

AN EXTENSION OF A THEOREM OF ZERMELO

JOUKO VÄÄNÄNEN

Abstract. We show that if (M, \in_1, \in_2) satisfies the first-order Zermelo–Fraenkel axioms of set theory when the membership relation is \in_1 and also when the membership relation is \in_2 , and in both cases the formulas are allowed to contain both \in_1 and \in_2 , then $(M, \in_1) \cong (M, \in_2)$, and the isomorphism is definable in (M, \in_1, \in_2) . This extends Zermelo’s 1930 theorem in [6].

Zermelo [6] proved that if (M, \in_1) and (M, \in_2) both satisfy the second-order Zermelo–Fraenkel axioms in which the Separation Schema and the Replacement Schema of *ZFC* are replaced by single second-order axioms, then $(M, \in_1) \cong (M, \in_2)$. We extend this as follows: Let us consider the vocabulary $\{\in_1, \in_2\}$, where both \in_1 and \in_2 are binary predicate symbols. Let $ZFC(\in_1)$ denote the *first-order* Zermelo–Fraenkel axioms of set theory when \in_1 is the membership relation but formulas are allowed to contain \in_2 too. Similarly, in $ZFC(\in_2)$ the membership relation is \in_2 but formulas are allowed to contain \in_1 too. We prove the following theorem:

THEOREM 1. *If $(M, \in_1, \in_2) \models ZFC(\in_1) \cup ZFC(\in_2)$, then $(M, \in_1) \cong (M, \in_2)$ via a definable class function.*

The result of Zermelo readily follows from our theorem. The important difference between our result and Zermelo’s result is that our theories $ZFC(\in_1)$ and $ZFC(\in_2)$ are *first-order* theories. It is important that we allow in these axiom systems formulas from the extended vocabulary $\{\in_1, \in_2\}$. Without this the result would be blatantly false as there are countable non-isomorphic models of *ZFC*, assuming there are models of *ZFC* at all. Since the isomorphism in Theorem 1 is definable, the result can be seen as a provable theorem of the first-order theory $ZFC(\in_1) \cup ZFC(\in_2)$.

Theorem 1 resembles the categoricity conclusion for set theory in [2, page 18]. There are two main differences: First, the axiomatization of set theory in [2] is informal, based on the Axiom of Extensionality and an informal full Comprehension Axiom, while our result is completely formal and in the context of *ZFC*. Secondly, it is assumed in [2] that the two \in -relations give rise to the same (informal) structure of the ordinals, owing to the uniqueness of the ordinal concept. We do not make this assumption but rather *prove* that the two \in -relations have isomorphic ordinals. Martin’s work has been

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extended to class theory in [5]. Theorem 1 was stated without proof in [3, page 104].

We call our theorem an *internal categoricity* result because it shows that one cannot have in one and the same domain two nonisomorphic membership-relations \in_1 and \in_2 if these relations can “talk” about each other.

Our theorem is a strong robustness result for set theory. Essentially, the model cannot be changed “internally”. To get a nonisomorphic model one has to go “outside” the model. Such robustness is important for set theory because set theory is already the “outside” of mathematics, the framework where mathematics is (or can be) built.

How are the numerous independence results in harmony with this internal categoricity? Let us take the Continuum Hypothesis CH as an example. CH is independent of ZFC in the sense that both $ZFC \cup \{CH\}$ and $ZFC \cup \{\neg CH\}$ are consistent, if ZFC itself is. Internal categoricity means simply that if (M, \in_1) satisfies CH and (M, \in_2) satisfies $\neg CH$, then either (M, \in_1) or (M, \in_2) does not satisfy the Separation Schema or the Replacement Schema if formulas are allowed to mention the other membership-relation. Such models cannot be internal to each other in the sense discussed.

In the below proof we will work in $ZFC(\in_1) \cup ZFC(\in_2)$ but in fact operate all the time in either \in_1 -set theory or in \in_2 -set theory. We have to keep the two set theories separate even though they also interact via the Separation and Replacement Schemas in the joint vocabulary $\{\in_1, \in_2\}$.

