CERTAIN BILATERAL HYPERGEOMETRIC IDENTITIES OF CAYLEY AND ORR TYPE

NIRMALA AGARWAL

1. Recently I **(1)** gave some new basic hypergeometric identities of the Cayley and Orr type with the help of a certain basic differential operator. The present paper deals with some bilateral generalizations of those identities together with certain new identities of the same type. In § 4 is indicated how the generalizations of Orr's identities given recently by Shukla **(8,** Theorems I, II**)** may be connected with each other. Later in § 5 certain general expansions of hypergeometric functions are deduced. The following notation has been used throughout this paper:

$$(q^a; n) = (a; n) = (1 - q^a)(1 - q^{a+1}) \dots (1 - q^{a+n-1}), \ (a; o) = 1$$

$$(q^a; -n) = (a; -n) = (-)^n q^{\frac{1}{2}n(n+1)}/q^{na}(q^{1-a}; n), \qquad |q| < 1$$

$$(a)_{-n} = (-)^n/(1-a)_n,$$

$${}_r\Psi_r\left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_r \end{matrix}; x\right) = \sum_{n=-\infty}^{\infty} \frac{(a_1; n)(a_2; n) \dots (a_r; n)\, x^n}{(b_1; n)(b_2; n) \dots (b_r; n)},$$

where, for convergence, $|x| < 1$, $|b_1 b_2 \dots b_r| < |a_1 a_2 \dots a_r x| < 1$.

$${}_rH_r\left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_r \end{matrix}; x\right) = \sum_{n=-\infty}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_r)_n}{(b_1)_n (b_2)_n \dots (b_r)_n} x^n, \qquad |x| = 1$$

$${}_r\Phi_s\left(\begin{matrix} a_1, a_2, \dots a_r \\ b_1, b_2, \dots, b_s \end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(a_1; n)(a_2; n) \dots (a_r; n)}{(1; n)(b_1; n)(b_2; n) \dots (b_s; n)} x^n, \qquad |x| < 1$$

$${}_rF_s\left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_r)_n}{n!(b_1)_n (b_2)_n \dots (b_s)_n} x^n, \qquad |x| < 1$$

$$\Gamma\left(\begin{matrix} a_1, a_2, \dots, a_m \\ b_1, b_2, \dots, b_m \end{matrix};\right) = \frac{\Gamma(a_1)\Gamma(a_2) \dots \Gamma(a_m)}{\Gamma(b_1)\Gamma(b_2) \dots \Gamma(b_m)},$$

$$\Gamma_m\left(\begin{matrix} a, b \\ c, d \end{matrix};\right) = \frac{(a)_m (b)_m}{(c)_m (d)_m},$$

$$(\delta; n) = (1 - q^\delta)(1 - q^{\delta+1}) \dots (1 - q^{\delta+n-1}),$$

where $\delta \equiv x\ \partial/\partial x$, and

$$\Delta_q(h) = \frac{\Gamma_q(q^{\delta+h})}{\Gamma_q(q^h)} = \frac{\Gamma_q(\delta + h)}{\Gamma_q(h)}$$

Received December 15, 1958.

https://doi.org/10.4153/CJM-1960-003-4 Published online by Cambridge University Press

where $\Gamma_q(x)$ is a basic gamma function defined by Jackson **(7)** and

$$\Delta(h) = \frac{\Gamma(\delta + h)}{\Gamma(h)}.$$

2. Three bilateral hypergeometric identities. We now proceed to prove the following three identities that:

$$(2.1)\quad (c;m)(d-b;m)(a+c-2m;m)\,\Delta_q(c+m) \times {}_1\Psi_1\binom{d-b+m;x}{1+m}\,{}_2\Psi_2\binom{a+c-m,\,b;\,xq^{-a-b+2m}}{1+m,\,c}$$

= the same expression with c and d interchanged.

