, \mathbb{H})$.

Proof. We want to study the reducibility of

$$\pi = \eta'(y + r, y) \times \eta'(y + r + \alpha, y + \alpha)$$

for $\alpha > 0$. The support of this representation is regular if and only if $y + r$, y, $y + r + \alpha$ and $y + \alpha$ are distinct; but since

$$y + r + \alpha > y + \alpha > y \quad \text{and} \quad y + r + \alpha > y + r > y,$$

the support is regular except when $r = \alpha$. The representation π is the standard representation attached to the character

$$\Lambda = \Lambda((y + r + \alpha, y + \alpha), (y + r, y)).$$

If $\alpha \notin \mathbb{Z}$, then the support is regular, all integral roots are imaginary compact for Λ, and then π is irreducible.

If $\alpha = 1$ and $r \neq 1$, then the support is regular, all the roots are integral for

$$\Lambda((y + r + 1, y + 1), (y + r, y)),$$

and $e_1 - e_3$ is a complex root, simple in

$$R_\Lambda^+ = \{e_1 - e_3, e_1 - e_2, e_1 - e_4, e_3 - e_2, e_3 - e_4, e_2 - e_4\}$$

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and satisfying the reducibility criterion since

$$(\sigma \cdot (e_1 - e_3))(y+r+1, y+1, y+r, y) = (e_2 - e_4)(y+r+1, y+1, y+r, y) = 1 > 0.$$

The only element smaller than Λ in the Bruhat G-order is

$$\Lambda' = \Lambda((y+r, y+1), (y+r+1, y)),$$

and we get

$$\begin{aligned}&\eta'(y+r, y) \times \eta'(y+r+1, y+1)\\ &\quad = \mathrm{Lg}(\eta'(y+r, y), \eta'(y+r+1, y+1)) + \eta'(y+r, y+1) \times \eta'(y+r+1, y).\end{aligned} \tag{9.8}$$

If $\alpha = 1$ and $r = 1$, the support is singular. Applying Zuckerman translation functors (see, e.g., [KV95]), we get

$$\begin{aligned}&\eta'(y+2, y) \times \eta'(y+3, y+1)\\ &\quad = \mathrm{Lg}(\eta'(y+2, y), \eta'(y+3, y+1)) + \eta'(y+1, y+1) \times \eta'(y+2, y).\end{aligned}$$

But, according to our convention, $\eta'(y+1, y+1) = 0$ (this is really what we get upon applying the translation functor to the wall), and thus

$$\eta'(y+1, y) \times \eta'(y+2, y+1) = \mathrm{Lg}(\eta'(y+1, y), \eta'(y+2, y+1))$$

is irreducible.

The next possibility of reducibility for $r = 1$ is $\alpha = 2$, but then the support is regular and we see as above that π is reducible; more precisely,

$$\begin{aligned}&\eta'(y+3, y+2) \times \eta'(y+1, y)\\ &\quad = \mathrm{Lg}(\eta'(y+3, y+2), \eta'(y+1, y)) + \eta'(y+2, y+1) \times \eta'(y+3, y).\end{aligned} \tag{9.9}$$

This completes the proof of the proposition. □

10. Hypothesis $U(3)$ for $A = \mathbb{H}$

We follow [Tad09], which gives a proof of $U(3)$ for $A = \mathbb{C}$ or $\mathbb{R}$, to deal with the case where $A = \mathbb{H}$.

Theorem 10.1. *Let $\delta = \eta'(y+r, y) \in D$ for $y \in \mathbb{C}$ and $r \in \mathbb{N}^\times$, and let $n \in \mathbb{N}^\times$. Then $u(\delta, n)$ is a prime in the ring $\mathcal{R}$.*

Proof. We know that δ is prime in $\mathcal{R}$, so we start with $n \geqslant 2$. Let us first deal with $r = 1$. In this case $\nu_\delta = 2$ and we have

$$\begin{aligned}a_0 = a(\delta, n) &= (\nu_\delta^{(n-1)/2}\delta, \nu_\delta^{((n-1)/2)-1}\delta, \ldots, \nu_\delta^{-(n-1)/2}\delta)\\ &= (\eta'(y+n, y+n-1), \eta'(y+n-2, y+n-3), \ldots \eta'(y-n+2, y-n+1)).\end{aligned}$$

Set $a_0 = a(\delta, n) = (X_1, \ldots, X_n)$ with

$$X_i = \gamma(y+n+2-2i, y+n+1-2i) \quad \text{for } i = 1, \ldots, n.$$

Remark 10.2. The support of $u(\delta, n)$ is the multiset

$$(y+n+2-2i, y+n+1-2i)_{i=1,\ldots n}.$$

This support is regular.

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Suppose that $u(\delta, n)$ is not prime in $\mathcal{R}$. Then there exist polynomials P and Q in the variables $d \in D$ which are non-invertible and such that $u(\delta, n) = PQ$. Since $u(\delta, n)$ is homogeneous in $\mathcal{R}$ for the natural graduation, the same holds for P and Q.

Let us write

$$P = \sum_{c \in M(D)} m(c, P)\lambda(c), \quad Q = \sum_{d \in M(D)} m(d, Q)\lambda(d). \tag{10.1}$$

Set $S_P = \{a \in M(D) \mid m(a, P) \neq 0\}$ and $S_Q = \{a \in M(D) \mid m(a, Q) \neq 0\}$. We get

$$\mathrm{Lg}(a_0) = X_1 \times X_2 \cdots \times X_n + \sum_{a \in M(D), a < a_0} M(a, a_0)\, \lambda(a).$$

Thus there exist $c_0 \in S_P$ and $d_0 \in S_Q$ such that

$$c_0 + d_0 = a_0 = (X_1, \ldots, X_n).$$

Since $\deg P > 0$ and $\deg Q > 0$, c_0 and d_0 are not empty and the polynomials P and Q are not constant. Denote by S_1 the set of X_i such that $X_i \in c_0$, and denote by S_2 the set of X_i such that $X_i \in d_0$. We get a partition of the X_i into two non-empty disjoint sets. Therefore we can find $1 \leqslant i \leqslant n-1$ such that

$$\{X_i, X_{i+1}\} \not\subset S_j \quad \text{for } j = 1, 2$$

and, without any loss of generality, we may suppose that $X_i \in S_1$ and $X_{i+1} \in S_2$. Furthermore, we have

$$|S_1| = \deg P, \quad |S_2| = \deg Q, \quad \deg P + \deg Q = n.$$

We get from (9.9) that $X_i \times X_{i+1}$ is reducible; more precisely,

$$X_i \times X_{i+1} = \mathrm{Lg}(X_i, X_{i+1}) + Y_i \times Y_{i+1}$$

where

$$Y_i = \eta'(y + n + 2 - 2i, y + n - 1 - 2i), \quad Y_{i+1} = \eta'(y + n + 1 - 2i, y + n - 2i).$$

We have $a_1 := (Y_i, Y_{i+1}) \prec (X_i, X_{i+1})$. Set

$$a_{i,i+1} = a_1 + (X_1, \ldots, X_{i-1}, X_{i+2}, \ldots, X_n).$$

Then $a_{i,i+1} \prec a_0$ and hence $M(a_1, a_0) \neq 0$ by Proposition 6.2. Therefore, there exist non-empty $c_1 \in S_P$ and $d_1 \in S_Q$ such that

$$c_1 + d_1 = a_{i,i+1}.$$

Suppose now that Y_i divides $\lambda(c_1)$ in $\mathcal{R}$. The case where Y_i divides $\lambda(d_1)$ is similar.

Suppose that $\lambda(Y_{i+1})$ divides $\lambda(c_1)$ also. We get a partition of the X_j, where $j \neq i, i+1$, into two non-empty sets S_1' and S_2' such that

$$c_1 = \{X_j : j \in S_1'\} + Y_i + Y_{i+1}, \quad d_1 = \{X_j : j \in S_2'\}.$$

As the polynomials P and Q are homogeneous, we get

$$\deg(P) = |S_1'| + 2, \quad \deg(Q) = |S_2'|.$$

We see that $X_{i+1} \notin T := S_1 \cup S_2'$ and thus $\{X_1, \ldots, X_n\} \not\subset T$. For $r \in \mathcal{R}$, denote by $\deg_T(r)$ the degree of r in the variables $X_j \in T$. We get $\deg_T P \geqslant |S_1| = \deg P$ and $\deg_T Q \geqslant |S_2'| = \deg Q$, so $\deg_T(\mathrm{Lg}(a_0)) \geqslant n$. But the fact that the total degree of $\mathrm{Lg}(a_0)$ is n implies $\deg_T(\mathrm{Lg}(a_0)) = n$.

The expression of $\mathrm{Lg}(a_0)$ in the basis $\lambda(b)$, $b \leqslant a_0$, shows that we can find $b_0 \in M(D)$ such that $M(b_0, a_0) \neq 0$, $\deg(b_0) = n$ and $\deg_T \lambda(b_0) = n$. Furthermore, $\lambda(b_0)$ can be written as

$$\lambda(b_0) = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} \quad \text{for } \alpha_j \in \mathbb{N} \text{ with } \alpha_1 + \cdots + \alpha_n = n.$$

Since $T \neq \{X_1, \ldots, X_n\}$, there exists j such that $\alpha_j > 1$. But then X_j appears with multiplicity at least two in b_0. Since $\mathrm{Supp}(b_0) = \mathrm{Supp}(a_0)$ is regular, we get a contradiction.

Suppose now that $\lambda(Y_{i+1})$ does not divide $\lambda(c_1)$. We get a partition of the X_j, where $j \neq i, i+1$, into two non-empty sets S_1' and S_2' such that

$$c_1 = \{X_j : j \in S_1'\} + Y_i, \quad d_1 = \{X_j : j \in S_2'\} + Y_{i+1}.$$

Now set $T = S_1' \cup S_2$ and observe that X_{i+1} does not belong to T; thus $\{X_1, \ldots, X_n\} \not\subset T$. For $r \in \mathcal{R}$, denote by $\deg_T(r)$ the degree of r in the variables $X_j \in T$ and Y_i. As above, we get that $\deg_T(\mathrm{Lg}(a_0)) = n$ and that there exists $b_0 \in M(D)$ such that $M(b_0, a_0) \neq 0$, $\deg(b_0) = n$ and $\deg_T(\lambda(b_0)) = n$. We can write

$$\lambda(b_0) = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} Y_i^{\alpha} \quad \text{for } \alpha_j \in \mathbb{N} \text{ with } \alpha_1 + \cdots + \alpha_n + \alpha = n.$$

Since $\{X_1, \ldots, X_n\} \not\subset T$, there exists j such that $\alpha_j = 0$. If $\alpha = 0$, we get a contradiction as before; thus $\alpha \geqslant 1$. But since multiplicities in $\mathrm{Supp}(a_0)$ are at most one, we get $\alpha = 1$, $\alpha_j = 1$ if $j \neq i+1, i$, and $\alpha_i = \alpha_{i+1} = 0$, which still gives a contradiction. This finishes the proof for the $r = 1$ case.

Let us deal with briefly the case where $r > 1$. Here $\nu_\delta = \nu$ and

$$\begin{aligned} a(\delta, n) &= (\nu^{(n-1)/2}\delta, \nu^{((n-1)/2)-1}\delta, \ldots, \nu^{-(n-1)/2}\delta) \\ &= \biggl(\eta'\biggl(x + \frac{n-1}{2}, y + \frac{n-1}{2}\biggr), \eta'\biggl(x + \frac{n-1}{2} - 1, y + \frac{n-1}{2} - 1\biggr), \\ &\qquad \ldots, \eta'\biggl(x - \frac{n-1}{2}, y - \frac{n-1}{2}\biggr)\biggr). \end{aligned}$$

Set $a_0 = a(\delta, n) = (X_1, \ldots, X_n)$ with

$$X_i = \gamma\biggl(y + r + \frac{n-1}{2} + 1 - i, y + \frac{n-1}{2} + 1 - i\biggr) \quad \text{for } i = 1, \ldots, n.$$

We proceed as above, now using formula (9.8) for the reducibility of $\lambda(X_i, X_{i+1})$:

$$\lambda(X_i, X_{i+1}) = \mathrm{Lg}(X_i, X_{i+1}) + \lambda(Y_i, Y_{i+1})$$

where

$$\begin{aligned} Y_i &= \eta'\biggl(y + r + \frac{n-1}{2} + 1 - i, y + \frac{n-1}{2} - i\biggr), \\ Y_{i+1} &= \eta'\biggl(y + r + \frac{n-1}{2} - i, y + \frac{n-1}{2} + 1 - i\biggr). \end{aligned}$$

In all cases, we get contradictions by inspecting multiplicities in the support. We leave the details to the reader. $\square$

11. Hypothesis $U(1)$ for the archimedean case

We recall briefly the arguments for $A = \mathbb{C}$ and $\mathbb{R}$, even though these are well known and presented elsewhere, because we will need the notation anyway. We give the complete argument for the $A = \mathbb{H}$ case.

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11.1 $A = \mathbb{C}$

This case is easy because for $\gamma = \gamma(x, y)$, with $x, y \in \mathbb{C}$ such that $x - y \in \mathbb{Z}$ is a character of $\mathbb{C}^\times$, we have

$$u(\gamma, n) = \gamma \circ \det.$$

Representations $u(\gamma, n)$ are thus one-dimensional representations of $\mathrm{GL}(n, \mathbb{C})$. Furthermore, if γ is unitary (i.e. $\mathrm{Re}(x + y) = 0$), then $u(\gamma, n)$ is unitary.