Let $tr_i(x)$ be the formula $\forall t \in_i x \forall w \in_i t (w \in_i x)$. Let $TC_i(x)$ be the unique u such that $tr_i(u) \wedge \forall v \in_i x (v \in_i u) \wedge \forall v ((tr_i(v) \wedge \forall w \in_i x (w \in_i v)) \rightarrow \forall w \in_i u (w \in_i v))$ (“ u is the \in_i -transitive closure of x ”). When we write $TC_i(\{x\})$, we mean by $\{x\}$ the singleton $\{x\}$ in the sense of \in_i . Let $\phi(x, y)$ be the formula $\exists f \psi(x, y, f)$, where $\psi(x, y, f)$ is the conjunction of the following formulas (where $f(t)$, $f(w)$ and $f(x)$ are in the sense of \in_1):

- (i) In the sense of \in_1 , the set f is a function with $TC_1(\{x\})$ as its domain.
- (ii) $\forall t \in_1 TC_1(x) (f(t) \in_2 TC_2(y))$.
- (iii) $\forall t \in_2 TC_2(y) \exists w \in_1 TC_1(x) (t = f(w))$.
- (iv) $\forall t \in_1 TC_1(x) \forall w \in_1 TC_1(\{x\}) (t \in_1 w \leftrightarrow f(t) \in_2 f(w))$.
- (v) $f(x) = y$.

We prove a sequence of lemmas about the formulas $\phi(x, y)$ and $\psi(x, y, f)$:

LEMMA 2. *If $\psi(x, y, f)$ and $\psi(x, y, f')$, then $f = f'$.*

PROOF. To prove $f = f'$ assume $w \in_1 TC_1(\{x\})$. We show $f(w) = f'(w)$. W.l.o.g. $f(s) = f'(s)$ for $s \in_1 w$. Suppose $t \in_2 f(w)$. Clearly, $t \in_2 TC_2(\{y\})$. By (iii), $t = f(s)$ for some $s \in_1 TC_1(\{x\})$. By (iv), $s \in_1 w$. By (iv) again, $f'(s) \in_2 f'(w)$. By assumption, $f(s) = f'(s)$. Hence $t \in_2 f'(w)$. Thus $\forall t (t \in_2 f(w) \rightarrow t \in_2 f'(w))$. By symmetry, $f(w) = f'(w)$. \dashv

LEMMA 3. *Suppose $\psi(x, y, f)$. If $x' \in_1 x$, then $\phi(x', f(x'))$. If $y' \in_2 y$, then there is $x' \in_1 x$ such that $f(x') = y'$ and $\phi(x', y')$.*

PROOF. Let $y' = f(x')$ and $f' = f \upharpoonright TC_1(\{x'\})$. Clearly now $\psi(x', y', f')$. Hence $\phi(x', f(x'))$. The other claim is proved similarly. \dashv

LEMMA 4. *If $\phi(x, y)$ and $\phi(x, y')$, then $y = y'$. If $\phi(x, y)$ and $\phi(x', y)$, then $x = x'$.*

PROOF. We may assume the claim holds for all \in_1 -elements of x . Suppose $\psi(x, y, f)$ and $\psi(x, y', f')$. We prove $y = y'$. Let $s \in_2 y$. By Lemma 3 there is $t \in_1 x$ such that $f(t) = s$ and $\phi(t, s)$. By (iv), $s \in_1 x$. Let $s' = f'(t)$. By (iv), $s' \in_2 y'$. By Lemma 3 again, $\phi(t, s')$. By the Induction Hypothesis, $s = s'$. We have proved $\forall s(s \in_2 y \rightarrow s \in_2 y')$. The converse follows from symmetry. Now to the second claim. We may assume the claim holds for all \in_2 -elements of y . Suppose $\psi(x, y, f)$ and $\psi(x', y, f')$. We prove $x = x'$. Let $s \in_1 x$. Thus $f(s) \in_2 y$. There is $s' \in_1 \text{TC}_1(\{x'\})$ such that $f'(s') = f(s)$. Now $\phi(s, f(s))$ and $\phi(s', f(s))$ by Lemma 3. Since $f(s) \in_2 y$, $s = s'$. Hence $s \in_1 x'$. We have proved $\forall s(s \in_1 x \rightarrow s \in_1 x')$. The converse follows from symmetry. \dashv

LEMMA 5. *If $\phi(x, y)$ and $\phi(x', y')$, then $x \in_1 x' \leftrightarrow y \in_2 y'$.*