$$(2.2)\quad (f-b;m)(d;m)(1-a;m)(e-c;m)\,\Delta_q(d+m) \times {}_1\Psi_1\binom{f-b+m;x}{1+m}\,{}_3\Psi_3\binom{a,\,b,\,e-c+m;\,xq^{f-b}}{1+m,\,d,\,e}$$

$$= (d-b;m)(f;m)(1-c;m)(e-a;m)\,\Delta_q(f+m) \times {}_1\Psi_1\binom{d-b+m;x}{1+m}\,{}_3\Psi_3\binom{c,\,b,\,e-a+m;\,xq^{d-b}}{1+m,\,e,\,f}$$

provided $a + f = c + d$.

$$(2.3)\quad \frac{(c;m)(c';m)}{(c-a;m)(c'-b';m)}\,\Delta_q(c+m)\Delta_q(c'+m) \times {}_2\Psi_2\binom{a,\,b;\,xq^{c-a-b+m}}{1+m,\,c}\,{}_2\Psi_2\binom{a',\,b';\,x}{1+m,\,c'}$$

$$= \frac{(1-a-a';m)(1-b-b';m)}{(1-a';m)(1-b;m)}\,q^{(a+b')m}\,\Delta_q(a+a')\,\Delta_q(b+b') \times {}_2\Psi_2\binom{b',\,c-a+m;\,x}{1+m,\,b+b'-m}\,{}_2\Psi_2\binom{c'-b'+m,\,a;\,x\,q^{a'+b'-c'-m}}{1+m,\,a+a'-m}$$

provided $a + a' + b + b' = c + c' + 2m$.

The identities (2.1, 2.2, and 2.3) are generalizations of certain earlier results (**1**, 2.1, 2.2, 2.3).

Proof of 2.1. Consider the known identity (**1**, 2.1), namely, that

$$(2.4)\qquad \Delta_q(c)\,{}_1\Phi_0(d-b;x)\,{}_2\Phi_1\binom{a+c,\,b;\,xq^{-a-b}}{c}$$

equals the same expression with c and d interchanged.

Comparing the coefficients of x^n on both the sides we get the transformation

$$(2.5)\quad (c;n)(d-b;n)\,{}_3\Phi_2\binom{a+c,\,b,\,-n;\,q^{1-a-d}}{c,\,1+b-d-n} = (d;n)(c-b;n)\,{}_3\Phi_2\binom{a+d,\,b,\,-n;\,q^{1-a-c}}{d,\,1+b-c-n}.$$

Replacing n, a, b, c, and d respectively by $2m+n$, $a-m$, $b-m$, $c-m$, $d-m$ in (2.5) we find that

$$(2.6)\quad (c;m+n)(d-b;m+n)(a+c-2m;m) \times {}_3\Psi_3\binom{a+c-m,\,b,\,-m-n;\,q^{1-a-d+2m}}{1+m,\,c,\,1+b-d-m-n}$$

equals the same expression with c and d interchanged.* Hence on comparing the coefficients of x^n and using the relation (2.6) we find that

$$(c;m)(d-b;m)(a+c-2m;m)\,\Delta_q(c+m) \times {}_1\Psi_1\binom{d-b+m;\,x}{1+m}\,{}_2\Psi_2\binom{a+c-m,\,b;\,x\,q^{-a-b+2m}}{1+m,\,c}$$

equals the same expression with c and d interchanged, which proves (2.1). Putting $m=0$ (2.1) gives (2.4).

Proceeding exactly in the above manner we can prove the identities (2.2) and (2.3). Putting $m=0$ in (2.2) and (2.3) we get back to the known identities due to Agarwal (**1**, 2.2, and 2.3).