11.2 $A = \mathbb{R}$

There are two cases to consider. The first is where $\delta = \delta(\alpha, \epsilon) \in D_1$, with $\alpha \in \mathbb{C}$ and $\epsilon \in \{0, 1\}$. This is similar to the $A = \mathbb{C}$ case above, since

$$u(\delta, n) = \delta \circ \det.$$

Representations $u(\delta, n)$ are one-dimensional representations of $\mathrm{GL}(n, \mathbb{R})$. Furthermore, if δ is unitary (i.e. $\mathrm{Re}(\alpha) = 0$), then $u(\delta, n)$ is unitary.

The second case is where $\delta = \eta(x, y) \in D_2$, with $x, y \in \mathbb{C}$ such that $x - y = r \in \mathbb{N}^\times$. We have already mentioned, without giving any details, that $\eta(x, y)$ is obtained by cohomological induction from the character $\gamma(x, y)$ of the Cartan subgroup $\mathbb{C}^\times$ of $\mathrm{GL}(2, \mathbb{R})$. Let us be more precise now. The cohomological induction functors considered here are normalized as in [KV95, (11.150)]: suppose that $(\mathfrak{g}_\mathbb{C}, K)$ is a reductive pair associated to a real reductive group G, $\mathfrak{q}_\mathbb{C} = \mathfrak{l}_\mathbb{C} \oplus \mathfrak{u}_\mathbb{C}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}_\mathbb{C}$ with Levi factor $\mathfrak{l}_\mathbb{C}$, and L is the normalizer in G of $\mathfrak{q}_\mathbb{C}$; then we define the cohomological induction functor to be

$$\begin{gathered}\mathcal{R}_{\mathfrak{q}_\mathbb{C}} : \mathcal{M}(\mathfrak{l}_\mathbb{C}, K \cap L) \longrightarrow \mathcal{M}(\mathfrak{g}_\mathbb{C}, K)\\ X \mapsto \Gamma^S \circ \mathrm{pro}(X \otimes \tilde{\tau})\end{gathered}$$

where $S = \dim(\mathfrak{u}_\mathbb{C} \cap \mathfrak{k}_\mathbb{C})$, Γ^S is the Sth Zuckerman derived functor from $\mathcal{M}(\mathfrak{g}_\mathbb{C}, K \cap L)$ to $\mathcal{M}(\mathfrak{g}_\mathbb{C}, K)$, pro is the parabolic induction functor from $\mathcal{M}(\mathfrak{l}_\mathbb{C}, K \cap L)$ to $\mathcal{M}(\mathfrak{g}_\mathbb{C}, K \cap L)$, and $\tilde{\tau}$ is a character of L, the square root of the character $\bigwedge^{\mathrm{top}}(\mathfrak{u}_\mathbb{C}/\mathfrak{u}_\mathbb{C} \cap \mathfrak{k}_\mathbb{C})$ (note that such a square root is usually defined only on a double cover of L, but for the cases of interest here, i.e. products of $G = \mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{C})$ or $\mathrm{GL}(n, \mathbb{H})$, we can find such a square root on L). This normalization preserves infinitesimal character.

With this notation, for $G = \mathrm{GL}(2, \mathbb{R})$, $L \simeq \mathbb{C}^\times$ and $\mathfrak{u}_\mathbb{C} = \mathfrak{g}_\mathbb{C}^{e_1 - e_2}$ we get

$$\mathcal{R}_{\mathfrak{q}_\mathbb{C}}(\gamma(x, y)) = \eta(x, y) \quad \text{for } x, y \in \mathbb{C} \text{ with } x - y \in \mathbb{N}.$$

Recall the convention $\eta(x, x) = \delta(x, 0) \times \delta(x, 1)$ for limits of discrete series; so this formula is also valid when $x = y$.

Set $a_0 = a(\eta(x, y), n) \in M(D)$. The standard representation $\lambda(a_0)$ is obtained by parabolic induction from the representation

$$\eta = \eta\left(x + \frac{n-1}{2}, y + \frac{n-1}{2}\right) \otimes \eta\left(x + \frac{n-3}{2}, y + \frac{n-3}{2}\right) \otimes \cdots \otimes \eta\left(x - \frac{n-1}{2}, y - \frac{n-1}{2}\right)$$

of $\mathrm{GL}(2, \mathbb{R}) \times \cdots \times \mathrm{GL}(2, \mathbb{R})$; from what has just been said, the representation η is obtained by cohomological induction from the character

$$\gamma = \gamma\left(x + \frac{n-1}{2}, y + \frac{n-1}{2}\right) \otimes \gamma\left(x + \frac{n-3}{2}, y + \frac{n-3}{2}\right) \otimes \cdots \otimes \gamma\left(x - \frac{n-1}{2}, y - \frac{n-1}{2}\right)$$

of $\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$. Furthermore, $u(\eta(x, y), n)$ is the unique irreducible quotient of $\lambda(a_0)$.

Results on independence of polarization in [KV95, ch. 11] show that the standard representation $\lambda(a_0)$ could also be obtained from the character γ of $(\mathbb{C}^\times)^n$ in the following way: first, use parabolic induction from $(\mathbb{C}^\times)^n$ to $\mathrm{GL}(n,\mathbb{C})$ (with respect to the usual upper-triangular Borel subgroup) to get the standard representation

$$\gamma\biggl(x+\frac{n-1}{2},y+\frac{n-1}{2}\biggr)\times\gamma\biggl(x+\frac{n-3}{2},y+\frac{n-3}{2}\biggr)\times\cdots\times\gamma\biggl(x-\frac{n-1}{2},y-\frac{n-1}{2}\biggr) \tag{11.1}$$

whose unique irreducible quotient is $u(\gamma(x,y),n)$; then, use the cohomological induction functor $\mathcal{R}_{\mathfrak{q}_\mathbb{C}}$ from $\mathrm{GL}(n,\mathbb{C})$ to $\mathrm{GL}(2n,\mathbb{R})$ (the reader can guess which θ-stable parabolic subalgebra $\mathfrak{q}_\mathbb{C}$ we use). This shows also that $u(\delta,n)$ is the unique irreducible quotient of $\mathcal{R}_{\mathfrak{q}_\mathbb{C}}(u(\gamma(x,y),n))$. Now, irreducibility and unitarizability theorems from [KV95] also imply, since the character $u(\gamma(x,y),n)$ of $\mathrm{GL}(n,\mathbb{C})$ is in the weakly good range, that $\mathcal{R}_{\mathfrak{q}_\mathbb{C}}(u(\gamma(x,y),n))$ is irreducible and unitary if $u(\gamma(x,y),n)$ is unitary. Thus we get

$$\mathcal{R}_{\mathfrak{q}_\mathbb{C}}(u(\gamma(x,y),n))=u(\eta(x,y),n),$$

and this representation is unitary if and only if $\mathrm{Re}(x+y)=0$.

In the degenerate case of $x=y$ (see (9.4)), we get

$$\mathcal{R}_{\mathfrak{q}_\mathbb{C}}(u(\gamma(x,y),n))=u(\delta(x,0),n)\times u(\delta(x,1),n).$$

11.3 $A=\mathbb{H}$

Let $\delta=\eta'(x,y)$, where $x,y\in\mathbb{C}$ with $x-y\in\mathbb{N}^\times$, be an irreducible representation of $\mathbb{H}^\times$. Consider the representation $u(\eta'(x,y),n)$ and recall the invariant ν_δ of Definition 7.1. We have seen that $\nu_\delta=\nu$ when $x-y>1$ and $\nu_\delta=\nu^2$ when $x-y=1$. In the former situation, the discussion concerning unitarizability of $u(\eta'(x,y),n)$ is exactly the same as that in the $A=\mathbb{R}$ case: the standard representation $\lambda(a_0)$ whose unique irreducible quotient is $u(\eta'(x,y),n)$ is obtained by cohomological induction from $\mathrm{GL}(n,\mathbb{C})$ to $\mathrm{GL}(n,\mathbb{H})$ of the representation γ defined in (11.1). Furthermore, $u(\eta'(x,y),n)$ is the unique irreducible quotient of $R_{\mathfrak{q}'_\mathbb{C}}(u(\gamma(x,y),n))$ and is unitary if and only if $\mathrm{Re}(x+y)=0$.

When $\nu_\delta=\nu^2$, i.e. $x-y=1$, we get the same results, not for $u(\eta'(x,y),n)$ but rather for $\bar{u}(\eta'(x,y),n)$, the Langlands quotient of the standard representation

$$\begin{aligned}\eta'\biggl(x+\frac{n-1}{2},y+\frac{n-1}{2}\biggr)&\times\eta'\biggl(x+\frac{n-3}{2},y+\frac{n-3}{2}\biggr)\times\cdots\times\eta'\biggl(x-\frac{n-1}{2},y-\frac{n-1}{2}\biggr)\\ &=\nu^{(n-1)/2}\eta'(x,y)\times\nu^{(n-3)/2}\eta'(x,y)\times\cdots\times\nu^{-(n-1)/2}\eta'(x,y).\end{aligned}$$

Recall that $u(\eta'(x,y),n)$ is the Langlands quotient of

$$\begin{aligned}\nu_\delta^{(n-1)/2}\eta'(x,y)&\times\nu_\delta^{(n-3)/2}\eta'(x,y)\times\cdots\times\nu_\delta^{-(n-1)/2}\eta'(x,y)\\ &=\nu^{n-1}\eta'(x,y)\times\nu^{n-3}\eta'(x,y)\times\cdots\times\nu^{-(n-1)}\eta'(x,y).\end{aligned}$$

From the two conditions $x-y=1$ and $\mathrm{Re}(x+y)=0$ we see that, up to a twist by a unitary character, we only have to study the case of $u(\eta',n)$ with $\eta'=\eta'(1/2,-1/2)$. Unitarity of $u(\eta',n)$ can be deduced from unitarity of the $\bar{u}(\eta',k)$ as in [BR04], using the fact that

$$\bar{u}(\eta',2n+1)=u(\eta',n+1)\times u(\eta',n). \tag{11.2}$$

For further reference, note that we also have

$$\bar{u}(\eta',2n)=\nu^{\frac{1}{2}}u(\eta',n)\times\nu^{-\frac{1}{2}}u(\eta',n). \tag{11.3}$$

One can also observe (as was done by the referee) that $u(\eta', n)$ is the trivial representation, and therefore is certainly unitarizable.

12. Vogan's classification and $U(0)$ in the archimedean case

As we have already said, hypothesis $U(0)$ has been established in the case where $A = \mathbb{R}$ or $\mathbb{C}$ by work of M. Baruch that filled the serious technical gap remaining in Kirillov's treatment of Bernstein approach [Kir62]. It is also possible to establish $U(0)$ from Vogan's classification, and this will work for $A = \mathbb{H}$ as well. Of course, this might seem a rather convoluted and unnatural approach to achieve the final goal of proving the classification of the unitary dual in Tadić's form, since a direct comparison between the classifications is possible. However, let us point out the following issues.

- One of the main difficulties in Vogan's paper is to prove some special cases of $U(0)$ (another difficult point is the exhaustion of the list of unitary almost spherical representations). The rest of his paper uses only standard and general techniques of the representation of real reductive groups, mainly cohomological induction.
- The argument which allows comparison between the two classifications ('independence of polarizations') is also the one that leads to $U(0)$ from Vogan's classification.
- There is still some hope of finding an uniform proof of $U(0)$ for all A.

In this section, we give a brief overview of Vogan's paper [Vog86] and how it implies $U(0)$. Here, $A = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Let us fix a unitary character

$$\delta : \mathrm{GL}(1, A) \simeq A^\times \to \mathbb{C}^\times.$$

It extends canonically to a family of unitary characters

$$\delta_n : \mathrm{GL}(n, A) \to \mathbb{C}^\times$$

by composition with the determinant $\mathrm{GL}(n, A) \to \mathrm{GL}(1, A)$ (the non-commutative determinant of Dieudonné if $F = \mathbb{H}$).

The *basic blocks* of Vogan's classification are the representations

$$\nu^{i\beta}\delta_n, \quad \beta \in \mathbb{R}$$

(in Tadić's notation we have $\nu^{i\beta}\delta_n = u(\nu^{i\beta}\delta, n)$, a unitary character of $\mathrm{GL}(n, A)$) and the representations

$$\pi(\nu^{i\beta}\delta, n; \alpha) = \nu^{-\alpha}\nu^{i\beta}\delta_n \times \nu^{\alpha}\nu^{i\beta}\delta_n, \quad 0 < \alpha < \tfrac{1}{2}$$

of $\mathrm{GL}(2n, F)$. These are Stein's complementary series.

