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Classifying Spaces for Monoidal Categories Through Geometric Nerves

M. Bullejos and A. M. Cegarra

Abstract. The usual constructions of classifying spaces for monoidal categories produce CW-complexes with many cells that, moreover, do not have any proper geometric meaning. However, geometric nerves of monoidal categories are very handy simplicial sets whose simplices have a pleasing geometric description: they are diagrams with the shape of the 2-skeleton of oriented standard simplices. The purpose of this paper is to prove that geometric realizations of geometric nerves are classifying spaces for monoidal categories.

1 Introduction

The theory of classifying spaces of categorical structures has become an essential part of the machinery of algebraic topology and algebraic K-theory, and one of the main reasons for this is that the classifying space constructions transport categorical coherence to homotopical coherence.

The classifying space $|\mathcal{C}|$ of a small category $\mathcal{C}$ is the geometric realization of the simplicial set nerve of $\mathcal{C}$, and the classifying space $\|\mathcal{S}\|$ of a simplicial category $\mathcal{S}\colon \Delta^{\mathrm{op}} \to \mathbf{Cat}$ is defined as Segal's realization [16] of the simplicial space $|\mathcal{S}|\colon \Delta^{\mathrm{op}} \to \mathbf{Top}; [n] \mapsto |\mathcal{S}_n|$. An alternative construction of $\|\mathcal{S}\|$ proceeds by the so-called Grothendieck category $\Delta^{\mathrm{op}}\textstyle\int\mathcal{S}$ [7, 6], since from [15, Theorem 1.2] the existence of a natural homotopy equivalence $\|\mathcal{S}\| \simeq |\Delta^{\mathrm{op}}\textstyle\int\mathcal{S}|$ follows . Indeed, Grothendieck's construction is usually used to define the classifying space of any pseudo-simplicial category, that is, any pseudo-functor $\mathcal{S}\colon \Delta^{\mathrm{op}} \rightsquigarrow \mathbf{Cat}$ [15, 9]. The resulting space $\|\mathcal{S}\| = |\Delta^{\mathrm{op}}\textstyle\int\mathcal{S}|$ is homotopy equivalent to the classifying space $\|\tilde{\mathcal{S}}\|$ of the rectified simplicial category $\tilde{\mathcal{S}}\colon \Delta^{\mathrm{op}} \to \mathbf{Cat}$ obtained by applying Street's first construction [18], to the pseudo-simplicial category $\mathcal{S}$ [13, §3].

If $\mathcal{M}^{\otimes} = (\mathcal{M}, \otimes, \boldsymbol{a}, e, \ell, r)$ is a monoidal category [12], its *classifying space* $\|\mathcal{M}^{\otimes}\|$ is defined as the classifying space $\|\overline{W}\mathcal{M}^{\otimes}\|$ of the pseudo-simplicial category $\overline{W}\mathcal{M}^{\otimes}$ that $\mathcal{M}^{\otimes}$ defines by the familiar bar construction. That is, the category of n-simplices

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is $\overline{W}_n\mathcal{M}^{\otimes} = \mathcal{M}^n$, $(\overline{W}_0\mathcal{M}^{\otimes} = \{e\})$. The face and degeneracy functor are defined on objects by

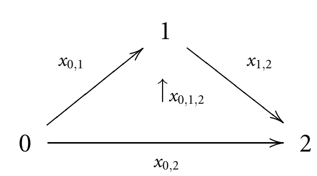
$$d_i(x_1,\dots,x_n) = \begin{cases} (x_2,\dots,x_n) & i=0,\\ (x_1,\dots,x_{i+1}\otimes x_i,\dots,x_n) & 0<i<n,\\ (x_1,\dots,x_{n-1}) & i=n,\end{cases}$$

$$s_i(x_1,\dots,x_n) = (x_1,\dots,x_i,e,x_{i+1},\dots,x_n),\ 0\leq i\leq n,$$

and similarly on arrows, and the natural isomorphisms $d_id_i \cong d_id_{i+1}$, $d_is_i \cong Id$ and $d_{i+1}s_i \cong Id$ are those arising from the associativity and unit constraints of $\mathcal{M}^{\otimes}$ (see [8, 9]). In particular, the space at level 1 of the spectrum associated to a symmetric monoidal category $(\mathcal{M}^{\otimes}, c)$ is the space $\|\mathcal{M}^{\otimes}\|$. Furthermore, let us remark that when $\mathcal{M}^{\otimes}$ is a strict monoidal category, that is, when it is an internal monoid in **Cat**, then $\overline{W}\mathcal{M}^{\otimes}$ is a genuine simplicial category whose classifying space $\|\mathcal{M}^{\otimes}\|$ is just the classifying space of the topological monoid with underlying space $|\mathcal{M}|$, the classifying space of the underlying category $\mathcal{M}$, and composition-law induced by the tensor functor $\otimes$.