3. Certain new identities and their generalizations. In this section we prove three new identities and later deduce their bilateral generalizations. The identities are

$$(3.1)\quad \Delta(1+a-c)\,{}_1F_0(e-c;x)\,{}_3F_2\binom{a,\,b,\,c;\,x}{1+a-b,\,1+a-c} = \Delta(e)\,{}_1F_0(1+a-2c;x)\,{}_4F_3\binom{a-2b,\,1+\tfrac{1}{2}a-b,\,-b,\,c;\,x}{\tfrac{1}{2}a-b,\,1+a-b,\,e}$$

and

$$(3.2)\quad \Delta(1+a-c)\,{}_1F_0(e-c;x)\,{}_4F_3\binom{a,\,\tfrac{1}{2}a+1,\,b,\,c;\,x}{\tfrac{1}{2}a,\,1+a-b,\,1+a-c} = \Delta(e)\,{}_1F_0(1+a-2c;x)\,{}_3F_2\binom{a-2b,\,-b,\,c;\,x}{1+a-b,\,e},$$

*The limiting case as $q\to 1$ of (2.6) can be obtained directly by putting $c=m+n+1$ and replacing $2-e$, $2-f$, b, d, and a respectively by c, d, $1-b$, $1-m$, and $1+a-2m$ in a known result due to M. Jackson (**6**, p. 34).

provided $1 + a - c - e = 2b$ in (3.1) and (3.2), and

$$(3.3)\quad \Delta(1+a-c)\ {}_1F_0(e-c;x)\ {}_4F_3\left(\begin{matrix} a, \tfrac{1}{2}a+1, b, c; x \\ \tfrac{1}{2}a, 1+a-b, 1+a-c \end{matrix}\right)$$

$$= \Delta(e)\ {}_1F_0(1+a-2c;x)\ {}_4F_3\left(\begin{matrix} a-2b-1, \tfrac{1}{2}a+\tfrac{1}{2}-b, -b-1, c; x \\ \tfrac{1}{2}a-\tfrac{1}{2}-b, 1+a-b, e \end{matrix}\right)$$

provided $a - c - e = 2b$.

Proof of (3.1). It is easy to see that for suitably restricted parameters we have **(4)**

$$\frac{\Gamma(\delta+c)\Gamma(d-c)}{\Gamma(\delta+d)} f(x) = \int_0^1 u^{c-1}(1-u)^{d-c-1} f(xu)\,du.$$

Let us replace c and d respectively by $1 + a - c$ and e and take

$$f(x) = {}_1F_0(e-c;x)\ {}_3F_2\left(\begin{matrix} a, b, c; x \\ 1+a-b, 1+a-c \end{matrix}\right).$$

Then the right-hand side becomes

$$\int_0^1 u^{a-c}(1-u)^{e+c-a-2}(1-ux)^{c-e}\ {}_3F_2\left(\begin{matrix} a, b, c; xu \\ 1+a-b, 1+a-c \end{matrix}\right) du.$$

Expanding the ${}_3F_2$ series and interchanging the order of integration and summation (which is easily justifiable), we have

$$\sum_{r=0}^{\infty} \frac{(a)_r(b)_r(c)_r}{r!(1+a-b)_r(1+a-c)_r} x^r \int_0^1 u^{a+r-c}(1-u)^{e+c-a-2}(1-ux)^{c-e}\,du$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(c)_r x^r}{r!(1+a-b)_r(1+a-c)_r} \cdot \frac{\Gamma(1+a-c+r)\Gamma(e+c-a-1)}{\Gamma(e+r)}$$

$${}_2F_1\left(\begin{matrix} e-c, 1+a-c+r; x \\ e+r \end{matrix}\right).$$

Using Euler's identity (Tract 1, 1.2, 2.) we get

$$(3.4)\quad \frac{\Gamma(1+a-c)\Gamma(e+c-a-1)}{\Gamma(e)}\ {}_1F_0(1+a-2c;x)$$

$$\sum_{r=0}^{\infty} \frac{(a)_r(b)_r(c)_r}{r!(1+a-b)_r(e)_r} x^r\ {}_2F_1\left(\begin{matrix} c+r, e+c-a-1; x \\ e+r \end{matrix}\right).$$