Vogan first considers parabolically induced representations of the form

$$\tau = \tau_1 \times \tau_2 \times \cdots \times \tau_r, \tag{12.1}$$

where each τ_j is either a unitary character

$$\tau_j = \nu^{\beta_j}\delta_{n_j}, \quad \beta_j \in i\mathbb{R},$$

or a Stein's complementary series

$$\tau_j = \pi(\nu^{\beta_j}\delta, n_j; \alpha), \quad \beta_j \in i\mathbb{R},\ 0 < \alpha < \tfrac{1}{2}.$$

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The reason for imposing these conditions is the following. Recall our choices of maximal compact subgroups $K(n, A)$ of $\mathrm{GL}(n, A)$ for $A = \mathbb{R}, \mathbb{C}$ and $\mathbb{H}$:

$$\mathrm{O}(n), \quad \mathrm{U}(n) \quad \text{and} \quad \mathrm{Sp}(n),$$

respectively, and denote by μ_n the restriction of δ_n to $K(n, A)$. We say that μ_n is a *special* one-dimensional representation of $K(n, A)$. If $A = \mathbb{R}$, then since μ_n factorizes through the determinant, there are two special representations of $\mathrm{O}(n)$, namely the trivial representation and the sign of the determinant. If $A = \mathbb{C}$, then special representations of $\mathrm{U}(n)$ are obtained by composing the determinant (with values in $\mathrm{U}(1)$) with a character of $\mathrm{U}(1)$ (given by an integer). Finally, if $A = \mathbb{H}$, then the only special representation of $\mathrm{Sp}(n)$ is the trivial one. A representation of $\mathrm{GL}(n, A)$ is said to be *almost spherical* (of type μ_n) if it contains the special K-type μ_n. This generalizes spherical representations. The characters $\delta_n \nu^\beta$ are exactly the ones whose restriction to $K(n, A)$ is μ_n. Thus the τ_i above are either almost spherical unitary characters of type μ (the family $\mu = (\mu_n)_n$ is fixed) or almost spherical Stein's complementary series of type μ.

Vogan showed the following [Vog86, Theorem 3.8].

Theorem 12.1. *The representations $\tau = \tau_1 \times \tau_2 \times \cdots \times \tau_r$ are:*

(i) *unitary;*

(ii) *irreducible.*

Furthermore, every irreducible, almost spherical of type μ, unitary representation is obtained in this way, and two irreducible, almost spherical of type μ, unitary representations

$$\tau = \tau_1 \times \tau_2 \times \cdots \times \tau_r$$

and

$$\tau' = \tau'_1 \times \tau'_2 \times \cdots \times \tau'_s$$

are equivalent if and only if the multisets $\{\tau'_i\}$ and $\{\tau_j\}$ are equal.

Note that this theorem contains a special case of $U(0)$, i.e. point (ii). It can be proved using [Bad08, Proposition 2.13] and results of Sahi [Sah95, Theorem 3A].

Furthermore, the classification of irreducible, almost spherical, unitary representations given by this theorem coincides with Tadić's classification. (One has to notice that an irreducible, almost spherical, unitary representation is such with respect to a unique special K-type: special K-types are minimal, and minimal K-types for $\mathrm{GL}(n, A)$ are unique and appear with multiplicity one.)

Vogan's classification of the unitary dual of $\mathrm{GL}(n, A)$ reduces matters to this particular case of almost spherical representations by using cohomological induction functors that preserve irreducibility and unitarity. More precisely, let us recall some material about Vogan's classification of the admissible dual of a real reductive group G by minimal K-types [Vog79]. To each irreducible representation of G is attached a finite number of minimal K-types. As mentioned above, for $G = \mathrm{GL}(n, A)$ the minimal K-type is unique and appears with multiplicity one. This gives a partition (which can be explicitly given in terms of the Langlands classification) of the admissible dual of $\mathrm{GL}(n, A)$.

Vogan's classification of the unitary dual deals with each term of this partition separately. To each irreducible representation μ of the compact group $K(n, A)$ is attached a subgroup L of $\mathrm{GL}(n, A)$ with maximal compact subgroup $K_L := K(n, A) \cap L$ and an irreducible

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representation μ_L of K_L. The subgroup L is a product of groups of the form $\mathrm{GL}(n_i, A_i)$,

$$K(n, A) \cap L \simeq \prod_i K(n_i, A_i),$$

and μ_L is a tensor product of special representations of the $K(n_i, A_i)$.

As opposed to Tadić's classification, which uses only parabolic induction functors, Vogan's classification of $\mathrm{GL}(n, \mathbb{R})$, for instance, uses classification of the almost spherical unitary dual of groups $\mathrm{GL}(k, \mathbb{C})$. More precisely:

- for $F = \mathbb{R}$, the subgroups L are products of $\mathrm{GL}(k, \mathbb{R})$ and $\mathrm{GL}(m, \mathbb{C})$;
- for $F = \mathbb{C}$, the subgroups L are products of $\mathrm{GL}(k, \mathbb{C})$;
- for $F = \mathbb{H}$, the subgroups L are products of $\mathrm{GL}(k, \mathbb{H})$ and $\mathrm{GL}(m, \mathbb{C})$.

A combination of parabolic and cohomological induction functors then defines a functor

$$\mathcal{I}_L^G$$

from $\mathcal{M}(L)$ to $\mathcal{M}(\mathrm{GL}(n, A))$ with the following properties.

- $\mathcal{I}_L^G$ sends an irreducible (respectively, unitary) representation of L with minimal K_L-type μ_L to an irreducible (respectively, unitary) representation of $\mathrm{GL}(n, F)$ with minimal K-type μ.
- $\mathcal{I}_L^G$ realizes a bijection between equivalence classes of irreducible unitary representations of L with minimal K_L-type μ_L and equivalence classes of irreducible unitary representations of $\mathrm{GL}(n, F)$ with minimal K-type μ.

From this point of view, to establish hypothesis $U(0)$, the first thing to do is to check that products of representations of the form (12.1) for different families of special K-types μ are irreducible. For $F = \mathbb{H}$, there is nothing to check since there is only one family of special K-types $\mu = (\mu_n)_n$. For $F = \mathbb{R}$, there are two families of special K-types, the trivial and sign characters of the determinant of $\mathrm{O}(n)$. The relevant result is then [Vog86, Lemma 16.1]. For $F = \mathbb{R}$, we have now obtained all irreducible unitary representations which are products of $u(\delta, k)$ and $\pi(\delta, k; \alpha)$ with δ being any unitary character of $\mathrm{GL}(1, \mathbb{R}) = \mathbb{R}^\times$.

The $A = \mathbb{C}$ case is simpler and is dealt with as follows. Notice first that since square integrable modulo center representations of $\mathrm{GL}(n, \mathbb{C})$ exist only for $n = 1$, the above assertion implies that we get all representations of Tadić's classification, and this establishes $U(0)$. In that case, the subgroups L from which we use cohomological induction are of the form

$$L = \mathrm{GL}(n_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(n_r, \mathbb{C}).$$

The cohomological induction setting requires that $\mathfrak{l}_{\mathbb{C}} = \mathrm{Lie}(L)_{\mathbb{C}}$ be a Levi factor of a θ-stable parabolic subalgebra $\mathfrak{q}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}} = \mathrm{Lie}(\mathrm{GL}(n, \mathbb{C}))_{\mathbb{C}}$. But L is also a Levi factor of a parabolic subgroup of $\mathrm{GL}(n, \mathbb{C})$. Thus there are two ways of inducing from L to $\mathrm{GL}(n, \mathbb{C})$: parabolic and cohomological induction. An 'independence of polarization' result, [Vog86, Theorem 17.6] (see [KV95, ch. 11] for a proof), asserts that the two coincide. This finishes the argument in the case of $A = \mathbb{C}$.

We now proceed to discuss the cases of $A = \mathbb{R}$ and $A = \mathbb{H}$. The representations from Tadić's classification which are still missing are the ones built from the $u(\delta, k)$ and $\pi(\delta, k; \alpha)$ with δ a square integrable modulo center representation of $\mathrm{GL}(2, \mathbb{R})$ or $\mathbb{H}^\times$. As we have seen in § 11.2, a square integrable modulo center representation of $\mathrm{GL}(2, \mathbb{R})$ or $\mathbb{H}^\times$ is obtained by cohomological induction from the subgroup $L \simeq \mathbb{C}^\times$ of $\mathrm{GL}(2, \mathbb{R})$ or $\mathrm{GL}(1, \mathbb{H}) = \mathbb{H}^\times$. This indicates why cohomological induction will produce the missing representations. Let us explain this in more detail.

For $F=\mathbb{R}$, we start with representations of the form

$$u(\chi_a,k_a),\ \pi(\chi_b,k_b;\alpha_b),\ u(\chi_c,k_c),\ \pi(\chi_d,k_d;\alpha_d),\ u(\chi_e,k_e),\ \pi(\chi_f,k_f;\alpha_f)$$

where the $u(\chi_a,k_a)$ are unitary characters of $\mathrm{GL}(k_a,\mathbb{C})$, the $\pi(\chi_b,k_b;\alpha_b)$ are Stein complementary series of $\mathrm{GL}(2k_b,\mathbb{C})$, the $u(\chi_c,k_c)$ are unitary characters of $\mathrm{GL}(k_c,\mathbb{R})$ of trivial type μ, the $\pi(\chi_d,k_d;\alpha_d)$ are Stein complementary series of $\mathrm{GL}(2k_d,\mathbb{R})$ of trivial type μ, the $u(\chi_e,k_e)$ are unitary characters of $\mathrm{GL}(k_c,\mathbb{R})$ of type $\mu=\mathrm{sgn}$, and the $\pi(\chi_f,k_f;\alpha_f)$ are Stein complementary series of $\mathrm{GL}(2k_f,\mathbb{R})$ of type $\mu=\mathrm{sgn}$.

The tensor product

$$\Big(\bigotimes_a u(\chi_a,k_a)\Big)\otimes\Big(\bigotimes_b \pi(\chi_b,k_b;\alpha_b)\Big)\otimes\Big(\bigotimes_c u(\chi_c,k_c)\Big)$$
$$\otimes\Big(\bigotimes_d \pi(\chi_d,k_d;\alpha_d)\Big)\otimes\Big(\bigotimes_e u(\chi_e,k_e)\Big)\otimes\Big(\bigotimes_f \pi(\chi_f,k_f;\alpha_f)\Big)$$

is a representation of the Levi subgroup

$$\prod_a \mathrm{GL}(k_a,\mathbb{C})\prod_b \mathrm{GL}(2k_b,\mathbb{C})\prod_c \mathrm{GL}(k_c,\mathbb{R})\prod_d \mathrm{GL}(2k_d,\mathbb{R})\prod_e \mathrm{GL}(k_e,\mathbb{R})\prod_f \mathrm{GL}(2k_f,\mathbb{R})$$

of $\mathrm{GL}(n,\mathbb{R})$, where $n=\sum_a 2k_a+\sum_b 4k_b+\sum_c k_c+\sum_d 2k_d+\sum_e k_e+\sum_k 2k_f$.

As seen previously, we first form almost spherical representations of a given type by parabolic induction. Thus we induce

$$\Big(\bigotimes_c u(\chi_c,k_c)\Big)\otimes\Big(\bigotimes_d \pi(\chi_d,k_d;\alpha_d)\Big)$$

from

$$\prod_c \mathrm{GL}(k_c,\mathbb{R})\prod_d \mathrm{GL}(2k_d,\mathbb{R})$$

to $\mathrm{GL}(q_0,\mathbb{R})$ where $q_0=\sum_c k_c+\sum_d 2k_d$, obtaining an irreducible unitary spherical representation π_0; similarly, we induce

$$\Big(\bigotimes_e u(\chi_e,k_e)\Big)\otimes\Big(\bigotimes_f \pi(\chi_f,k_f;\alpha_f)\Big)$$

from

$$\prod_e \mathrm{GL}(k_e,\mathbb{R})\prod_d \mathrm{GL}(2k_f,\mathbb{R})$$

to $\mathrm{GL}(q_1,\mathbb{R})$ where $q_1=\sum_e k_e+\sum_f 2k_f$, obtaining an irreducible unitary almost spherical representation of type $\mu=\mathrm{sgn}$.

Then we mix spherical representations and almost spherical representations of type $\mu=\mathrm{sgn}$, parabolically inducing $\pi_0\times\pi_1$ from $\mathrm{GL}(q_0,\mathbb{R})\times\mathrm{GL}(q_1,\mathbb{R})$ to $\mathrm{GL}(q_0+q_1,\mathbb{R})$. We thus get an irreducible unitary representation π of $\mathrm{GL}(q_0+q_1,\mathbb{R})$.

The group $\prod_a \mathrm{GL}(k_a,\mathbb{C})\prod_b \mathrm{GL}(2k_b,\mathbb{C})\times\mathrm{GL}(q_0+q_1,\mathbb{R})$ is denoted by L_θ in [Vog86]. Applying the cohomological induction functor $\mathcal{I}_{L_\theta}^G$ to the representation

$$\Big(\bigotimes_a u(\chi_a,k_a)\Big)\otimes\Big(\bigotimes_b \pi(\chi_b,k_b;\alpha_b)\Big)\otimes\pi$$

of L_θ, we get an irreducible unitary representation ρ of $\mathrm{GL}(n,\mathbb{R})$.

Independence of polarization theorems (see [Vog86, Theorems 17.6, 17.7 and 17.9] and [KV95, ch. 11]) allow us to reverse the order of the two types of induction. We could in fact start with cohomological induction, inducing each

$$u(\chi_a, k_a)$$

from $\mathrm{GL}(k_a, \mathbb{C})$ to $\mathrm{GL}(2k_a, \mathbb{R})$. In the non-degenerate case, following the terminology of [Vog86, Definition 17.3], we get representations $u(\delta_a, 2k_a)$ where δ_a is a square integrable modulo center irreducible representation of $\mathrm{GL}(2, \mathbb{R})$. In the degenerate case, δ_a is a limit of discrete series (9.4). These are almost spherical representations of the kind we had before (see [Vog86, Proposition 17.10]).