The classifying space $\|\mathcal{M}^{\otimes}\|$ of a monoidal category $\mathcal{M}^{\otimes}$ gives a CW-complex, but its cells do not enjoy any proper geometric meaning. However, there is another convincing way of associating a space to $\mathcal{M}^{\otimes}$. This way goes through what Duskin [3] called the *geometric nerve* $\Delta\mathcal{M}^{\otimes}$ of the monoidal category and it was developed (even in the more general context of bicategories) by Street and Duskin himself (*cf.* [19]). This geometric nerve $\Delta\mathcal{M}^{\otimes}$ is a simplicial set that encodes the entire monoidal and categorical structure of $\mathcal{M}^{\otimes}$, and whose simplices have the following pleasing geometrical description:

There is only one 0-simplex in $\Delta\mathcal{M}^{\otimes}$, say e. Its 1-simplices are the objects of $\mathcal{M}$ that are placed on edges $0 \xrightarrow{x_{0,1}} 1$. Its 2-simplices are arrows in $\mathcal{M}$ of the form $x_{0,1,2}\colon x_{0,2} \to x_{1,2}\otimes x_{0,1}$ that are placed on triangles



For $n \geq 3$, an n-simplex of $\Delta\mathcal{M}^{\otimes}$ can be thought of as the 2-skeleton of an oriented

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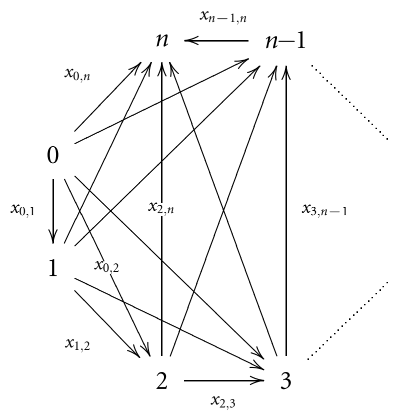
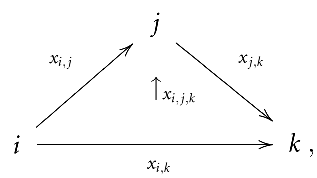
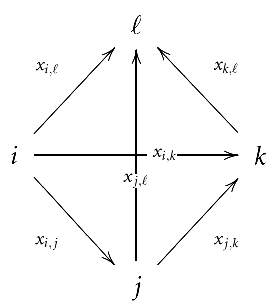
standard n-simplex with objects $x_{i,j}$ placed on the edges $i \to j$, for $0 \le i < j \le n$,



and arrows $x_{i,j,k} : x_{i,k} \to x_{j,k} \otimes x_{i,j}$ placed on the inside of the triangles



for $0 \le i < j < k \le n$. These data are required to satisfy that each tetrahedron



$$x_{i,j,k} : x_{i,k} \to x_{j,k} \otimes x_{i,j}$$
$$x_{i,j,\ell} : x_{i,\ell} \to x_{j,\ell} \otimes x_{i,j}$$
$$x_{i,k,\ell} : x_{i,\ell} \to x_{k,\ell} \otimes x_{i,k}$$
$$x_{j,k,\ell} : x_{j,\ell} \to x_{k,\ell} \otimes x_{j,k}$$

for $0 \le i < j < k < \ell \le n$, is commutative in the sense that the following diagram

$$\begin{array}{ccccc} x_{i,j} & \xrightarrow{x_{i,j,\ell}} & x_{j,\ell} \otimes x_{i,j} & \xrightarrow{x_{j,k,l} \otimes Id} & (x_{k,\ell} \otimes x_{j,k}) \otimes x_{i,j} \\ \downarrow{\scriptstyle x_{i,k,\ell}} & & & & \wr \uparrow a \\ x_{k,\ell} \otimes x_{i,k} & & \xrightarrow{Id \otimes x_{i,j,k}} & & x_{k,\ell} \otimes (x_{j,k} \otimes x_{i,j}) \end{array}$$

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commutes. The simplicial operators

$$\Delta_{n+1}\mathcal{M}^{\otimes} \xleftarrow{s_i} \Delta_n\mathcal{M}^{\otimes} \xrightarrow{d_1} \Delta_{n-1}\mathcal{M}^{\otimes}$$

are defined as for the usual nerve of an ordinary category, thus the face operator d_i deletes the data in which the index i appears. This simplicial set $\Delta\mathcal{M}^{\otimes}$ becomes coskeletal in dimensions greater than 3.