Now, we have the transformation

$${}_4F_3\left(\begin{matrix} a-2b, \tfrac{1}{2}a+1-b, -b, c; x \\ \tfrac{1}{2}a-b, 1+a-b, e \end{matrix}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(c)_r}{r!(1+a-b)_r(e)_r} x^r\ {}_2F_1\left(\begin{matrix} c+r, e+c-a-1; x \\ e+r \end{matrix}\right)$$

https://doi.org/10.4153/CJM-1960-003-4 Published online by Cambridge University Press

provided $1 + a - c - e = 2b$, which can easily be obtained by collecting the coefficients of x^n and using the known summation theorem **(2**, §§ 4.5, 1.2**)**. Hence using this transformation in (3.4) we have the required identity (3.1).

To prove (3.2) and (3.3) we proceed exactly as above with

$$f(x) = {}_1F_0(e - c; x)\, {}_4F_3\begin{pmatrix} a, \frac{1}{2}a + 1, b, c; x \\ \frac{1}{2}a, 1 + a - b, 1 + a - c \end{pmatrix},$$

and use the transformations **(2**, §§ 4.5, 1.3, 1.4**)** giving the sum of a nearly-poised ${}_4F_3$.

We can also obtain the basic analogue of the identity (3.3) in the form

$$(3.5)\quad \Delta_q(1 + a - c)\, {}_1\Phi_0(q^{e-c}; x)\, {}_5\Phi_4\begin{pmatrix} q^a, q^{\frac{1}{2}a+1}, -q^{\frac{1}{2}a+1}, q^b, q^c; xq^{a-2b-2c} \\ q^{\frac{1}{2}a}, -q^{\frac{1}{2}a}, q^{1+a-b}, q^{1+a-c} \end{pmatrix}$$

$$= \Delta_q(e)\, {}_1\Phi_0(q^{1+a-2c}; x)\, {}_5\Phi_4\begin{pmatrix} q^{a-2b-1}, q^{\frac{1}{2}a+\frac{1}{2}-b}, -q^{\frac{1}{2}a+\frac{1}{2}-b}, q^{-b-1}, q^c; xq^{1+a-2c} \\ q^{\frac{1}{2}a-\frac{1}{2}-b}, -q^{\frac{1}{2}a-\frac{1}{2}-b}, q^{1+a-b}, q^e \end{pmatrix}$$

provided $a - c - e = 2b$.

To prove (3.5) we use the basic integral

$$(3.6)\quad \frac{\Gamma_q(\delta + c)\Gamma_q(d - c)}{\Gamma_q(\delta + d)}\, \Phi(x) = \int_0^1 u^{c-1}(1 - uq)^{d-c-1}\Phi(xu)\, d(qu)$$

used in an earlier paper as well **(1)**, where $(1 - q^a x)^{-a}$ means the basic binomial expansion

$$1 + \frac{(1 - q^a)}{(1 - q)} x + \frac{(1 - q^a)(1 - q^{a+1})}{(1 - q)(1 - q^2)} x^2 + \dots,$$

or ${}_1\Phi_0\,(a; x)$.

Replace c and d respectively by $1 + a - c$ and e in (3.6) and take

$$\Phi(x) = {}_1\Phi_0(q^{e-c}; x)\, {}_5\Phi_4\begin{pmatrix} q^a, q^{\frac{1}{2}a+1}, -q^{\frac{1}{2}a+1}, q^b, q^c; xq^{a-2b-2c} \\ q^{\frac{1}{2}a}, -q^{\frac{1}{2}a}, q^{1+a-b}, q^{1+a-c} \end{pmatrix}.$$

Proceeding as for (3.1) we obtain, on using a known summation theorem due to Bailey **(3**, § 3 (3)**)**, the required identity (3.5).