PROOF. Suppose $\psi(x, y, f)$ and $\psi(x', y', f')$. Suppose $x \in_1 x'$. Then $z = f'(x) \in_2 y'$. By Lemma 3, $\phi(x, z)$. We have $\phi(x, y)$ and $\phi(x, z)$. By Lemma 4, $y = z$. Hence $y \in_2 y'$. The converse is similar. \dashv

Let $\text{On}_1(x)$ be the \in_1 -formula saying that x is an ordinal, i.e., a transitive set of transitive sets, and similarly $\text{On}_2(x)$. For $\text{On}_1(\alpha)$ let V_α^1 be the α^{th} level of the cumulative hierarchy in the sense of \in_1 , and similarly V_y^2 when $\text{On}_2(y)$.

LEMMA 6. *If $\phi(\alpha, y)$, then $\text{On}_1(\alpha)$ if and only if $\text{On}_2(y)$. If α is a limit ordinal then so is y , i.e., if $\forall u \in_1 \alpha \exists v \in_1 \alpha (u \in_1 v)$, then $\forall u \in_2 y \exists v \in_2 y (u \in_2 v)$, and vice versa.*

PROOF. Let us fix y . Suppose $\psi(\alpha, y, f)$. We prove that y is a transitive set of transitive sets. Suppose $w \in_2 s \in_2 y$. There are $t \in_1 \alpha$ and $u \in_1 t$ such that $f(t) = s$ and $f(u) = w$. Now $w \in_2 y$ follows from $u \in_1 \alpha$. This shows that $\text{tr}_2(y)$. Similarly one proves that all $s \in_2 y$ satisfy $\text{tr}_2(y)$. This ends the proof of the first claim. The second claim is proved similarly. \dashv

LEMMA 7. *Suppose $\psi(\alpha, y, f)$. If $\text{On}_1(\alpha)$ (or equivalently $\text{On}_2(y)$), then there is $\bar{f} \supseteq f$ such that $\psi(V_\alpha^1, V_y^2, \bar{f})$.*

PROOF. We use induction on α . Suppose the claim holds for α . We prove the claim for $\alpha + 1$. Suppose to this end $\psi(\alpha + 1, y + 1, f)$. We construct \bar{f} such that $\psi(V_{\alpha+1}^1, V_{y+1}^2, \bar{f})$. From $\psi(\alpha + 1, y + 1, f)$ we obtain $\psi(\alpha, y, f \upharpoonright \alpha + 1)$. By assumption there is $g \supseteq f \upharpoonright \alpha + 1$ such that $\psi(V_\alpha^1, V_y^2, g)$. Let $\theta(u, v)$ be the formula

$$\forall w (w \in_2 v \leftrightarrow (w \in_2 V_y^2 \wedge \exists t \in_1 u (g(t) = w))).$$

It follows from the Separation Schema of $ZFC(\in_2)$ that for all $u \in_1 V_{\alpha+1}^1$ there is v such that $\theta(u, v)$. By the Replacement Schema of $ZFC(\in_1)$, we can let \bar{f} be a function (in the sense of $\{\in_1\}$) such that for all $u \in_1 V_{\alpha+1}^1$ we have $\theta(u, \bar{f}(u))$. It is easy to see, using the Separation Schema of $ZFC(\in_1)$, that $\psi(V_{\alpha+1}^1, V_{y+1}^2, \bar{f})$.

Suppose then the claim holds for all $\beta < \alpha = \bigcup \alpha$. For each $\beta < \alpha$ there is thus some g_β such that $\psi(V_\beta^1, V_{f(\beta)}^2, g_\beta)$. By the Replacement Schema of

$ZFC(\in_1)$ we can form the \in_1 -set $\bar{f} = \bigcup_{\beta < \alpha} g_\beta \cup \{(V_\alpha^1, V_y^2)\}$. It is easy to see that $\psi(V_\alpha^1, V_y^2, \bar{f})$. ⊢

LEMMA 8. $\forall x \exists y \phi(x, y)$ and $\forall y \exists x \phi(x, y)$.