In the same way, we induce all

$$\pi(\chi_b, k_b; \alpha_b)$$

from $\mathrm{GL}(2k_b, \mathbb{C})$ to $\mathrm{GL}(4k_b, \mathbb{R})$. In the non-degenerate case, we get representations $\pi(\delta_b, 2k_b; \alpha_b)$, where δ_b is as above. In the degenerate case, we still get almost spherical representations.

The parabolically induced representation from

$$\prod_a \mathrm{GL}(2k_a, \mathbb{R}) \prod_b \mathrm{GL}(4k_b, \mathbb{R}) \times \mathrm{GL}(q_0 + q_1, \mathbb{R})$$

to $\mathrm{GL}(n, \mathbb{R})$ of

$$\left(\bigotimes_a u(\delta_a, k_a)\right) \otimes \left(\bigotimes_b \pi(\delta_b, k_b; \alpha_b)\right) \otimes \pi$$

is ρ (and thus irreducible); see [Vog86, Theorem 17.6].

This finishes the comparison of the two classifications. The $A = \mathbb{H}$ case is entirely similar.

We deduce hypothesis $U(0)$ by using independence of polarization again. We want to show that $\rho = \rho_1 \times \rho_2$ is irreducible if ρ_1 and ρ_2 are irreducible and unitary. We write ρ_1 and ρ_2 as above using, first, cohomological induction and, then, parabolic induction. Using parabolic induction in stages, we see that $\rho_1 \times \rho_2$ can also be written in this form. Using independence of polarization again, we write ρ as a parabolically and then cohomologically induced representation; we see that, as such, this is a representation appearing in Vogan's classification, which must therefore be irreducible.

13. Jacquet–Langlands correspondence in the archimedean case

The ideas in this section are taken from [AH97], which deals with a similar problem (Kazhdan–Patterson lifting).

13.1 Jacquet–Langlands correspondence and coherent families

Since we need to consider the $A = \mathbb{R}$ and $A = \mathbb{H}$ cases simultaneously, we shall add superscripts to the notation when needed, as in § 4. We have already noted that the Jacquet–Langlands correspondence between essentially square integrable modulo center irreducible representations of $\mathrm{GL}(2, \mathbb{R})$ and irreducible representations of $\mathbb{H}^\times$ is given at the level of Grothendieck groups by

$$\mathbf{LJ}(\eta(x, y)) = -\eta'(x, y).$$

Representations in $D_1^{\mathbb{R}}$ are sent to 0. We extend this linearly to an algebra morphism

$$\mathcal{R}^{\mathbb{R}} \to \mathcal{R}^{\mathbb{H}}.$$

Lemma 13.1. *Jacquet–Langlands correspondence preserves supports.*

Proof. Let $a = (\eta(x_1, y_1), \dots, \eta(x_r, y_r)) \in M(D)$. We then have

$$\mathbf{LJ}(\lambda(a)) = (-1)^r \lambda(a')$$

where $a = (\eta'(x_1, y_1), \dots, \eta'(x_r, y_r))$. The support of a is $(x_1, y_1, \dots, x_r, y_r)$, and this is also the support of a'. □

We now recall the definition of a coherent family of Harish-Chandra modules.

Definition 13.2. Let G be a real reductive group and H a Cartan subgroup, and let $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$ be the respective complexifications of their Lie algebras. Let Λ be the lattice of weights of H in finite-dimensional representations of G. A coherent family of (virtual) Harish-Chandra modules based at $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ is a family

$$\{\pi(\lambda + \mu) \mid \mu \in \Lambda\}$$

(note that $\lambda + \mu$ is merely a formal expression, since the two terms are not actually in the same group) in the Grothendieck group $\mathcal{R}(G)$ such that the following hold.

- The infinitesimal character of $\pi(\lambda + \mu)$ is given by $\lambda + d\mu$.
- For any finite-dimensional representation F of G we have, with $\Delta(F)$ denoting the set of weights of H in F, the following identity in $\mathcal{R}(G)$:

$$\pi(\lambda + \mu) \otimes F = \sum_{\gamma \in \Delta(F)} \pi(\lambda + \mu + \gamma).$$

Jacquet–Langlands correspondence preserves coherent families.

Lemma 13.3. *Let us identify two Cartan subgroups H and H' of* $\mathrm{GL}(2n, \mathbb{R})$ *and* $\mathrm{GL}(n, \mathbb{H})$, *respectively, isomorphic to $(\mathbb{C}^\times)^n$. Let $\pi(\lambda + \mu)$ be a coherent family of Harish-Chandra modules for* $\mathrm{GL}(2n, \mathbb{R})$ *based at $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$. Then $\mathbf{LJ}(\pi(\lambda + \mu))$ is a coherent family for* $\mathrm{GL}(n, \mathbb{H})$.

Proof. The first property of coherent families is satisfied by $\mathbf{LJ}(\pi(\lambda + \mu))$ because of the previous lemma. For the second property, let us remark first that since $\mathrm{GL}(2n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{H})$ are two real forms of $\mathrm{GL}(2n, \mathbb{C})$, a finite-dimensional representation F of one of these two groups is in fact the restriction of a finite-dimensional representation of $\mathrm{GL}(2n, \mathbb{C})$. We get that, for any regular element g' of $\mathrm{GL}(n, \mathbb{H})$ corresponding to an element g in $\mathrm{GL}(2n, \mathbb{R})$,

$$\begin{aligned}\sum_{\gamma \in \Delta(F)} \Theta_{\mathbf{LJ}(\pi(\lambda+\mu+\gamma))}(g') &= \sum_{\gamma \in \Delta(F)} \Theta_{\pi(\lambda+\mu+\gamma)}(g) = \Theta_{\pi(\lambda+\mu)\otimes F}(g) \\ &= \Theta_{\pi(\lambda+\mu)}(g)\, \Theta_F(g) = \Theta_{\mathbf{LJ}(\pi(\lambda+\mu))}(g')\Theta_F(g') = \Theta_{\mathbf{LJ}(\pi(\lambda+\mu))\otimes F}(g')\end{aligned}$$

and so

$$\sum_{\gamma \in \Delta(F)} \mathbf{LJ}(\pi(\lambda + \mu + \gamma)) = \mathbf{LJ}(\pi(\lambda + \mu)) \otimes F.$$ □

13.2 Jacquet–Langlands correspondence and cohomological induction

The cohomological induction functor $\mathcal{R}_{\mathfrak{q}_{\mathbb{C}}}$ introduced in § 11.2 preserves irreducibility and unitarity when the infinitesimal character of the induced module satisfies certain positivity

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properties with respect to $\mathfrak{q}_{\mathbb{C}}$ ('weakly good range'). Furthermore, under the same conditions, other derived functors $\Gamma^i(\mathrm{pro}(\bullet \otimes \tilde{\tau}))$, $i \neq S$, vanish. This is not true in general, and is why we need to consider the Euler–Poincaré characteristic

$$\widehat{\mathcal{R}}_{\mathfrak{q}_{\mathbb{C}}} := \sum_i (-1)^i \Gamma^i(\mathrm{pro}(\bullet \otimes \tilde{\tau})).$$

This is no longer a functor between $\mathcal{M}(L)$ and $\mathcal{M}(G)$ but, rather, is simply a morphism between the Grothendieck groups $\mathcal{R}(L)$ and $\mathcal{R}(G)$.

LEMMA 13.4. *The morphism $\widehat{\mathcal{R}}_{\mathfrak{q}_{\mathbb{C}}} : \mathcal{R}(L) \to \mathcal{R}(G)$ preserves coherent families.*

Proof. The functors $\Gamma^i(\mathrm{pro}(\bullet \otimes \tilde{\tau}))$ are normalized in order to preserve infinitesimal character, thus the first property of coherent family is preserved.

Let $\pi(\lambda+\mu)$ be a coherent family of Harish-Chandra modules for $(\mathfrak{l}, L \cap K)$. We want to show that for any finite-dimensional representation F of G,

$$\widehat{\mathcal{R}}_{\mathfrak{q}_{\mathbb{C}}}(\pi(\lambda+\mu)) \otimes F = \sum_{\gamma \in \Delta(F)} \widehat{\mathcal{R}}_{\mathfrak{q}_{\mathbb{C}}}(\pi(\lambda+\mu+\gamma)). \tag{13.1}$$

However,

$$\begin{aligned}\sum_{\gamma \in \Delta(F)} \widehat{\mathcal{R}}_{\mathfrak{q}_{\mathbb{C}}}(\pi(\lambda+\mu+\gamma)) &= \widehat{\mathcal{R}}_{\mathfrak{q}_{\mathbb{C}}}\left(\sum_{\gamma \in \Delta(F)} \pi(\lambda+\mu+\gamma)\right) \\ &= \widehat{\mathcal{R}}_{\mathfrak{q}_{\mathbb{C}}}(\pi(\lambda+\mu) \otimes F).\end{aligned}$$

Hence it is enough to show that for any $(\mathfrak{l}, L \cap K)$-module X,

$$\widehat{\mathcal{R}}_{\mathfrak{q}_{\mathbb{C}}}(X) \otimes F = \widehat{\mathcal{R}}_{\mathfrak{q}_{\mathbb{C}}}(X \otimes F). \tag{13.2}$$

Let U be any $(\mathfrak{g}, K)$-module. Using adjunction properties of the functors involved, we compute:

$$\begin{aligned}&\mathrm{Hom}_{\mathfrak{g},K}(U, \Gamma(\mathrm{pro}((X \otimes F) \otimes \tilde{\tau}))) \simeq \mathrm{Hom}_{\mathfrak{l},L\cap K}(U, X \otimes F \otimes \tilde{\tau}) \\ &\quad \simeq \mathrm{Hom}_{\mathfrak{l},L\cap K}(U, X \otimes (F^*)^* \otimes \tilde{\tau}) \simeq \mathrm{Hom}_{\mathfrak{l},L\cap K}(U, \mathrm{Hom}_{\mathbb{C}}(F^*, X \otimes \tilde{\tau})) \\ &\quad \simeq \mathrm{Hom}_{\mathfrak{l},L\cap K}(U \otimes F^*, X \otimes \tilde{\tau}) \simeq \mathrm{Hom}_{\mathfrak{g},K}(U \otimes F^*, \Gamma(\mathrm{pro}(X \otimes \tilde{\tau}))) \\ &\quad \simeq \mathrm{Hom}_{\mathfrak{g},K}(U, \Gamma(\mathrm{pro}(X \otimes \tilde{\tau})) \otimes F).\end{aligned}$$

From this we deduce that $\Gamma(\mathrm{pro}(X \otimes \tilde{\tau} \otimes F)) \simeq \Gamma(\mathrm{pro}(X \otimes \tilde{\tau})) \otimes F$.

The same is true with Γ^i replacing Γ in the computation above. This can be seen by using general arguments and the exactness of the functor $\bullet \otimes F$. Thus, for all $i \geqslant 0$, $\Gamma^i(\mathrm{pro}(X \otimes \otimes \tilde{\tau} \otimes F)) \simeq \Gamma^i(\mathrm{pro}(X \otimes \tilde{\tau})) \otimes F$, which implies (13.2). □

Let us denote by $\widehat{\mathcal{R}}^{\mathbb{R}}_{\mathfrak{q}_{\mathbb{C}}}$ and $\widehat{\mathcal{R}}^{\mathbb{H}}_{\mathfrak{q}'_{\mathbb{C}}}$ the Euler–Poincaré morphisms of cohomological induction between $\mathrm{GL}(1,\mathbb{C})$ and, respectively, $\mathrm{GL}(2,\mathbb{R})$ and $\mathrm{GL}(1,\mathbb{H})$, where $\mathfrak{q}_{\mathbb{C}}$ and $\mathfrak{q}'_{\mathbb{C}}$ are as given in §§ 11.2 and 11.3.