It may be noted that as a consequence of these identities we get certain interesting relations between two terminating nearly-poised series. From (3.1), (3.2), and (3.3) respectively we get

$$(3.7)\quad {}_4F_3\begin{pmatrix} a, b, c, -n; \\ 1 + a - b, 1 + a - c, 1 + c - e - n \end{pmatrix}$$

$$= \frac{(1 + a - 2c)_n(e)_n}{(1 + a - c)_n(e - c)_n}\, {}_5F_4\begin{pmatrix} a - 2b, 1 + \frac{1}{2}a - b, -b, c, -n; \\ \frac{1}{2}a - b, 1 + a - b, e, 2c - a - n \end{pmatrix}$$

provided $1 + a - c - e = 2b$,

$$(3.8)\quad {}_5F_4\begin{pmatrix} a, \frac{1}{2}a + 1, b, c, -n; \\ \frac{1}{2}a, 1 + a - b, 1 + a - c, 1 + c - e - n \end{pmatrix}$$

$$= \frac{(1 + a - 2c)_n(e)_n}{(1 + a - c)_n(e - c)_n}\, {}_4F_3\begin{pmatrix} a - 2b, -b, c, -n; \\ 1 + a - b, e, 2c - a - n \end{pmatrix}$$

provided $1 + a - c - e = 2b$, and

$$(3.9)\quad {}_5F_4\left(\begin{matrix}a, \tfrac{1}{2}a+1, b, c, -n; \\ \tfrac{1}{2}a, 1+a-b, 1+a-c, 1+c-e-n\end{matrix}\right)$$

$$= \frac{(1+a-2c)_n(e)_n}{(1+a-c)_n(e-c)_n}\, {}_5F_4\left(\begin{matrix}a-2b-1, \tfrac{1}{2}a+\tfrac{1}{2}-b, -b-1, c, -n; \\ \tfrac{1}{2}a-\tfrac{1}{2}-b, 1+a-b, e, 2c-a-n\end{matrix}\right)$$

provided $a - c - e = 2b$.

The basic analogue of (3.9) may be written as (from 3.5)

$$(3.10)\quad {}_6\Phi_5\left(\begin{matrix}q^a, q^{\frac{1}{2}a+1}, -q^{\frac{1}{2}a+1}, q^b, q^c, q^{-n}; q \\ q^{\frac{1}{2}a}, -q^{\frac{1}{2}a}, q^{1+a-b}, q^{1+a-c}, q^{1+c-e-n}\end{matrix}\right)$$

$$= \frac{(q^{1+a-2c}; n)(q^e; n)}{(q^{1+a-c}; n)(q^{e-c}; n)}\, {}_6\Phi_5\left(\begin{matrix}q^{a-2b-1}, q^{\frac{1}{2}a+\frac{1}{2}-b}, -q^{\frac{1}{2}a+\frac{1}{2}-b}, q^{-b-1}, q^c, q^{-n}; q \\ q^{\frac{1}{2}a-\frac{1}{2}-b}, -q^{\frac{1}{2}a-\frac{1}{2}-b}, q^{1+a-b}, q^e, q^{2c-a-n}\end{matrix}\right)$$

provided $a - c - e = 2b$.

Next we deduce the bilateral generalizations of the identities (3.1), (3.2), and (3.3). In the known transformation due to Shukla (**8**, 2.2) let us take $M = 3, N = 1, a_1 = b_1 = 1$ and $c_4 = 0$. This gives us a relation between three non-terminating nearly-poised ${}_4H_4$ series and a terminating ${}_4F_3$ series viz.;

$$(3.11)\quad (1-E)(1-F)$$

$$\Gamma\left(\begin{matrix}1+E-D, 1+F-D, D-E, D-F; \\ 2+a-b-D, 2+a-c-D, 2+c-e-D, D-a, D-b, D-c\end{matrix}\right)$$

$$\times \frac{(D+e-c-1)_n}{(D)_n}$$

$${}_4H_4\left(\begin{matrix}1+a-D, 1+b-D, 1+c-D, 1-D-n; \\ 2-D, 2+a-b-D, 2+a-c-D, 2+c-e-D-n\end{matrix}\right)$$