PROOF. Let us first assume that both

$$\forall \alpha (\text{On}_1(\alpha) \rightarrow \exists y \phi(\alpha, y)) \tag{1}$$

and

$$\forall y (\text{On}_2(y) \rightarrow \exists \alpha \phi(\alpha, y)). \tag{2}$$

hold. In order to prove $\forall x \exists y \phi(x, y)$, suppose x is given. There is α such that $\text{On}_1(\alpha)$ and $x \in_1 V_\alpha^1$. By (1) there are v and f such that $\psi(\alpha, v, f)$. By Lemma 7 there is $\bar{f} \supseteq f$ such that $\psi(V_\alpha^1, V_v^2, \bar{f})$. By Lemma 3, $\phi(x, \bar{f}(x))$. Thus $\exists y \phi(x, y)$.

In order to prove $\forall y \exists x \phi(x, y)$, suppose y is given. There is v such that $\text{On}_2(v)$ and $y \in_2 V_v^2$. By (2) there are α and f such that $\psi(\alpha, v, f)$. By Lemma 7 there is $\bar{f} \supseteq f$ such that $\psi(V_\alpha^1, V_v^2)$. By condition (iii) of the definition of ψ there is $w \in_1 V_\alpha^1$ such that $\bar{f}(w) = y$. By Lemma 3, $\phi(w, \bar{f}(w))$. Thus $\exists x \phi(x, y)$.

Thus it suffices to show that the failure of (1) or (2) to hold leads to a contradiction.

CASE 1. $\neg(1) \wedge \neg(2)$. Let α be the \in_1 -least α such that $\text{On}_1(\alpha) \wedge \neg \exists y \phi(\alpha, y)$. Let y be the \in_2 -least y such that $\text{On}_2(y) \wedge \neg \exists \beta \phi(\beta, y)$. It is easy to see that $\phi(\alpha, y)$, a contradiction.

CASE 2. $(1) \wedge \neg(2)$. Let y be the \in_2 -least y such that $\text{On}_2(y) \wedge \neg \exists \alpha \phi(\alpha, y)$. Now, $\forall t \in_2 y \exists \alpha (\text{On}_1(\alpha) \wedge \phi(\alpha, t))$. Clearly, y is an \in_2 -limit ordinal. Suppose $z \in_2 V_t^2$, where $t \in_2 y$. Let α and f be such that $\text{On}_1(\alpha) \wedge \psi(\alpha, t, f)$. By Lemma 7 there is $\bar{f} \supseteq f$ such that $\psi(V_\alpha^1, V_t^2, \bar{f})$. There is $x \in_1 V_\alpha^1$ such that $\bar{f}(x) = z$. Thus $\phi(x, z)$ and hence

$$\forall z \in_2 V_y^2 \exists x \phi(x, z).$$

By the Replacement Schema in $ZFC(\in_2)$ there is c such that

$$\forall z \in_2 V_y^2 \exists x \in_2 c \phi(x, z). \tag{3}$$

Let α be such that $c \in_1 V_\alpha^1$. By (1) there are t and f such that $\phi(\alpha, t, f)$. Necessarily, $t \in_2 y$. By Lemma 7 there is $\bar{f} \supseteq f$ such that $\psi(V_\alpha^1, V_t^2, \bar{f})$. In particular, $\bar{f}(c) \in_2 V_y^2$. By (3) there is $b \in_2 c$ such that $\phi(b, \bar{f}(c))$. Since also $\phi(c, \bar{f}(c))$, Lemma 4 gives $c = b$. Thus $c \in_2 c$, a contradiction.

CASE 3. $\neg(1) \wedge (2)$. This case is analogous to Case 2. ⊢

PROPOSITION 9. *The class defined by $\phi(x, y)$ is an isomorphism between the \in_1 -reduct and the \in_2 -reduct.*

PROOF. By Lemmas 4, 5, and 8. ⊢

A similar result holds for first-order Peano arithmetic, extending the categoricity result of Dedekind [1] of second-order Peano arithmetic. The proof (see [4]) of this is similar, but somewhat easier.

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DEPARTMENT OF MATHEMATICS AND STATISTICS
PO BOX 64 (GUSTAF HÄLLSTRÖMIN KATU 2)
FI-00014 UNIVERSITY OF HELSINKI, FINLAND

and

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION
UNIVERSITY OF AMSTERDAM
SCIENCE PARK 107
1098 XG AMSTERDAM, NETHERLANDS

E-mail: jouko.vaananen@helsinki.fi