LEMMA 13.5. *With the above notation, for $x, y \in \mathbb{C}$ with $x - y \in \mathbb{Z}$ we have*

$$\mathbf{LJ}(\widehat{\mathcal{R}}^{\mathbb{R}}_{\mathfrak{q}_{\mathbb{C}}}(\gamma(x,y))) = -\widehat{\mathcal{R}}^{\mathbb{H}}_{\mathfrak{q}'_{\mathbb{C}}}(\gamma(x,y)).$$

Proof. When $x - y \geqslant 0$, we have

$$\widehat{\mathcal{R}}^{\mathbb{R}}_{\mathfrak{q}_{\mathbb{C}}}(\gamma(x,y)) = -\mathcal{R}^{\mathbb{R}}_{\mathfrak{q}_{\mathbb{C}}}(\gamma(x,y)) = -\eta(x,y)$$

and

$$\widehat{\mathcal{R}}^{\mathbb{H}}_{\mathfrak{q}'_{\mathbb{C}}}(\gamma(x,y)) = -\mathcal{R}^{\mathbb{H}}_{\mathfrak{q}'_{\mathbb{C}}}(\gamma(x,y)) = -\eta'(x,y).$$

The formula is thus true in this case. It is also true for the case where $x-y<0$, because $\mathbf{LJ}(\widehat{\mathcal{R}}^{\mathbb{R}}_{\mathfrak{q}_{\mathbb{C}}}(\gamma(x-n,y+n)))$ and $\widehat{\mathcal{R}}^{\mathbb{H}}_{\mathfrak{q}'_{\mathbb{C}}}(\gamma(x-n,y+n))$ are two coherent families which coincide for $n \geqslant 0$ and are therefore equal. $\square$

Theorem 13.6. *Let $\mathcal{R}^{\mathbb{R}}_{\mathfrak{q}_{\mathbb{C}}}$ and $\mathcal{R}^{\mathbb{H}}_{\mathfrak{q}'_{\mathbb{C}}}$ be the cohomological induction functors from $\mathrm{GL}(n,\mathbb{C})$ to $\mathrm{GL}(2n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{H})$, respectively. Then*

$$\mathbf{LJ} \circ \widehat{\mathcal{R}}^{\mathbb{R}}_{\mathfrak{q}_{\mathbb{C}}} = (-1)^n \widehat{\mathcal{R}}^{\mathbb{H}}_{\mathfrak{q}'_{\mathbb{C}}}.$$

Proof. It is enough to show that the formula holds on the basis $\lambda(a)$, $a \in M(D)$, of $\mathcal{R}^{\mathbb{C}}$. Let $a = (\gamma(x_1,y_1),\ldots,\gamma(x_r,y_r)) \in M(D)$. We compute

$$\begin{aligned}
\mathbf{LJ} \circ \widehat{\mathcal{R}}^{\mathbb{R}}_{\mathfrak{q}_{\mathbb{C}}}(\lambda(a)) &= \mathbf{LJ} \circ \widehat{\mathcal{R}}^{\mathbb{R}}_{\mathfrak{q}_{\mathbb{C}}}(\gamma(x_1,y_1) \times \cdots \times \gamma(x_r,y_r)) \\
&= \mathbf{LJ}(i^{\mathrm{GL}(2r,\mathbb{R})}_{\mathrm{GL}(2,\mathbb{R})^r} \circ \widehat{\mathcal{R}}^{\mathbb{R}}_{\mathfrak{q}_{\mathbb{C}}}(\gamma(x_1,y_1) \otimes \cdots \otimes \gamma(x_r,y_r))) \\
&= i^{\mathrm{GL}(r,\mathbb{H})}_{\mathrm{GL}(1,\mathbb{H})^r} \circ \mathbf{LJ}(\widehat{\mathcal{R}}^{\mathbb{R}}_{\mathfrak{q}_{\mathbb{C}}}(\gamma(x_1,y_1) \otimes \cdots \otimes \gamma(x_r,y_r))) \\
&= (-1)^r i^{\mathrm{GL}(r,\mathbb{H})}_{\mathrm{GL}(1,\mathbb{H})^r} \circ \widehat{\mathcal{R}}^{\mathbb{H}}_{\mathfrak{q}'_{\mathbb{C}}}(\gamma(x_1,y_1) \otimes \cdots \otimes \gamma(x_r,y_r)) \\
&= (-1)^r \widehat{\mathcal{R}}^{\mathbb{H}}_{\mathfrak{q}'_{\mathbb{C}}}(\gamma(x_1,y_1) \times \cdots \times \gamma(x_r,y_r)) \\
&= (-1)^r \widehat{\mathcal{R}}^{\mathbb{H}}_{\mathfrak{q}'_{\mathbb{C}}}(\lambda(a)).
\end{aligned}$$

We have used the independence of polarization theorem from [KV95, ch. 11] to replace a part of the cohomological induction by parabolic induction; we have also used the fact that $\mathbf{LJ}$ commutes with parabolic induction. $\square$

Corollary 13.7. *Recall the representations $\bar{u}(\eta',n)$ introduced in § 11.3. We have*

$$\mathbf{LJ}(u(\eta(x,y),n) = (-1)^n\, \bar{u}(\eta'(x,y),n),$$

for $x,y \in \mathbb{C}$ such that $x-y \in \mathbb{N}$.

Recall that when $x-y \neq 1$, $\bar{u}(\eta'(x,y),n) = u(\eta'(x,y),n)$ (see § 11.3).

Proof. This follows from the previous theorem and the formulas $\mathcal{R}^{\mathbb{R}}_{\mathfrak{q}_{\mathbb{C}}}(u(\gamma(x,y))) = u(\eta(x,y),n)$ and $\mathcal{R}^{\mathbb{H}}_{\mathfrak{q}'_{\mathbb{C}}}(u(\gamma(x,y))) = \bar{u}(\eta'(x,y),n)$ obtained in §§ 11.2 and 11.3. $\square$

In order to compute the transfer to $\mathrm{GL}(n,\mathbb{H})$ of any irreducible unitary representation of $\mathrm{GL}(2n,\mathbb{R})$, we need to compute the transfer of the $u(\delta,k)$ when $\delta \in D^{\mathbb{R}}_1$. But in this case, if $\delta = \delta(\alpha,\epsilon)$, then

$$u(\delta(\alpha,\epsilon),2k) = \delta(\alpha,\epsilon) \circ \det,$$

and we know from [DKV84] that the transfer of this character is the character

$$\delta(\alpha,\epsilon) \circ \mathrm{RN}$$

(where RN is the reduced norm), which is

$$u(\eta'(\alpha + \tfrac{1}{2}, \alpha - \tfrac{1}{2}), k).$$

From this, we get the next theorem.

Theorem 13.8. *Let u be an irreducible unitary representation of* $\mathrm{GL}(2n,\mathbb{R})$. *Then* $\mathbf{LJ}(u)$ *is either* 0 *or, up to a sign, an irreducible unitary representation of* $\mathrm{GL}(n,\mathbb{H})$. *For representations* $u(\delta,k)$, *we get that:*

– *if* $\delta=\delta(\alpha,\epsilon)\in D_1^{\mathbb{R}}$, *then*

$$\mathbf{LJ}(u(\delta(\alpha,\epsilon),2k))=u(\eta'(\alpha+\tfrac{1}{2},\alpha-\tfrac{1}{2}),k);$$

– *if* $\delta=\eta(x,y)\in D_2^{\mathbb{R}}$, *then*

$$\mathbf{LJ}(u(\eta(x,y)),k)=(-1)^k\bar{u}(\eta'(x,y),k).$$

To make it simple, a character is sent by $\mathbf{LJ}$ to the corresponding character, while if $\delta\in D_2^F$ and $\delta'=\mathbf{C}(\delta)=-\mathbf{LJ}(\delta)$, then $\mathbf{LJ}(u(\delta,k))=(-1)^k\bar{u}(\delta',k)$.

In the first case, note that we are dealing with a slightly different situation from non-archimedean fields, since the reduced norm of $\mathbb{H}$ is *not* surjective but has image in $\mathbb{R}_+^*$. In particular, if s is the character sign of the determinant on $\mathrm{GL}_{2k}(\mathbb{R})$, then $\mathbf{LJ}(s)$ is the trivial character of $\mathrm{GL}_k(\mathbb{H})$. In the non-archimedean case, it is easy to check that $\mathbf{LJ}$ is injective on the set of representations $u(\delta,k)$.

The above theorem gives a correspondence between irreducible unitary representations of $\mathrm{GL}(2n,\mathbb{R})$ and those of $\mathrm{GL}(n,\mathbb{H})$, by forgetting the signs. As in the introduction, we denote this correspondence by $|\mathbf{LJ}|$. Using (11.2) and (11.3), we can easily reformulate the result as that given in the introduction.

14. Character formulas and ends of complementary series

From Tadić's classification of the unitary dual and the character formula for induced representations, the character of any irreducible unitary representation of $\mathrm{GL}(n,A)$ can be computed from the characters of the $u(\delta,n)$, with $\delta\in D$ and $n\in\mathbb{N}$. It is remarkable that the characters of the $u(\delta,n)$ can be computed or, more precisely, expressed in terms of characters of square integrable modulo center representations. We also give composition series for the ends of complementary series. This information is important for the topology of the unitary dual (see [Tad87]).

14.1 $A=\mathbb{C}$

Let $\gamma=\gamma(x,y)$ be a character of $\mathbb{C}^\times$, where $x,y\in\mathbb{C}$ with $x-y=r\in\mathbb{Z}$. The representation $u(\gamma(x,y),n)$ is the character

$$\det\circ\gamma$$

of $\mathrm{GL}(n,\mathbb{C})$. There is a formula, due to Zuckerman, for the trivial character of any real reductive group; it is obtained from a finite-length resolution of the trivial representation by standard modules in the category $\mathcal{M}(G)$.

For $\mathrm{GL}(n,\mathbb{C})$ this formula is

$$\mathbf{1}_{\mathrm{GL}(n,\mathbb{C})}=u(\gamma(0,0),n)=\sum_{w\in\mathfrak{S}_n}(-1)^{l(w)}\prod_{i=1}^{n}\gamma\Big(\frac{n-1}{2}-i+1,\frac{n-1}{2}-w(i)+1\Big),$$

where $\mathbf{1}_{\mathrm{GL}(n,\mathbb{C})}$ denotes the trivial representation.

From this we obtain, upon tensoring with $\gamma(x,y)$,

$$u(\gamma(x,y),n)=\sum_{w\in\mathfrak{S}_n}(-1)^{l(w)}\prod_{i=1}^{n}\gamma\left(x+\frac{n-1}{2}-i+1,y+\frac{n-1}{2}-w(i)+1\right). \tag{14.1}$$

Set

$$\gamma_{i,j}=\gamma\left(x+\frac{n-1}{2}-i+1,y+\frac{n-1}{2}-j+1\right)\in\mathcal{R}.$$

The formula above then becomes

$$u(\gamma(x,y),n)=\det((\gamma_{i,j})_{1\leqslant i,j\leqslant n}). \tag{14.2}$$

Using the Lewis Carroll identity [CR08], we deduce easily from this a formula for composition series of ends of complementary series. Such a formula was obtained previously by Tadić [Tad95] using partial results of Sahi [Sah95], but the proof was complicated. For an easy formula, set

$$\gamma(x,y)=\delta(\beta,r)$$

with $r=x-y$ and $2\beta=x+y$.

Proposition 14.1. *With the above notation, for $n\geqslant 2$ we have*

$$\begin{aligned}\nu^{-\frac{1}{2}}u(\delta(\beta,r),n)\times\nu^{\frac{1}{2}}u(\delta(\beta,r),n)&=u(\delta(\beta,r),n+1)\times u(\delta(\beta,r),n-1)\\&\quad+u(\delta(\beta,r+1),n)\times u(\delta(\beta,r-1),n).\end{aligned} \tag{14.3}$$

14.2 $A=\mathbb{R}$

Let $\eta(x,y)$ be an essentially square integrable modulo center representation of $\mathrm{GL}(2,\mathbb{R})$, with $x,y\in\mathbb{C}$ such that $x-y=r\in\mathbb{N}^\times$. Since

$$u(\eta(x,y),n)=-\widehat{\mathcal{R}}^{\mathbb{R}}_{\mathfrak{q}_\mathbb{C}}(u(\gamma(x,y))),$$

we get from (14.1) that

$$u(\eta(x,y),n)=-\sum_{w\in\mathfrak{S}_n}(-1)^{l(w)}\prod_{i=1}^{n}\widehat{\mathcal{R}}^{\mathbb{R}}_{\mathfrak{q}_\mathbb{C}}\left(\gamma\left(x+\frac{n-1}{2}-i+1,y+\frac{n-1}{2}-w(i)+1\right)\right).$$

We observed from the proof of Lemma 13.5 that $-\widehat{\mathcal{R}}^{\mathbb{R}}_{\mathfrak{q}_\mathbb{C}}(\gamma(x-n,y+n))$ is a coherent family of representations of $\mathrm{GL}(2,\mathbb{R})$ such that $-\widehat{\mathcal{R}}^{\mathbb{R}}_{\mathfrak{q}_\mathbb{C}}(\gamma(x-n,y+n))=\eta(x-n,y+n)$ when $x-n>y+n$. Set $\tilde{\eta}(x-n,y+n)=-\widehat{\mathcal{R}}^{\mathbb{R}}_{\mathfrak{q}_\mathbb{C}}(\gamma(x-n,y+n))$. Then we get

$$u(\eta(x,y),n)=(-1)^{n+1}\sum_{w\in\mathfrak{S}_n}(-1)^{l(w)}\prod_{i=1}^{n}\tilde{\eta}\left(x+\frac{n-1}{2}-i+1,y+\frac{n-1}{2}-w(i)+1\right).$$

Set

$$\tilde{\eta}_{i,j}=\tilde{\eta}\left(x+\frac{n-1}{2}-i+1,y+\frac{n-1}{2}-j+1\right).$$

The formula above becomes

$$u(\eta(x,y),n)=(-1)^{n+1}\det((\tilde{\eta}_{i,j})_{1\leqslant i,j\leqslant n}). \tag{14.4}$$

Again by using the Lewis Carroll identity [CR08], we deduce easily a formula for composition series of ends of complementary series.