$$+ \text{ idem } (D; E, F)$$

$$= \Gamma\left(\begin{matrix}D, E, F, 2-D, 2-E, 2-F; \\ 1+a-b, 1+a-c, 1+c-e, 1-a, 1-b, 1-c\end{matrix}\right)$$

$$\times \frac{(e-c)_n}{(1)_n}\, {}_4F_3\left(\begin{matrix}a, b, c, -n; \\ 1+a-b, 1+a-c, 1+c-e-n\end{matrix}\right).$$

Transform the ${}_4F_3$ series on the right by (3.7) and then replace n, a, b, c, and e respectively by $2m + n$, $a - 2m$, $b - m$, $c - m$, and $e - m$. After some simplification we find that (3.11) may be written as

$$(3.12)\quad (1-E)(1-F)$$

$$\Gamma\left(\begin{matrix}1+E-D, 1+F-D, D-E, D-F; \\ 2+a-b-D-m, 2+a-c-D-m, 2+c-e-D, \\ D-a+2m, D-b+m, D-c+m\end{matrix}\right)$$

$$\times \frac{(D+e-c-1)_{2m}}{(D)_{2m}}\, {}_1H_1\left(\begin{matrix}D+e-c-1+2m; x \\ D+2m\end{matrix}\right)\times$$

https://doi.org/10.4153/CJM-1960-003-4 Published online by Cambridge University Press

$$
{}_3H_3\begin{pmatrix} 1+a-D-2m,\ 1+b-D-m,\ 1+c-D-m;\ x \\ 2-D,\ 2+a-b-D-m,\ 2+a-c-D-m \end{pmatrix}
$$

$$
+ \text{ idem } (D; E, F)
$$

$$
= \Gamma\begin{pmatrix} D, E, F, 2-D, 2-E, 2-F; \\ 1+a-b-m,\ 1+a-c-m,\ 1+c-e,\ 1-a+2m,\ 1-b+m,\ 1-c+m \end{pmatrix}
$$

$$
\times \Gamma_m\begin{pmatrix} e,\ a-2b,\ \tfrac{1}{2}a+1-b,\ 1-c,\ -b+m,\ 1+a-2c; \\ 1,\ \tfrac{1}{2}a-b,\ b-a,\ c-a,\ 1+a-c \end{pmatrix}
$$

$$
\times \frac{\Delta(e+m)}{\Delta(1+a-c+m)}
$$

$$
\times {}_4H_4\begin{pmatrix} a-2b+m,\ \tfrac{1}{2}a+1-b+m,\ -b+2m,\ c;\ x \\ 1+m,\ \tfrac{1}{2}a-b+m,\ 1+a-b,\ e \end{pmatrix}
$$

$$
{}_1H_1\begin{pmatrix} 1+a-2c+m \\ 1+m \end{pmatrix};x\Big),
$$

provided $1+a-c-e = 2b-2m$. (3.12) reduces to (3.1) when $D = 1$ and $m = 0$.

In exactly the same manner bilateral generalizations of the identities (3.2) and (3.3) may be written in the form

$$
(3.13) \quad (1-E)(1-F)(1-G)
$$

$$
\Gamma\begin{bmatrix} 1+E-D,\ 1+F-D,\ 1+G-D,\ D-E,\ D-F,\ D-G; \\ 1+\tfrac{1}{2}a-D-m, 2+a-b-D-m, 2+a-c-D-m, 2+c-e-D,\ D-a+2m,\ D-\tfrac{1}{2}a-1+m,\ D-b+m,\ D-c+m \end{bmatrix}
$$

$$
\times \frac{(D+e-c-1)_{2m}}{(D)_{2m}}\ {}_1H_1\begin{pmatrix} D+e-c-1+2m;\ x \\ D+2m \end{pmatrix}
$$