The purpose of this article is to prove the following:

Theorem 1 *For any monoidal category $\mathcal{M}^{\otimes} = (\mathcal{M}, \otimes, \boldsymbol{a}, e, \ell, r)$, there is a homotopy equivalence*

$$\|\mathcal{M}^{\otimes}\| \simeq |\Delta\mathcal{M}^{\otimes}|.$$

To our knowledge, the above theorem has only been stated in the case in which every arrow in $\mathcal{M}$ is invertible and every object has a quasi-inverse with respect to the tensor product, that is, when $\mathcal{M}^{\otimes}$ is a categorical group (cf. [4]). Indeed, when $\mathcal{M}^{\otimes}$ is a strict categorical group (that is, when $\mathcal{M}^{\otimes}$ is an internal group object in **Cat** or equivalently an internal category in the category of groups, cf. [10, Section 3]) then it is easy to see that the geometric nerve $\Delta\mathcal{M}^{\otimes}$ is isomorphic to the simplicial set obtained as the Kan classifying complex [11] of the simplicial group nerve of $\mathcal{M}$.

2 Proof of Theorem 1

Throughout $\mathcal{M}^{\otimes} = (\mathcal{M}, \otimes, \boldsymbol{a}, e, \ell, r)$ is any fixed small monoidal category.

The simplicial category Δ is regarded here as the full subcategory of **Cat**, the category of small categories, whose objects are the categories defined by the ordered sets $[n] = \{0 \leq 1 \leq \cdots \leq n\},\ n \geq 0$.

Since any monoidal category $\mathcal{M}^{\otimes}$ can be considered as a bicategory [1] with only one object [19, Example 2] and any category is a bicategory whose 2-cells are all identities, it makes complete sense to consider the set of (strictly unitary) morphisms of bicategories from a small category to a small monoidal category. Furthermore, it is not difficult to see that the data for an n-simplex $\mathbf{x}$ of the geometric nerve $\Delta\mathcal{M}^{\otimes}$, as described in the introduction, is the same as the data for a morphism of bicategories $\mathbf{x}\colon [n] \to \mathcal{M}^{\otimes}$. That is, $\mathbf{x}$ consists of a family

$$\mathbf{x} = \{x_{i,j}, x_{i,j,k}\}_{0\leq i\leq j\leq k\leq n},$$

with $x_{i,j}$ and $x_{i,j,k}\colon x_{i,k} \to x_{j,k} \otimes x_{i,j}$ objects and arrows in $\mathcal{M}$, respectively, such that:

- $x_{i,i} = e$,
- $x_{i,j,j} = \ell\colon x_{i,j} \to e \otimes x_{i,j}, \quad x_{i,i,j} = r\colon x_{i,j} \to x_{i,j} \otimes e$,
- $(x_{j,k,l} \otimes Id_{x_{i,j}})\, x_{i,j,l} = \boldsymbol{a}_{x_{k,l},x_{j,k},x_{i,j}}\, (Id_{x_{k,l}} \otimes x_{i,j,k})\, x_{i,k,l}$,

for any $0 \leq i \leq j \leq k \leq l \leq n$.

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Thus, the geometric nerve of $\mathcal{M}^{\otimes}$ can be described as the simplicial set

$$\Delta\mathcal{M}^{\otimes} = \mathrm{Mor}(-, \mathcal{M}^{\otimes}) \colon \Delta^{\mathrm{op}} \to \mathbf{Set},$$

which takes each ordered set $[n]$ to the set $\mathrm{Mor}([n], \mathcal{M}^{\otimes})$ of bicategory morphisms from $[n]$ to $\mathcal{M}^{\otimes}$ (cf. [19, p. 573] and [3, Section 2]).

We now note that the geometric nerve $\Delta\mathcal{M}^{\otimes}$ is the simplicial set of objects of the simplicial category

$$\underline{\Delta}\mathcal{M}^{\otimes} = \underline{\mathrm{Mor}}(-, \mathcal{M}^{\otimes}) \colon \Delta^{\mathrm{op}} \to \mathbf{Cat},$$

whose category of n-simplices is $\underline{\Delta}_n\mathcal{M}^{\otimes} = \underline{\mathrm{Mor}}([n], \mathcal{M}^{\otimes})$, the category of bicategory morphisms form $[n]$ to $\mathcal{M}^{\otimes}$ with (strictly unitary) transformations between them. That is, an arrow $\mathbf{f} \colon \mathbf{x} \to \mathbf{x}'$ in $\underline{\Delta}_n\mathcal{M}^{\otimes}$ consists of a family $\mathbf{f} = \{f_{i,j} \colon x_{i,j} \to x'_{i,j}\}_{0 \le i \le j \le n}$ of arrows in $\mathcal{M}$, that satisfy the equations:

$$f_{i,i} = Id_e \text{ and } (f_{j,k} \otimes f_{i,j})\, x_{i,j,k} = x'_{i,j,k}\, f_{i,k}, \tag{1}$$

for any $0 \le i \le j \le k \le n$.