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Proposition 14.2. *With the above notation, for $n \geqslant 2$ and $x - y > 1$ we have*

$$\begin{aligned}
&\nu^{-\frac{1}{2}} u(\eta(x,y),n) \times \nu^{\frac{1}{2}} u(\eta(x,y),n) \\
&\quad = u(\eta(x,y),n+1) \times u(\eta(x,y),n-1) \\
&\qquad + u(\eta(x+\tfrac{1}{2}, y-\tfrac{1}{2}),n) \times u(\eta(x-\tfrac{1}{2}, y+\tfrac{1}{2}),n).
\end{aligned} \tag{14.5}$$

If $x = y + 1$, recalling the convention that

$$\eta(x-\tfrac{1}{2}, x-\tfrac{1}{2}) = \delta(x-\tfrac{1}{2},0) \times \delta(x-\tfrac{1}{2},1),$$

we get

$$\begin{aligned}
&\nu^{-\frac{1}{2}} u(\eta(x,x-1),n) \times \nu^{\frac{1}{2}} u(\eta(x,x-1),n) \\
&\quad = u(\delta(x,x-1),n+1) \times u(\eta(x,x-1),n-1) \\
&\qquad + u(\eta(x+\tfrac{1}{2}, x-\tfrac{3}{2}),n) \times [u(\delta(x-\tfrac{1}{2},0),n) \times u(\delta(x-\tfrac{1}{2},1),n)].
\end{aligned} \tag{14.6}$$

Remark 14.3. We cannot deduce by our method the composition series of the ends of complementary series for $u(\delta, n)$ when $\delta \in D_1$. There is still a formula for the character of $u(\delta, n)$, since $u(\delta, n) = \delta \circ \det$ is a one-dimensional representation (Zuckerman); but there is no interpretation for the right-hand side of this formula as a determinant, so we cannot apply the Lewis Carroll identity.

14.3 $A = \mathbb{H}$

The discussion is similar to that in the real case for $u(\eta'(x,y),n)$ when $x - y \geqslant 2$.

Proposition 14.4. *With the above notation, for $n \geqslant 2$ and $x - y \geqslant 2$ we have*

$$\begin{aligned}
&\nu^{-\frac{1}{2}} u(\eta'(x,y),n) \times \nu^{\frac{1}{2}} u(\eta'(x,y),n) \\
&\quad = u(\eta'(x,y),n+1) \times u(\eta'(x,y),n-1) \\
&\qquad + u(\eta'(x+\tfrac{1}{2}, y-\tfrac{1}{2}),n) \times u(\eta'(x-\tfrac{1}{2}, y+\tfrac{1}{2}),n).
\end{aligned} \tag{14.7}$$

If $y = x - 1$, we get the same kind of character formulas but for the $\bar{u}(\eta'(x,y),n)$ instead:

$$\bar{u}(\eta'(x,x-1),n) = (-1)^{n+1} \det((\tilde{\eta}'_{i,j})_{1 \leqslant i,j \leqslant n}), \tag{14.8}$$

where $\tilde{\eta}'_{i,j} = \tilde{\eta}'(x + (n-1)/2 - i + 1,\ y + (n-1)/2 - j + 1)$ and $\tilde{\eta}'$ denotes the coherent family coinciding with η when $x - y$ is positive, as in the real case.

Again from the Lewis Carroll identity, we deduce the following (with $2n$ in place of n):

$$\begin{aligned}
&\nu^{-\frac{1}{2}} \bar{u}(\eta'(x,x-1),2n) \times \nu^{\frac{1}{2}} \bar{u}(\eta'(x,x-1),2n) \\
&\quad = \bar{u}(\eta'(x,x-1),2n+1) \times \bar{u}(\eta(x,x-1),2n-1) \\
&\qquad + \bar{u}(\eta'(x+\tfrac{1}{2}, x-\tfrac{1}{2}),2n) \times \bar{u}(\eta(x-\tfrac{1}{2}, x-\tfrac{3}{2}),2n).
\end{aligned} \tag{14.9}$$

The representations $\bar{u}(\eta'(\cdot,\cdot),\cdot)$ in this expression can be expressed as products of $u(\eta'(\cdot,\cdot),\cdot)$ explicitly in the following way:

$$\begin{aligned}
\bar{u}(\eta'(x,x-1),2n) &= u(\eta'(x+\tfrac{1}{2}, x-\tfrac{1}{2}),n) \times u(\eta'(x-\tfrac{1}{2}, x-\tfrac{3}{2}),n), \\
\bar{u}(\eta'(x,x-1),2n+1) &= u(\eta'(x,x-1),n+1) \times u(\eta'(x,x-1),n).
\end{aligned}$$

Substituting these into (14.9) and using the fact that the ring $\mathcal{R}$ is a domain, we get the following result.

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Proposition 14.5.

$$\begin{aligned}&\nu^{-1}u(\eta'(x,x-1),n)\times\nu u(\eta'(x,x-1),n)\\&\quad=u(\eta'(x,x-1),n+1)\times u(\eta'(x,x-1),n-1)\\&\qquad+u(\eta'(x+\tfrac{1}{2},x-\tfrac{1}{2}),n)\times u(\eta'(x-\tfrac{1}{2},x-\tfrac{3}{2}),n).\end{aligned}\tag{14.10}$$

15. Compatibility and further comments

Let F be a local field (archimedean or non-archimedean of any characteristic), and let A be a central division algebra of dimension d^2 over F (if F is archimedean, then $d\in\{1,2\}$). If $g\in G_{nd}^F$ is a regular semisimple element, we say that g *transfers* if there exists an element g' of G_n^A which corresponds to g (see § 4). Then g transfers if and only if its characteristic polynomial decomposes into a product of irreducible polynomials of degrees divisible by d. We say that $\pi\in\mathcal{R}(G_{nd}^F)$ is *d-compatible* if $\mathbf{LJ}(\pi)\neq 0$. In other words, π is d-compatible if and only if its character does not vanish identically on the set of elements of G_{nd}^F which transfer. This justifies the dependence of the definition on d only (and not on D). We then have the following results.

Proposition 15.1. *Let $\pi_i\in\mathrm{Irr}_{n_i}^F$, $1\leqslant i\leqslant k$, with $\sum_i n_i=n$. Then $\pi_1\times\pi_2\times\cdots\times\pi_k$ is d-compatible if and only if for all $1\leqslant i\leqslant k$, d divides n_i and π_i is d-compatible.*

Proof. If an element $g\in G_n^F$ is conjugated with an element of a Levi subgroup of G_n^F, say $(g_1,g_2,\ldots,g_k)\in G_{(n_1,n_2,\ldots,n_k)}$ with $g_i\in G_{n_i}^F$, then the characteristic polynomial of g is the product of the characteristic polynomials of the g_i. It follows that if g is semisimple regular, it transfers if and only if $d|n_i$ for all i and each g_i transfers.

It is a general fact that for a fully induced representation of a group G from a Levi subgroup M, the character is zero on regular semisimple elements which are not conjugated in G to some element in M. Moreover, one has a precise formula for the character of the fully induced representation in terms of the character of the inducing representation (see [Har70] and [Clo84, Proposition 3] for non-archimedean F and [Kna01, § 13] for archimedean F). The proposition follows. ☐

We now define an order $\ll$ that is finer than the Bruhat order $<$ on Irr_n^A. If $\pi=\mathrm{Lg}(\delta_1,\delta_2,\ldots,\delta_k)$ and $\pi'=\mathrm{Lg}(\delta_1',\delta_2',\ldots,\delta_{k'}')$ are in Irr_n^A, we set $\pi\ll\pi'$ if

$$\mathrm{Lg}(\mathbf{C}^{-1}(\delta_1),\mathbf{C}^{-1}(\delta_2),\ldots,\mathbf{C}^{-1}(\delta_k))<\mathrm{Lg}(\mathbf{C}^{-1}(\delta_1'),\mathbf{C}^{-1}(\delta_2'),\ldots,\mathbf{C}^{-1}(\delta_{k'}'))$$

in Irr_{nd}^F.

Proposition 15.2. *Let $\delta_i\in D_{n_i}^F$ for $1\leqslant i\leqslant k$. Assume that for all $1\leqslant i\leqslant k$ we have $d|n_i$, and set $\delta_i'=\mathbf{C}(\delta_i)\in D_{n_{i/d}}^A$. Then $\mathrm{Lg}(\delta_1,\delta_2,\ldots,\delta_k)$ is compatible and one has*

$$\mathbf{LJ}(\mathrm{Lg}(\delta_1,\delta_2,\ldots,\delta_k))=(-1)^{nd-n}\mathrm{Lg}(\delta_1',\delta_2',\ldots,\delta_k')+\sum_{j\in J}m_j\pi_j',$$

where J is empty or finite, $m_j\in\mathbb{Z}^*$, $\pi_j'\in\mathrm{Irr}_{\sum n_{i/d}}^A$ and $\pi_j'\ll\mathrm{Lg}(\delta_1',\delta_2',\ldots,\delta_k')$ for all $j\in J$.

Proof. Apply Proposition 6.1 and induction on the number of representations smaller than $\mathrm{Lg}(\delta_1,\delta_2,\ldots,\delta_k)$. See [Bad07, Proposition 3.10]. ☐

Proposition 15.3. *If $\delta \in D_n^F$, set $\deg(\delta) = n$ and let $l(\delta)$ be the length of* $\mathrm{Supp}(\delta)$ *(note that $l(\delta)$ divides $\deg(\delta)$). Then the following hold.*

(a) *$u(\delta, k)$ is d-compatible if and only if either $d|\deg(\delta)$ or $d|(k\deg(\delta)/l(\delta))$.*

(b) *There exists $k_\delta \in \mathbb{N}^*$ such that $u(\delta, k)$ is d-compatible if and only if $k_\delta|k$; moreover, $k_\delta|d$.*

Proof. The proof of (a) is in [Bad08, § 3.5] for the non-archimedean case. In the archimedean case it follows from Theorem 13.8.

Assertion (b) follows easily from (a). For the archimedean (non-trivial, i.e. $A = \mathbb{H}$) case, $d = 2$ and the transfer theorem, Theorem 13.8, shows that:

– if $\deg(\delta) = 2$, then $u(\delta, k)$ is 2-compatible for all k (hence $k_\delta = 1$);

– if $\deg(\delta) = 1$, then $u(\delta, k)$ is 2-compatible if (and *only* if, because of the dimension of G_k^F) k is even (hence $k_\delta = 2$). □

Let γ be an irreducible generic unitary representation of G_n^F. As γ is generic, it is fully induced from an essentially square integrable representation (see [Zel80] for non-archimedean fields and § 8 for archimedean fields). Then as γ is unitary, thanks to the classification of the unitary spectrum (see [Tad86, Vog86] and § 8 of the present paper), γ is an irreducible product $\sigma_1 \times \sigma_2 \times \cdots \times \sigma_p \times \pi_1 \times \pi_2 \times \cdots \times \pi_l$, where for $1 \leqslant i \leqslant p$, $\sigma_i \in D^{u,F}$, and for $1 \leqslant j \leqslant l$, $\pi_j = \pi(\delta_j, 1; \alpha_j)$ for some $\delta_j \in D^{u,F}$ and some $\alpha_i \in]0, 1/2[$.

Using the Langlands classification, it is easy to see that the representation

$$\nu^{(k-1)/2}\gamma \times \nu^{(k-1)/2-1}\gamma \times \cdots \times \nu^{-(k-1)/2}\gamma$$

has a unique quotient $u(\gamma, k)$, and one has

$$u(\gamma, k) = u(\sigma_1, k) \times u(\sigma_2, k) \times \cdots \times u(\sigma_p, k) \times \pi(\delta_1, k; \alpha_1) \times \pi(\delta_2, k; \alpha_2) \times \cdots \times \pi(\delta_l, k; \alpha_l)$$

(see, for instance, [Bad07, § 4.1]). The local components of cuspidal automorphic representations of GL_n over adeles of global fields are unitary generic representations [Sha74]. According to the classification of the residual spectrum [MW89], it follows that local component of residual automorphic representations of the linear group are of type $u(\gamma, k)$.

Proposition 15.4. *Let γ be a unitary generic representation of G_n^F for some $n \in \mathbb{N}^\times$. There exists k_γ such that $u(\gamma, k)$ is d-compatible if and only if $k_\gamma|k$. Moreover, $k_\gamma|d$.*

Proof. The (easy) proof given in [Bad08, § 3.5] for non-archimedean fields works also for archimedean fields. If

$$u(\gamma, k) = u(\sigma_1, k) \times u(\sigma_2, k) \times \cdots \times u(\sigma_p, k) \times \pi(\delta_1, k; \alpha_1) \times \pi(\delta_2, k; \alpha_2) \times \cdots \times \pi(\delta_l, k; \alpha_l),$$

then $u(\gamma, k)$ is d-compatible if and only if all the $u(\sigma_i, k)$ and $u(\delta_j, k)$ are compatible (Proposition 15.1). Then Proposition 15.3 implies Proposition 15.4. If $F = \mathbb{R}$, then $k_\gamma = 1$ if and only if all the σ_i and δ_j are in D_2. If not, then $k_\gamma = 2$. □

16. Notation for the global case

Let F be a global field of *characteristic zero* and let D be a central division algebra over F of dimension d^2. Let $n \in \mathbb{N}^*$. Set $A = M_n(D)$. For each place v of F, let F_v be the completion of F at v and set $A_v = A \otimes F_v$. For every place v of F, A_v is isomorphic to $M_{r_v}(D_v)$ for some positive integer r_v and some central division algebra D_v of dimension d_v^2 over F_v such that $r_v d_v = nd$.

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We fix once and for all an isomorphism $A_v \simeq M_{r_v}(D_v)$ and identify these two algebras. We say that $M_n(D)$ is *split* at a place v if $d_v = 1$. The set V of places where $M_n(D)$ is not split is finite. For each v, d_v divides d and, moreover, d is the smallest common multiple of the d_v over all the places v.