$$
{}_4H_4\begin{pmatrix} 1+a-D-2m, 2+\tfrac{1}{2}a-D-m, 1+b-D-m, 1+c-D-m;\ x \\ 2-D,\ 1+\tfrac{1}{2}a-D-m,\ 2+a-b-D-m,\ 2+a-c-D-m \end{pmatrix}
$$

$$
+ \text{ idem } (D; E, F, G)
$$

$$
= \Gamma\begin{pmatrix} D, E, F, G, 2-D, 2-E, 2-F, 2-G; \\ 1+a-b-m,\ 1+a-c-m,\ 1+c-e,\ \tfrac{1}{2}a-m,\ 1-a+2m,\ -\tfrac{1}{2}a+m,\ 1-b+m,\ 1-c+m \end{pmatrix}
$$

$$
\times \Gamma_m\begin{pmatrix} e,\ a-2b,\ -b+m,\ 1-c,\ 1+a-2c \\ 1,\ b-a,\ c-a,\ 1+a-c \end{pmatrix} \frac{\Delta(e+m)}{\Delta(1+a-c+m)}
$$

$$
\times {}_1H_1\begin{pmatrix} 1+a-2c+m;\ x \\ 1+m \end{pmatrix} {}_3H_3\begin{pmatrix} a-2b+m,\ -b+2m,\ c;\ x \\ 1+m,\ 1+a-b,\ e \end{pmatrix}
$$

https://doi.org/10.4153/CJM-1960-003-4 Published online by Cambridge University Press

provided $1 + a - c - e = 2b - 2m$,

(3.14)

$$= \Gamma\begin{pmatrix} D, E, F, G, 2-D, 2-E, 2-F, 2-G; \\ 1+a-b-m, 1+a-c-m, 1+c-e, \tfrac{1}{2}a-m, 1-a+2m, \\ -\tfrac{1}{2}a+m, 1-b+m, 1-c+m \end{pmatrix}$$

$$\times \Gamma_m\begin{pmatrix} e, 1+a-2c, a-2b-1, -b-1+m, 1-c, \tfrac{1}{2}a+\tfrac{1}{2}-b \\ 1, b-a, c-a, 1+a-c, \tfrac{1}{2}a-\tfrac{1}{2}-b \end{pmatrix}$$

$$\frac{\Delta(e+m)}{\Delta(1+a-c+m)}$$

$$\times {}_1H_1\begin{pmatrix} 1+a-2c+m; x \\ 1+m \end{pmatrix}$$

$$\times {}_4H_4\begin{pmatrix} a-2b+m-1, \tfrac{1}{2}a+\tfrac{1}{2}-b+m, -b-1+2m, c \\ 1+m, \tfrac{1}{2}a-\tfrac{1}{2}-b+m, 1+a-b, e \end{pmatrix}; x$$

provided $a - c - e = 2b - 2m$. Putting $D = 1$ and $m = 0$ in (3.13) and (3.14) we get back to the identities (3.1) and (3.2).

4. Next we show how the bilateral generalizations due to Shukla (**8**, Theorems I, II) of Orr's identities (Tract, § 10.1 (2 and 3)) may be deduced from each other by the use of the following identity:

(4.1)

$$(1-a)_m(1-b)_m(1-a')_m(1-b')_m {}_2H_2\begin{pmatrix} a, b \\ 1+m, c \end{pmatrix}; x\Big) {}_2H_2\begin{pmatrix} a', b' \\ 1+m, c' \end{pmatrix}; x\Big)$$

$$= (c-a)_m(c-b)_m(c'-a')_m(c'-b')_m$$

$${}_2H_2\begin{pmatrix} c-a+m, c-b+m \\ 1+m, c \end{pmatrix}; x\Big) {}_2H_2\begin{pmatrix} c'-a'+m, c'-b'+m \\ 1+m, c' \end{pmatrix}; x\Big)$$

with $a + a' + b + b' = c + c' + 2m$.