Observe that $\underline{\Delta}\mathcal{M}^{\otimes}$ is a reduced simplicial category, that is, $\underline{\Delta}_0\mathcal{M}^{\otimes} \cong \{e\}$ is the discrete category with only one object. Moreover, $\mathcal{M} \cong \underline{\Delta}_1\mathcal{M}^{\otimes}$, by means of the identification

$$E_1 \colon (x_{0,1} \xrightarrow{f_{0,1}} x'_{0,1}) \mapsto (\mathbf{x} \xrightarrow{\mathbf{f}} \mathbf{x}').$$

In general we have:

Lemma 1 *For every $n \ge 2$, there is a full embedding of categories*

$$E_n \colon \mathcal{M}^n \hookrightarrow \underline{\Delta}_n\mathcal{M}^{\otimes}.$$

Proof The functor $E = E_n$ takes the object $\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{M}^n$ to the morphism $E\mathbf{x} \colon [n] \to \mathcal{M}^{\otimes}$ defined inductively by the equations

$$(E\mathbf{x})_{i,j+1} = \begin{cases} x_{j+1} & i = j \\ x_{j+1} \otimes (E\mathbf{x})_{i,j} & i < j, \end{cases}$$

and

$$(E\mathbf{x})_{i,j,k+1} = \begin{cases} Id_{(E\mathbf{x})_{i,k+1}} & i \le j = k \\ a_{x_{k+1},(E\mathbf{x})_{j,k},(E\mathbf{x})_{i,j}}\ \big(Id_{x_{k+1}} \otimes (E\mathbf{x})_{i,j,k}\big) & i \le j < k. \end{cases}$$

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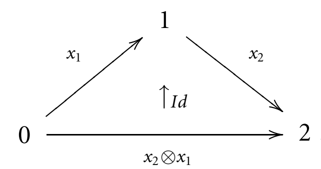
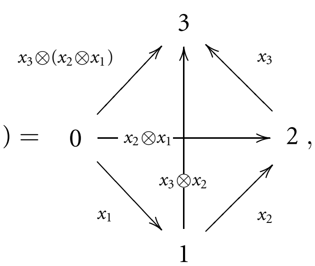
Thus, for example,

$E_2(x_1, x_2) =$



$Id\colon x_2 \otimes x_1 \to x_2 \otimes x_1,$

$E_2(x_1, x_2, x_3) =$



$$\begin{aligned} &Id\colon x_2 \otimes x_1 \to x_2 \otimes x_1 \\ &Id\colon x_3 \otimes (x_2 \otimes x_1) \to x_3 \otimes (x_2 \otimes x_1) \\ &Id\colon x_3 \otimes x_2 \to x_3 \otimes x_2 \\ &\mathfrak{a}\colon x_3 \otimes (x_2 \otimes x_1) \to (x_3 \otimes x_2) \otimes x_1, \end{aligned}$$

and so on.

It is a straightforward consequence of Mac Lane's coherence theorem [12] that $E\mathbf{x}$ is actually an object in $\underline{\Delta}_n\mathcal{M}^{\otimes}$. Further, the functor E on an arrow $\mathbf{f} = (f_1, \dots, f_n)\colon \mathbf{x} \to \mathbf{x}'$ in $\mathcal{M}^n$ is inductively given by

$$(E\mathbf{f})_{i,j+1} = \begin{cases} f_{j+1} & i = j, \\ f_{j+1} \otimes (E\mathbf{f})_{i,j} & i < j. \end{cases}$$

It is clear, from its construction, that E is a faithful and injective on objects functor. That E is full follows from the equalities (1) and the fact that every arrow $(E\mathbf{x}')_{i,j,k}$ is an isomorphism. ■

Note that the pseudo-simplicial category $\overline{W}\mathcal{M}^{\otimes}$ is a pseudo-simplicial subcategory of the simplicial category $\underline{\Delta}\mathcal{M}^{\otimes}$ via the embeddings $E_n\colon \mathcal{M}^n \hookrightarrow \underline{\Delta}_n\mathcal{M}^{\otimes}$ in Lemma 1, with the natural isomorphisms $d_mE_n \cong E_{n-1}d_m$ and $E_ns_m \cong s_mE_{n+1}$ being canonically induced by the associativity and unit constraints $\mathfrak{a}, \ell$ and r of $\mathcal{M}^{\otimes}$ (see Remark below Lemma 2). Lemma 2 below will imply that the simplicial category $\underline{\Delta}\mathcal{M}^{\otimes}$ defines, by realization, a classifying space for the monoidal category $\mathcal{M}^{\otimes}$.