Let $G'(F)$ be the group $A^\times = \mathrm{GL}_n(D)$. For every finite place v of F, set $G'_v = A_v^\times = \mathrm{GL}_{r_v}(D_v)$. For every finite place v of F, set $K_v = \mathrm{GL}_{r_v}(O_v)$ where O_v is the ring of integers of D_v. Let $\mathbb{A}$ be the ring of adeles of F. We define the group $G'(\mathbb{A})$ of adeles of $G'(F)$ to be the restricted product of the G'_v over all of the v, with respect to the family of open compact subgroups K_v with v finite.

Let G'_∞ be the direct product of G'_v over the set of infinite places of F and let G'_f be the restricted product of G'_v over the finite places, with respect to the open compact subgroups K_v. The group $G'(\mathbb{A})$ decomposes into the direct product

$$G'(\mathbb{A}) = G'_\infty \times G'_f.$$

Fix maximal compact subgroups K_v at archimedean places v as before: $K_v = \mathrm{O}(n)$, $\mathrm{U}(n)$ or $\mathrm{Sp}(n)$ according to whether G'_v is $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$ or $\mathrm{GL}_n(\mathbb{H})$. Let K_∞ (respectively, K_f) be the compact subgroup of G_∞ (respectively, of G'_f) which is the direct product of K_v over the infinite places (respectively, finite places) v. Let K be $K_\infty \times K_f$ as a (compact) subgroup of $G'(\mathbb{A})$. Let $\mathfrak{g}_\infty$ be the Lie algebra of G_∞.

An *admissible* $G'(\mathbb{A})$*-module* is a linear space V which is both a $(\mathfrak{g}_\infty, K_\infty)$-module and a smooth G'_f-module such that the actions of $(\mathfrak{g}_\infty, K_\infty)$ and G'_f commute and, for every irreducible equivalence class of continuous representations π of K, the π-isotypic component of V is of finite dimension. It is *irreducible* if it has no proper $G'(\mathbb{A})$-submodule and *unitary* if it admits a hermitian product which is invariant under the actions of both $(\mathfrak{g}_\infty, K_\infty)$ and G'_f.

If V is an irreducible admissible $G'(\mathbb{A})$-module, then V is isomorphic to a tensor product $V_\infty \otimes V_f$, where V_∞ is an irreducible $(\mathfrak{g}_\infty, K_\infty)$-module and V_f is an irreducible smooth representation of V_f.

If (π, H) is a unitary irreducible admissible G_f-module, then π decomposes into a restricted tensor product $\bigotimes_{v \text{ finite}} \pi_v$ where π_v is a unitary irreducible representation of G'_v (see [GGP90, JL70, Lan80] or [Fla79]). For almost all v, π_v has a fixed vector under the maximal compact subgroup K_v. Such a representation is said to be *spherical*. The π_v are determined by π. Such a π_v is called the *local component* of π at the place v. The set of local components π_v determines π.

Let $Z(F)$ be the center of $G'(F)$ and, for each place v, let Z_v be the center of G'_v. Then we identify the center $Z(\mathbb{A})$ of $G'(\mathbb{A})$ with the restricted product of the Z_v, with respect to the open compact subgroups $Z_v \cap K_v$ at finite places. For any finite v, we fix a Haar measure dg_v on G'_v such that the volume of K_v is one and a Haar measure dz_v on Z_v such that the volume of $Z_v \cap K_v$ is one. The set of measures $\{dg_v\}_{v \text{ finite}}$ induces a well-defined Haar measure on the locally compact group G'_f, and the set $\{dz_v\}_{v \text{ finite}}$ induces a well-defined measure on its center (see, for instance, [RV99], where measures on restricted products are explained).

For the archimedean groups we choose Duflo and Vergne's normalization, which is defined as follows. Let G be a reductive group (complex or real), and pick a G-invariant, symmetric, non-degenerate bilinear form κ on the Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ will be endowed with the Lebesgue measure dX such that the volume of a parallelotope supported by a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ is equal to $|\det(\kappa(X_i, X_j))|^{1/2}$, and G will be endowed with the Haar measure tangent to dX. If G'

is a closed subgroup of G such that κ is non-degenerate on its Lie algebra $\mathfrak{g}'$, we endow G' with the Haar measure determined by κ as above. This gives measures on G'_∞ and its center.

We now fix the measure dg on $G'(\mathbb{A}) = G'_\infty \times G'_f$ (respectively, dz on $Z(\mathbb{A})$) to be the product of measures chosen previously for the infinite and finite parts. We fix a measure on $Z(\mathbb{A})\backslash G'(\mathbb{A})$, i.e. the quotient measure $dz\backslash dg$.

We view $G'(F)$ as a subgroup of $G'(\mathbb{A})$ via the diagonal embedding. As $G'(F) \cap Z(\mathbb{A})\backslash G'(F)$ is a discrete subgroup of $Z(\mathbb{A})\backslash G'(\mathbb{A})$, $dz\backslash dg$ defines a measure on the quotient space $Z(\mathbb{A})G'(F)\backslash G'(\mathbb{A})$. The measure of the space $Z(\mathbb{A})G'(F)\backslash G'(\mathbb{A})$ is finite.

Fix a unitary continuous character ω of $Z(\mathbb{A})$ that is trivial on $Z(F)$.

Let $L^2(Z(\mathbb{A})G'(F)\backslash G'(\mathbb{A}); \omega)$ be the space of classes of functions f defined on $G'(\mathbb{A})$ with values in $\mathbb{C}$ such that:

(i) f is left invariant under $G'(F)$;

(ii) f satisfies $f(zg) = \omega(z)f(g)$ for all $z \in Z(\mathbb{A})$ and almost all $g \in G'(\mathbb{A})$;

(iii) $|f|^2$ is integrable over $Z(\mathbb{A})G'(F)\backslash G'(\mathbb{A})$.

Let R'_ω be the representation of $G'(\mathbb{A})$ in $L^2(Z(\mathbb{A})G'(F)\backslash G'(\mathbb{A}); \omega)$ by right translations. As explained in [BJ79], each irreducible subspace of $L^2(Z(\mathbb{A})G'(F)\backslash G'(\mathbb{A}); \omega)$ gives rise to a unique unitary irreducible admissible $G'(\mathbb{A})$-module. We call such a $G'(\mathbb{A})$-module a *discrete series of* $G'(\mathbb{A})$.

Every discrete series of $G'(\mathbb{A})$ with central character ω appears in R'_ω with a finite multiplicity [GGP90].

Let $R'_{\omega,\mathrm{disc}}$ be the subrepresentation of R'_ω generated by the discrete series. If π is a discrete series, we call the multiplicity with which π appears in $R'_{\omega,\mathrm{disc}}$ the *multiplicity of* π *in the discrete spectrum.*

Notation. Fix n and D as before. The same constructions work, starting with $A = \mathrm{GL}_{nd}(F)$ instead of $A = \mathrm{GL}_n(D)$. We denote by $G(\mathbb{A})$ the group of invertible elements of A and modify all the notation accordingly.

17. Further insight into some local results

We would like to point out that some of the archimedean results presented in this paper can be proved by global methods and local tricks as in the non-archimedean case [Bad07, Bad08], avoiding any reference to cohomological induction. These are hypothesis $U(1)$ for $\mathrm{GL}(n, \mathbb{H})$, the fact that products of representations in $\mathcal{U}_{\mathbb{H}}$ are irreducible, and the Jacquet–Langlands transfer of unitary representations (using $U(0)$ for $\mathrm{GL}(n, \mathbb{R})$, see [Bar03], but not for $\mathrm{GL}(n, \mathbb{H})$). Here we sketch these proofs.

17.1 Hypothesis $U(1)$ and the transfer of $u(\delta, k)$

Let $\mathbf{LJ} : \mathcal{R}^{\mathbb{R}}_{2n} \to \mathcal{R}^{\mathbb{H}}_n$ be the morphism between Grothendieck groups extending the classical Jacquet–Langlands correspondence for square integrable representations (§ 4). We give here an alternative proof of the following result.

Proposition 17.1.

(a) *If* $\chi \in D_1$, *then* $\mathbf{LJ}(u(\chi, 2n)) = \chi'_n$.

(b) *If* $\delta \in D_2$ *and* $\delta' = \mathbf{C}(\delta)$, *then* $\mathbf{LJ}(u(\delta, n)) = (-1)^n \bar{u}(\delta', n)$.

(c) *Hypothesis* $U(1)$, *i.e. the statement that the* $u(\delta', n)$ *are unitary, holds for* $\mathrm{GL}(n, \mathbb{H})$.

Assertion (a) is obvious since $u(\chi, 2n) = \chi_{2n}$ and the equality of characters can be checked directly. To prove (c), recall that we have

$$\mathbf{LJ}(u(\delta, n)) = (-1)^n\biggl(\bar{u}(\delta', n) + \sum_{i=1}^{k} a_i u_i\biggr), \tag{17.1}$$

where the u_i are irreducible non-equivalent representations of $\mathrm{GL}(n, \mathbb{H})$ that are non-equivalent to $\bar{u}(\delta', n)$ and the a_i are non-zero integers (Proposition 15.2).

We now claim that all the irreducible representations on the right-hand side of the equality are unitary and that the a_i are all positive. One may proceed as in [Bad07]: choose a global field F and a division algebra D over F such that if $G'(\mathbb{A})$ is the adele group of $D^\times$, then $G'_v = \mathrm{GL}_n(\mathbb{H})$ for some place v. As $\delta \in D_2$, there exists a cuspidal representation ρ of $G(\mathbb{A}) = \mathrm{GL}_{2n}(\mathbb{A})$ such that $\rho_v = \delta$. According to the classification of the residual spectrum for $G(\mathbb{A})$ (see [MW89]), there exists a residual representation π of $G(\mathbb{A})$ such that $\pi_v = u(\delta, n)$. Comparing the trace formula from [AC89] (or the simple trace formula from [Art88]) for $G(\mathbb{A})$ and for $G'(\mathbb{A})$, one obtains, using standard simplifications and multiplicity one on the $G(\mathbb{A})$ side, a local formula $\mathbf{LJ}(u(\delta, n)) = \pm\sum_{j=1}^{k} b_j w_i$ where the b_j are multiplicities of representations, hence positive, and the w_j are local components of global discrete series, hence unitary. By linear independence of characters on $\mathrm{GL}(n, \mathbb{H})$, this formula is the same as (17.1), which implies, in particular, that $\bar{u}(\delta', n)$ is unitary (see [Bad07, Corollary 4.8(a)]). This gives the assertion $U(1)$, because when δ' is not a character one has $\bar{u}(\delta', n) = u(\delta', n)$, whereas when δ' is a (unitary) character we know that $u(\delta', k)$ is the unitary character $\delta' \circ \mathrm{RN}$. So (c) is proved.

We now prove (b). We want to show that on the right-hand side of (17.1) there is just one term, $\bar{u}(\delta', n)$. If π is an irreducible unitary representation of $\mathrm{GL}(n, \mathbb{R})$, we say that π is *semirigid* if it is a product of representations $u(\delta, k)$. We already showed in the previous paragraph that all these representations $u(\delta, k)$ correspond via $\mathbf{LJ}$ to zero or a sum of unitary representations. As $\mathbf{LJ}$ commutes with products and a product of irreducible unitary representations is a sum of irreducible unitary representations, it follows that any sum of semirigid irreducible unitary representations of some $\mathrm{GL}(2n, \mathbb{R})$ correspond to either zero or a sum of unitary representations of $\mathrm{GL}(n, \mathbb{H})$. The relation (17.1) now shows that for all $\alpha \in \mathbb{R}$, $\mathbf{LJ}(\pi(\delta, n; \alpha)) = \nu'^{\alpha}(\sum_{i=0}^{k} a_i u_i) \times \nu'^{-\alpha}(\sum_{i=0}^{k} a_i u_i)$, where $a_0 = 1$ and $u_0 = \bar{u}(\delta', n)$. When $\alpha = 1/2$, on the left-hand side of the equality we obtain a sum of semirigid unitary representations (see Proposition 14.5 for the precise formula), so on the right-hand side we should have a sum of unitary representations. But this is impossible as soon as the sum $\sum_{i=1}^{k} a_i u_i$ contains a representation u_1, since then the mixed product $\nu'^{-1/2}u_0 \times \nu'^{1/2}u_1$ would contain a non-hermitian subquotient (the 'bigger' one for the Bruhat order, for example). This shows that there is only one u_i, with $i = 0$, and so $\mathbf{LJ}(u(\delta, n)) = (-1)^n \bar{u}(\delta', n)$. □

17.2 Irreducibility and transfer of all unitary representations

We now know that the representations in $\mathcal{U}_{\mathbb{H}}$ are all unitary. To show that their products remain irreducible, we can use the irreducibility trick in [Bad08, Proposition 2.13], which reduces the problem to showing that $u(\delta', k) \times u(\delta', k)$ is irreducible for all discrete series δ' of $\mathrm{GL}(1, \mathbb{H})$ and all $k \in \mathbb{N}^\times$. Let δ be a square integrable representation of $\mathrm{GL}(2, \mathbb{R})$ such that $\mathbf{LJ}(\delta) = \delta'$. It follows that $\mathbf{LJ}(u(\delta, k) \times u(\delta, k)) = \bar{u}(\delta', k) \times \bar{u}(\delta', k)$. On the left-hand side we have the irreducible representation $M = u(\delta, k) \times u(\delta, k)$. On the right-hand side we have a sum of unitary representations, the product $M' = \bar{u}(\delta', k) \times \bar{u}(\delta', k)$ (we already know that $\bar{u}(\delta', k)$ is unitary), which we want to show has actually only a single term. Apply the same α trick as before: we know that $\pi(M, \alpha)$ corresponds to $\pi(M', \alpha)$. For $\alpha = 1/2$, the first representation decomposes into a sum of semirigid unitary representations, while the second is a sum containing non-unitary representations unless M' contains a single term. Notice that the Langlands quotient theorem and hypothesis $U(4)$ guarantee that M' has a subquotient which appears with multiplicity one, so either M' is a sum containing two different terms, or it is irreducible. So the square of $\bar{u}(\delta', k)$ is irreducible for all k. If δ' is not a character, then $u(\delta', k) = \bar{u}(\delta', k)$ and so the square of $u(\delta', k)$ is irreducible. If δ' is a character, then we have seen that $\bar{u}(\delta', 2k+1) = u(\delta', k) \times u(\delta', k+1)$, so the result again follows.