Lemma 2 *For any $n \geq 2$, the embedding $E_n\colon \mathcal{M}^n \hookrightarrow \underline{\Delta}_n\mathcal{M}^{\otimes}$ is a right adjoint section to the Segal map projection*

$$P_n = \prod_{k=1}^{n} d_0 \dots d_{k-2}d_{k+1} \dots d_n\colon \underline{\Delta}_n\mathcal{M}^{\otimes} \to \mathcal{M}^n.$$

Proof Note that for any arrow $\mathbf{f}\colon \mathbf{x} \to \mathbf{y}$ in $\underline{\Delta}_n\mathcal{M}^{\otimes}$,

$$P_n(\mathbf{x} \xrightarrow{\mathbf{f}} \mathbf{y}) = (x_{0,1} \xrightarrow{f_{0,1}} y_{0,1}, \dots, x_{n-1,n} \xrightarrow{f_{n-1,n}} y_{n-1,n}).$$

Then, $P_nE_n = Id_{\mathcal{M}^n}$. Moreover, there is a natural transformation

$$\varepsilon_n \colon Id_{\underline{\Delta}_n\mathcal{M}^{\otimes}} \to E_nP_n \tag{2}$$

that takes a morphism $\mathbf{x} \colon [n] \to \mathcal{M}^{\otimes}$ to the transformation $\varepsilon_n\mathbf{x} \colon \mathbf{x} \to E_nP_n(\mathbf{x})$, inductively defined by:

$$(\varepsilon_n\mathbf{x})_{i,j+1} = \begin{cases} Id_{x_{j,j+1}} & i = j, \\ \big(Id_{x_{j,j+1}} \otimes (\varepsilon_n\mathbf{x})_{i,j}\big)\ x_{i,j,j+1} & i < j. \end{cases}$$

Since $P_n(\varepsilon_n\mathbf{x}) = Id_{P_n(\mathbf{x})}$, for any $\mathbf{x} \in \underline{\Delta}_n\mathcal{M}^{\otimes}$, and $\varepsilon_nE_n(\mathbf{x}) = Id_{E_n(\mathbf{x})}$, for any $\mathbf{x} \in \mathcal{M}^n$, it follows that E_n is a right adjoint to P_n, being the identity and ε_n the unit and the counit of the adjunction respectively. ■

Remark The entire data for the pseudo-functor $\overline{W}\mathcal{M}^{\otimes} \colon \Delta^{\mathrm{op}} \rightsquigarrow \mathbf{Cat}$ can be described with the help of the functor $\underline{\Delta}\mathcal{M}^{\otimes} \colon \Delta^{\mathrm{op}} \to \mathbf{Cat}$ and the adjoint functors E_n and P_n in Lemmas 1 and 2. In fact, for any arrow $\alpha \colon [m] \to [n]$ in Δ, the square

$$\begin{array}{ccc} \mathcal{M}^n & \xrightarrow{\alpha^*} & \mathcal{M}^m \\ {\scriptstyle E_n}\downarrow & & \uparrow{\scriptstyle P_m} \\ \underline{\Delta}_n\mathcal{M}^{\otimes} & \xrightarrow{\alpha^*} & \underline{\Delta}_m\mathcal{M}^{\otimes} \end{array}$$

commutes, that is, $\alpha^* = P_m\alpha^*E_n : \mathcal{M}^n \to \mathcal{M}^m$, and for any pair of composable arrows $[k]\xrightarrow{\beta}[m] \xrightarrow{\alpha} [n]$ in Δ, the natural isomorphism $(\alpha\beta)^* \cong \beta^*\alpha^* \colon \mathcal{M}^n \to \mathcal{M}^k$ is precisely realized by the natural transformation (2), $\varepsilon_m \colon Id_{\underline{\Delta}\mathcal{M}^{\otimes}} \to E_mP_m$, as

$$P_k\beta^*\varepsilon_m\alpha^*E_n \colon P_k(\alpha\beta)^*E_n \xrightarrow{\sim} (P_k\beta^*E_m)(P_m\alpha^*E_n)$$

(note that, for every $\mathbf{x} \in \underline{\Delta}_m\mathcal{M}^{\otimes}$, the arrow $\varepsilon_m\mathbf{x} \colon \mathbf{x} \to E_mP_m(\mathbf{x})$ is an isomorphism in $\underline{\Delta}_m\mathcal{M}^{\otimes}$ whenever every arrow $x_{i,j,k} \colon x_{i,k} \to x_{j,k} \otimes x_{i,j}$ is an isomorphism).

Furthermore, given any arrow $\alpha \colon [n] \to [m]$ in Δ, the natural isomorphisms $\alpha^*E_n \cong E_m\alpha^*$ that form part of the data for the pseudo-simplicial embedding $E \colon \overline{W}\mathcal{M}^{\otimes} \hookrightarrow \underline{\Delta}\mathcal{M}^{\otimes}$ are precisely

$$\varepsilon_m\alpha^*E_n \colon \alpha^*E_n \xrightarrow{\sim} E_mP_m\alpha^*E_n = E_m\alpha^*;$$

in particular, $\varepsilon_nd_iE_{n+1} \colon d_iE_{n+1} \xrightarrow{\sim} E_nd_i$ and $\varepsilon_{n+1}s_iE_n \colon s_iE_n \xrightarrow{\sim} E_{n+1}s_i,\ 0 \le i \le n$, are the primary ones.

As a consequence of Lemma 2 above, we can prove the following:

Proposition 3 *The pseudo-simplicial embedding $E \colon \overline{W}\mathcal{M}^{\otimes} \hookrightarrow \underline{\Delta}\mathcal{M}^{\otimes}$ induces a homotopy equivalence*

$$\|\mathcal{M}^{\otimes}\| \simeq \|\underline{\Delta}\mathcal{M}^{\otimes}\|.$$

Proof As we proved, every functor $E_n\colon \mathcal{M}^n \hookrightarrow \underline{\Delta}_n\mathcal{M}^{\otimes}$ has a left adjoint and therefore [14, Corollary 1] the induced map on classifying spaces $|E_n|\colon |\mathcal{M}^n| \to |\underline{\Delta}_n\mathcal{M}^{\otimes}|$ is a homotopy equivalence. Then, the proposition follows from [15, Corollary 3.3.1]. ■

It is remarkable that the simplicial space $|\underline{\Delta}\mathcal{M}^{\otimes}|\colon [n] \mapsto |\underline{\Delta}_n\mathcal{M}^{\otimes}|$ satisfies the hypothesis of Proposition 1.5 in [17], that is:

- the space $|\underline{\Delta}_0\mathcal{M}^{\otimes}|$ is contractible (since $\underline{\Delta}_0\mathcal{M}^{\otimes} = \{e\}$ is the trivial category),
- the maps $|P_n|\colon |\underline{\Delta}_n\mathcal{M}^{\otimes}| \to |\underline{\Delta}_1\mathcal{M}^{\otimes}|^n = |\mathcal{M}|^n$ are homotopy equivalences (as a consequence of Lemma 2).

Therefore, the already known result below follows:

Corollary 4 *The induced map $|\mathcal{M}| \to \Omega\|\underline{\Delta}\mathcal{M}^{\otimes}\| \simeq \Omega\|\mathcal{M}^{\otimes}\|$ is a homotopy equivalence if and only if the monoid of connected components of the monoidal category is a group. Hence, the classifying space of the underlying category $|\mathcal{M}|$ is, up to group completion, a loop space.*

Let us write $\mathrm{N}\mathcal{C}$ for the simplicial set nerve of any small category $\mathcal{C}$. Then, there is a natural homeomorphism (see [14, Lemma, p. 86] for example)

$$\|\underline{\Delta}\mathcal{M}^{\otimes}\| \cong |\mathrm{diag}\,\mathrm{N}\underline{\Delta}\mathcal{M}^{\otimes}|,$$

where $\mathrm{N}\underline{\Delta}\mathcal{M}^{\otimes}\colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathbf{Set}$ is the bisimplicial set defined by $([p],[q]) \mapsto \mathrm{N}_p\underline{\Delta}_q\mathcal{M}^{\otimes}$. More explicitly, an element $\chi \in \mathrm{N}_p\underline{\Delta}_q\mathcal{M}^{\otimes}$ can be described as a string

$$\chi = (\mathbf{x}^0 \xrightarrow{\mathbf{f}^1} \mathbf{x}^1 \to \cdots \to \mathbf{x}^{p-1} \xrightarrow{\mathbf{f}^p} \mathbf{x}^p) \tag{3}$$

of p composable arrows in the category $\underline{\Delta}_q\mathcal{M}^{\otimes}$. The vertical face and degeneracy operators

$$\mathrm{N}_p\underline{\Delta}_{q+1}\mathcal{M}^{\otimes} \xleftarrow{s^v_m} \mathrm{N}_p\underline{\Delta}_q\mathcal{M}^{\otimes} \xrightarrow{d^v_m} \mathrm{N}_p\underline{\Delta}_{q-1}\mathcal{M}^{\otimes}, \quad 0 \le m \le q$$

are induced by those of $\underline{\Delta}\mathcal{M}^{\otimes}$, that is

$$d^v_m(\chi) = (d_m\mathbf{x}^0 \xrightarrow{d_m f^1} d_m\mathbf{x}^1 \to \cdots \to d_m\mathbf{x}^{p-1} \xrightarrow{d_m f^p} d_m\mathbf{x}^p)$$

and

$$s^v_m(\chi) = (s_m\mathbf{x}^0 \xrightarrow{s_m f^1} s_m\mathbf{x}^1 \to \cdots \to s_m\mathbf{x}^{p-1} \xrightarrow{s_m f^p} s_m\mathbf{x}^p).$$