This implies that *if u is an irreducible unitary representation of* $\mathrm{GL}(2n, \mathbb{R})$, *then* $\mathbf{LJ}(u)$ *is either zero or plus or minus a irreducible unitary representation of* $\mathrm{GL}(n, \mathbb{H})$.

The proofs here are based on the trace formula and do not involve cohomological induction. However, the really difficult result is $U(0)$ on $\mathrm{GL}(n, \mathbb{H})$, which does rely on cohomological induction.

18. Global results

For all $v \in V$, denote by $\mathbf{LJ}_v$ (respectively, $|\mathbf{LJ}|_v$) the correspondence $\mathbf{LJ}$ (respectively, $|\mathbf{LJ}|$), as defined in §§ 4 and 13, applied to G_v and G'_v.

If π is a discrete series of $G(\mathbb{A})$, we say that π is *D-compatible* if for all $v \in V$, π_v is d_v-compatible. Then $\mathbf{LJ}(\pi_v) \neq 0$ and $|\mathbf{LJ}|_v(\pi_v)$ is an irreducible representation of G'_n.

Here are the Jacquet–Langlands correspondence, multiplicity-one and strong multiplicity-one theorems for $G'(\mathbb{A})$ (already known for $G(\mathbb{A})$; see [Pia79, Sha74]).

Theorem 18.1.

(a) *There exists a unique map* $\mathbf{G}$ *from the set of discrete series of* $G'(\mathbb{A})$ *into the set of discrete series of* $G(\mathbb{A})$ *such that* $\mathbf{G}(\pi') = \pi$ *implies* $|\mathbf{LJ}|_v(\pi_v) = \pi'_v$ *for all places* $v \in V$ *and* $\pi_v = \pi'_v$ *for all places* $v \notin V$. *The map* $\mathbf{G}$ *is injective and onto the set of D-compatible discrete series of* $G(\mathbb{A})$.

(b) *The multiplicity of every discrete series of* $G'(\mathbb{A})$ *in the discrete spectrum is one. If two discrete series of* $G'(\mathbb{A})$ *have isomorphic local components at almost every place, then they are equal.*

The proof is the same as that of [Bad08, Theorem 5.1], with the following minor changes. [Bad08, Lemma 5.2] is obviously still true when the inner form is not split at infinite places, using Proposition 15.1. For the finiteness property quoted in [Bad08, p. 417] as [BB], one has to replace this reference by [Bad05], which addresses the case of an inner form ramified at infinite places. We do not need here the claim (d) in [Bad08, Theorem 5.1], which is now a particular case of Tadić's classification of unitary representation for inner forms. At the bottom of [Bad08, pp. 417 and 419], the independence of characters on a product of connected p-adic groups is used. Here the product also involves real, sometimes non-connected, groups such as $\mathrm{GL}(n, \mathbb{R})$. The linear independence of characters on each of these GL_n is enough to ensure the linear independence of characters on the product, as at infinite places representations are Harish-Chandra modules so that for all these groups, real or p-adic, irreducible representations correspond to irreducible modules on a well-chosen algebra with idempotents.

As in [Bad08], the core of the proof is the powerful equality [AC89, (17.8)] (comparison of trace formulas for $G(\mathbb{A})$ and $G'(\mathbb{A})$).

Let us now show the classification of cuspidal representations of $G'(\mathbb{A})$ in terms of cuspidal representations of $G(\mathbb{A})$. Let ν (respectively, ν') be the global character of $G(\mathbb{A})$ (respectively, $G'(\mathbb{A})$) given by the product of local characters as before (i.e. the absolute value of the reduced norm). Recall that, according to the Moeglin–Waldspurger classification, every discrete series π of $G(\mathbb{A})$ is the unique irreducible quotient of an induced representation $\nu^{(k-1)/2}\rho \times \nu^{(k-3)/2}\rho \times \cdots \times \nu^{-(k-1)/2}\rho$ where ρ is cuspidal. Then k and ρ are determined by π, so π is cuspidal if and only if $k = 1$. We set $\pi = \mathrm{MW}(\rho, k)$.

Proposition 18.2.

(a) *Let $n \in \mathbb{N}^\times$ and let ρ be a cuspidal representation of $G_n(\mathbb{A})$. Then there exists k_ρ such that if $k \in \mathbb{N}^\times$, then $\mathrm{MW}(\rho, k)$ is D-compatible if and only if $k_\rho | k$. Moreover, $k_\rho | d$.*

(b) *Let π' be a discrete series of $G'(\mathbb{A})$ and let $\pi = \mathbf{G}(\pi')$. Then π' is cuspidal if and only if π is of the form $\mathrm{MW}(\rho, k_\rho)$.*

(c) *Let ρ' be a cuspidal representation of some $G'_n(\mathbb{A})$. Write $\mathbf{G}(\rho') = \mathrm{MW}(\rho, k_\rho)$ and set $\nu_{\rho'} = \nu^{k_\rho}$. For every $k \in \mathbb{N}^\times$, the induced representation*

$$\nu_{\rho'}^{(k-1)/2}\rho' \times \nu_{\rho'}^{(k-3)/2}\rho' \times \cdots \times \nu_{\rho'}^{-(k-1)/2}\rho'$$

has a unique irreducible quotient which we will denote by $\mathrm{MW}'(\rho', k)$. It is a discrete series, and all discrete series are obtained from some cuspidal ρ' in that way. If $\mathbf{G}(\rho') = \mathrm{MW}(\rho, k_\rho)$, we have $\mathbf{G}(\mathrm{MW}'(\rho', k)) = \mathrm{MW}(\rho, kk_\rho)$.

Proof.

(a) This follows from Proposition 15.4 and the fact that for all $v \in V$, $d_v | d$.

(b) This is [Bad08, Proposition 5.5], with 'cuspidal' in place of 'basic cuspidal', thanks to Grbac's appendix. Both the proof of the claim and the proof in the appendix work the same way here.

(c) When $G'_n(\mathbb{A})$ is split at infinite places, this assertion is claim (a) of [Bad08, Proposition 5.7]. We follow the same idea of reducing the problem to a local computation. As [Bad08] makes use of Zelevinsky involution, we have to provide here a proof for the archimedean case (in which the involution doesn't exist). First, to show that the induced representation

$$\nu_{\rho'}^{(k-1)/2}\rho' \times \nu_{\rho'}^{(k-3)/2}\rho' \times \cdots \times \nu_{\rho'}^{-(k-1)/2}\rho'$$

has a constituent which is a discrete series, we will show directly that $\mathbf{G}^{-1}(\mathrm{MW}(\rho, kk_\rho))$, which is indeed a discrete series, is a constituent of

$$\nu_{\rho'}^{(k-1)/2}\rho' \times \nu_{\rho'}^{(k-3)/2}\rho' \times \cdots \times \nu_{\rho'}^{-(k-1)/2}\rho'.$$

We will show this place by place, local component by local component. Fix a place v and let γ be the local component of ρ at the place v. It is an irreducible unitary generic representation, and we know that $u(\gamma, k_\rho)$ transfers. Set $\pi = \mathbf{LJ}(u(\gamma, k_\rho))$. What we want to prove is that $\mathbf{LJ}(u(\gamma, kk_\rho))$ is a subquotient of $\nu^{k_\rho(k-1)/2}\pi \times \nu^{k_\rho(k-3)/2}\pi \times \cdots \times \nu^{k_\rho(-(k-1)/2)}\pi$. The unitary generic representation γ may be written as $\gamma = (\times_i \sigma_i) \times (\times_j \pi(\tau_j, 1, \alpha_j))$, with σ_i and τ_j being square integrable representations and $\alpha_j \in]0, 1/2[$. So it is enough to prove the result for when γ is a square integrable representation. Let us suppose that γ is square integrable. To prove that $\pi = \mathbf{LJ}(u(\gamma, k_\rho))$ implies that $\mathbf{LJ}(u(\gamma, kk_\rho))$ is a quotient of $\nu^{k_\rho(k-1)/2}\pi \times \nu^{k_\rho(k-3)/2}\pi \times \cdots \times \nu^{k_\rho(-(k-1)/2)}\pi$, we would like to show that the essentially square integrable support of the representation $\mathbf{LJ}(u(\gamma, kk_\rho))$ is the union of the square integrable support of the representations $\{\nu^{k_\rho((k-1)/2-i)}\pi\}_{i\in\{0,1,\dots,k-1\}}$. Then, as the essentially square integrable support of $\times_{i=0}^{k-1}[\nu^{k_\rho((k-1)/2-i)}\pi]$ is in standard order, $\mathbf{LJ}(u(\gamma, kk_\rho))$ will be the unique quotient of the product.

If γ lives on a group of a size such that it transfers to some $\mathbf{C}(\gamma)$, then $\pi = \bar{u}(\mathbf{C}(\gamma), k_\rho)$ and $\mathbf{LJ}(u(\gamma, kk_\rho)) = \bar{u}(\mathbf{C}(\gamma), kk_\rho)$ (see [Bad08, Proposition 3.7(a)] and the second case of transfer in Theorem 13.8 of this paper), hence the result is straightforward. If not, then $u(\gamma, k_\rho)$ satisfies the 'twisted' case of transfer [Bad08, Proposition 3.7(b)] for a non-archimedean field or the first case of Theorem 13.8 for an archimedean field. In the non-archimedean case, one may compute more explicit formulas for the transfer (see [Bad08, (3.9)]) and verify that it works. In the archimedean case, γ is a character of $\mathrm{GL}_1(\mathbb{R})$, so $\pi = \gamma \circ \mathrm{RN}_{k_\rho/2}$ and $\mathbf{LJ}(u(\gamma, kk_\rho)) = \gamma \circ \mathrm{RN}_{kk_\rho/2}$. □

Let us recall the uniqueness of the cuspidal support for automorphic representations. According to a result of Langlands [Lan79] specialized to our case, we know that any automorphic representation of $G'(\mathbb{A})$ is a constituent of an induced representation of the form $\nu'^{a_1}\rho_1 \times \nu'^{a_2}\rho_2 \times \cdots \times \nu'^{a_k}\rho_k$, where the a_i are real numbers and the ρ_i are cuspidal representations. In [JS81] it was proved that, for $G(\mathbb{A})$, the pairs (ρ_i, a_i) are unique. In [Bad08] it is shown that the result is true (more or less by transfer) for the more general case $G'(\mathbb{A})$, if the inner form is split at infinite places. Using the previous results, the same proof now works with no condition on the infinite places.

19. L-functions, ϵ-factors and transfer

The fundamental work of Jacquet, Langlands and Godement on L-functions and ϵ-factors of linear groups over division algebras easily implies the following theorem. What we call ϵ'-factors, following [GJ72], are sometimes called γ-factors in the literature. The value of all functions depends on the choice of some additive non-trivial character ψ of $\mathbb{R}$ which is not relevant to the results.

THEOREM 19.1.

(a) *Let u be a 2-compatible irreducible unitary representation of $\mathrm{GL}_{2n}(\mathbb{R})$ and u' the irreducible unitary representation of $\mathrm{GL}_n(\mathbb{H})$ such that $\mathbf{LJ}(u) = \pm u'$. Then the ϵ' factors of u and u' are equal.*

(b) *Let $\delta \in D_2$ and set $\delta' = \mathbf{C}(\delta)$. Then for all $k \in \mathbb{N}^\times$ the L-functions of $u(\delta, k)$ and $\bar{u}(\delta', k)$ are equal and the ϵ-factors of $u(\delta, k)$ and $\bar{u}(\delta', k)$ are equal.*

(c) *If χ is a character of $\mathrm{GL}(2n, \mathbb{R})$ and $\chi' = \mathbf{LJ}(\chi)$, then the ϵ'-factors of χ and χ' are equal.*

Proof. If we establish (b) and (c), then (a) follows from [GJ72, Corollary 8.9] and the classifications of unitary representations in Tadić's setting as explained earlier in this paper.

Statement (b) is proved in [JL70] for $k = 1$. As a particular case of [Jac79, (5.4), p. 80], the L-function (respectively, ϵ-factor) of a Langlands quotient $u(\delta, k)$ is the product of the L-functions (respectively, ϵ-factors) of representations $\nu^{i-(k-1)/2}\delta$, $0 \leqslant i \leqslant k-1$. The same proof given there for $\mathrm{GL}_{2n}(\mathbb{R})$ works for $\mathrm{GL}_n(\mathbb{H})$ as well, so the $k = 1$ case implies the general case.

Assertion (c), when χ is the trivial character, is just [GJ72, Corollary 8.10, p. 121]. The general case follows easily by torsion with χ (or by reproducing the same proof). □

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