The horizontal face and degeneracy operators

$$\mathrm{N}_{p+1}\underline{\Delta}_q\mathcal{M} \xleftarrow{s^h_m} \mathrm{N}_p\underline{\Delta}_q\mathcal{M} \xrightarrow{d^h_m} \mathrm{N}_{p-1}\underline{\Delta}_q\mathcal{M}, \quad 0 \le m \le q$$

are those of the nerve $\mathrm{N}\underline{\Delta}_q\mathcal{M}$, that is,

$$d^h_m(\chi) = \begin{cases} (\mathbf{x}^1 \xrightarrow{\mathbf{f}^2} \mathbf{x}^2 \cdots \mathbf{x}^{p-1} \xrightarrow{\mathbf{f}^p} \mathbf{x}^p) & m = 0, \\ (\mathbf{x}^0 \xrightarrow{\mathbf{f}^1} \mathbf{x}^1 \cdots \mathbf{x}^{m-1} \xrightarrow{\mathbf{f}^{m+1}\mathbf{f}^m} \mathbf{x}^{m+1} \cdots \mathbf{x}^{p-1} \xrightarrow{\mathbf{f}^p} \mathbf{x}^p) & 0 < m < p, \\ (\mathbf{x}^0 \xrightarrow{\mathbf{f}^1} \mathbf{x}^1 \cdots \mathbf{x}^{p-2} \xrightarrow{\mathbf{f}^{p-1}} \mathbf{x}^{p-1}) & m = p, \end{cases}$$

$$s^h_m(\chi) = (\mathbf{x}^0 \to \cdots \mathbf{x}^{m-1} \xrightarrow{\mathbf{f}^m} \mathbf{x}^m \xrightarrow{Id} \mathbf{x}^m \xrightarrow{\mathbf{f}^{m+1}} \mathbf{x}^{m+1} \cdots \to \mathbf{x}^p),\ 0 \le m \le p.$$

The following lemma is the key to proving our theorem in this paper.

Lemma 5 *For any $p \ge 1$, the simplicial set $\mathrm{N}_{p-1}\underline{\Delta}\mathcal{M}^{\otimes}$ is a simplicial deformation retract of $\mathrm{N}_p\underline{\Delta}\mathcal{M}^{\otimes}$, via the simplicial injection*

$$s^h_0 \colon \mathrm{N}_{p-1}\underline{\Delta}\mathcal{M}^{\otimes} \hookrightarrow \mathrm{N}_p\underline{\Delta}\mathcal{M}^{\otimes}.$$

Proof Since $d^h_0 s^h_0 = Id$, to prove this lemma it suffices to exhibit a simplicial homotopy $H^p \colon Id \to s^h_0 d^h_0$. We first consider the case $p = 1$.

To define H^1 we begin by defining a simplicial homotopy $\mu \colon d^h_1 \to d^h_0$, between the simplicial maps $d^h_1, d^h_0 \colon \mathrm{N}_1\underline{\Delta}\mathcal{M}^{\otimes} \longrightarrow \mathrm{N}_0\underline{\Delta}\mathcal{M}^{\otimes} = \underline{\Delta}\mathcal{M}^{\otimes}$, as follows: for each $0 \le m \le q$, let $\mu_m \colon \mathrm{N}_1\underline{\Delta}_q\mathcal{M}^{\otimes} \to \underline{\Delta}_{q+1}\mathcal{M}^{\otimes}$ be the map that associates to each arrow $\mathbf{f} \colon \mathbf{x} \to \mathbf{y}$ in $\underline{\Delta}_q\mathcal{M}^{\otimes}$ the geometric $(q+1)$-simplex $\mu_m\mathbf{f}$ given by:

$$(\mu_m\mathbf{f})_{i,j} = \begin{cases} y_{i,j} & j \le m, \\ x_{i,j-1} & i \le m < j, \\ x_{i-1,j-1} & m < i, \end{cases}$$

and

$$(\mu_m\mathbf{f})_{i,j,k} = \begin{cases} y_{i,j,k} \colon y_{i,k} \to y_{j,k} \otimes y_{i,j} & k \le m, \\ (Id_{x_{j,k-1}} \otimes f_{i,j})\, x_{i,j,k-1} \colon x_{i,k-1} \to x_{j,k-1} \otimes y_{i,j} & j \le m < k, \\ x_{i,j-1,k-1} \colon x_{i,k-1} \to x_{j-1,k-1} \otimes x_{i,j-1} & i \le m < j, \\ x_{i-1,j-1,k-1} \colon x_{i-1,k-1} \to x_{j-1,k-1} \otimes x_{i-1,j-1} & m < i. \end{cases}$$

It is straightforward to check that the above definitions do indeed give a simplicial homotopy μ as predicted. Further, observe that $\mu_0 = s_0 d^h_1$ and that $\mu_m s^h_0 = s_m$ (*i.e.*, for $\mathbf{f} = Id_{\mathbf{x}}$, any identity arrow in $\underline{\Delta}_q\mathcal{M}^{\otimes}$, $\mu_m Id_{\mathbf{x}} = s_m\mathbf{x}$) for all $0 \le m \le q$.

The homotopy $H^1 \colon Id \to s^h_0 d^h_0$ we are going to define will satisfy that $d^h_1 H^1 = \mu$ and is determined by the maps $H^1_m \colon \mathrm{N}_1\underline{\Delta}_q\mathcal{M} \to \mathrm{N}_1\underline{\Delta}_{q+1}\mathcal{M}$, $0 \le m \le q$, which apply an arrow $\mathbf{f} \colon \mathbf{x} \to \mathbf{y}$ in $\underline{\Delta}_q\mathcal{M}^{\otimes}$ to the arrow $H^1_m\mathbf{f} \colon \mu\mathbf{f} \to s_m\mathbf{y}$ in $\underline{\Delta}_{q+1}\mathcal{M}^{\otimes}$ given by:

$$(H^1_m\mathbf{f})_{i,j} = \begin{cases} Id_{y_{i,j}} \colon y_{i,j} \to y_{i,j} & j \le m, \\ f_{i,j-1} \colon x_{i,j-1} \to y_{i,j-1} & i \le m < j, \\ f_{i-1,j-1} \colon x_{i-1,j-1} \to y_{i-1,j-1} & i < m. \end{cases}$$

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Observe that $H_0^1 = s_0^v \colon \mathrm{N}_1\underline{\Delta}_q\mathcal{M}^\otimes \to \mathrm{N}_1\underline{\Delta}_{q+1}\mathcal{M}^\otimes$, whence $d_0^v H_0^1 = Id$, and that $d_{q+1}^v H_q^1 \mathbf{f} = Id_\mathbf{y} = s_0^h d_0^h(\mathbf{f})$, for all $f \colon \mathbf{x} \to \mathbf{y}$ in $\underline{\Delta}_q\mathcal{M}^\otimes$. To check in full the remaining homotopy simplicial identities, needed for $H^1 \colon Id \to s_0^h d_0^h$ to be a simplicial homotopy, is again straightforward (though tedious) and we will leave it to the reader.

Finally, for an arbitrary $p \geq 2$, the homotopy $H^p \colon Id \to s_0^h d_0^h$ is defined by the maps $H_m^p \colon \mathrm{N}_p\underline{\Delta}_q\mathcal{M}^\otimes \to \mathrm{N}_p\underline{\Delta}_{q+1}\mathcal{M}^\otimes$, $0 \leq m \leq q$, which take an element χ as in (3) to

$$H_m^p(\chi) = (\mu_m \mathbf{f}^1 \xrightarrow{H_m^1\mathbf{f}} s_m\mathbf{x}^1 \xrightarrow{s_m\mathbf{f}^2} s_m\mathbf{x}^2 \to \cdots \to s_m\mathbf{x}^{p-1} \xrightarrow{s_m\mathbf{f}^p} s_m\mathbf{x}^p).$$

The simplicial identities that make of H^p a simplicial homotopy are at this stage much easier to verify. ■

We are now ready to complete the proof of Theorem 1:

Since $\Delta\mathcal{M}^\otimes = \mathrm{N}_0\underline{\Delta}\mathcal{M}^\otimes$, we have a bisimplicial map $\phi \colon \Delta\mathcal{M}^\otimes \hookrightarrow \mathrm{N}\underline{\Delta}\mathcal{M}^\otimes$, where $\Delta\mathcal{M}^\otimes$ is considered as a bisimplicial set that is constant in the horizontal direction, given by

$$\phi_{p,q}(\mathbf{x}) = s_0^h \overset{(p)}{\cdots} s_0^h(\mathbf{x}) = (\mathbf{x} \xrightarrow{Id} \mathbf{x} \to \cdots \to \mathbf{x} \xrightarrow{Id} \mathbf{x}) \in \mathrm{N}_p\underline{\Delta}_q\mathcal{M}^\otimes,$$

for each $\mathbf{x} \in \Delta_q\mathcal{M}^\otimes$. By an iterative application of Lemma 5 above, we see that the simplicial maps

$$\phi_{p,*} \colon \Delta\mathcal{M}^\otimes \hookrightarrow \mathrm{N}_p\underline{\Delta}\mathcal{M}^\otimes$$

induce, for all $p \geq 0$, a homotopy equivalence on realizations, $|\phi_{p,*}| \colon |\Delta\mathcal{M}^\otimes| \xrightarrow{\sim} |\mathrm{N}_p\underline{\Delta}\mathcal{M}^\otimes|$. It follows then that the simplicial map

$$\mathrm{diag}\phi \colon \Delta\mathcal{M}^\otimes \to \mathrm{diag}\mathrm{N}\underline{\Delta}\mathcal{M}^\otimes$$

also induces a homotopy equivalence on realizations (see [2, 1.2,4.3] for example). Thus,

$$|\Delta\mathcal{M}^\otimes| \simeq |\mathrm{diag}\mathrm{N}\underline{\Delta}\mathcal{M}^\otimes| \cong \|\underline{\Delta}\mathcal{M}^\otimes\|,$$

which together with Proposition 3, completes the proof of Theorem 1.

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