The identity (4.1) may be obtained from the known identity due to Chaundy (**4**, 25).

The identities due to Shukla can also be written in the form

(4.2)
$$(1-E)\,\Gamma\begin{pmatrix} 1+E-D, D-E; \\ 1+2c-D-2m, 3/2+a+b-c-D-m, \\ D-2b+2m, D-2a+2m \end{pmatrix}$$

$$\times \frac{(c+D-a-b+\tfrac{1}{2}+m)_{2m}}{(D)_{2m}} {}_1H_1\begin{pmatrix} c+D-a-b-\tfrac{1}{2}+3m; x \\ D+2m \end{pmatrix}$$

$${}_2H_2\begin{pmatrix} 1+2a-D-2m, 1+2b-D-2m; x \\ 2-D, 1+2c-D-2m \end{pmatrix} + \text{idem } (D; E)$$

https://doi.org/10.4153/CJM-1960-003-4 Published online by Cambridge University Press

$$= \Gamma\begin{bmatrix} D, E, 2-D, 2-E; \\ 2c-2m, \tfrac{1}{2}+a+b-c-m, 1-2a+2m, 1-2b+2m \end{bmatrix}$$

$$\Gamma_m\begin{bmatrix} \tfrac{1}{2}+c-a, \tfrac{1}{2}+c-b, 1+c, 1-a, 1-b; \\ 1, 1, \tfrac{1}{2}+c, \tfrac{1}{2}+c, 1-c, \tfrac{1}{2}-c \end{bmatrix}$$

$$\frac{\Delta(c+1+m)}{\Delta(c+\tfrac{1}{2}+m)}\, {}_2H_2\begin{pmatrix} a, b \\ 1+m, c \end{pmatrix}; x\Big)\, {}_2H_2\begin{pmatrix} c-a+\tfrac{1}{2}+m, c-b+\tfrac{1}{2}+m \\ 1+m, c+1 \end{pmatrix}; x\Big)$$

and

$$(4.3)\quad (1-E)\,\Gamma\begin{pmatrix} 1+E-D, D-E; \\ 2c-D-2m, 3/2+a+b-c-D-m, \\ 1+D-2a+2m, D-2b+2m \end{pmatrix}$$

$$\times \frac{(c+D-a-b-\tfrac{1}{2}+m)_{2m}}{(D)_{2m}}\, {}_1H_1\begin{pmatrix} c+D-a-b-\tfrac{1}{2}+3m \\ D+2m \end{pmatrix}; x\Big)$$

$${}_2H_2\begin{pmatrix} 2a-2m-D, 2b+1-D-2m \\ 2-D, 2c-2m-D \end{pmatrix}; x\Big) + \text{idem } (D; E)$$

$$= \Gamma\begin{pmatrix} D, E, 2-D, 2-E; \\ 2c-2m-1, \tfrac{1}{2}+a+b-c-m, 2-2a+2m, 1-2b+2m \end{pmatrix}$$

$$\Gamma_m\begin{pmatrix} \tfrac{1}{2}+c-a, c-\tfrac{1}{2}-b, 1-a, 1-b; \\ 1, 1, 3/2-c, c-\tfrac{1}{2}, 1-c \end{pmatrix} \frac{\Delta(c+m)}{\Delta(c-\tfrac{1}{2}+m)}\, {}_2H_2\begin{pmatrix} a, b \\ 1+m, c \end{pmatrix}; x\Big)$$

$${}_2H_2\begin{pmatrix} c-a+\tfrac{1}{2}+m, +c-b-\tfrac{1}{2}+m \\ 1+m, c \end{pmatrix}; x\Big).$$

Now let $q \to 1$ in (2.3) and then use the identity (4.1); to transform the right-hand side put $a + a' = b + b' = c + \tfrac{1}{2} + m$, and $c' = c + 1$. This gives us the transformation

$$\Gamma_m(c, c+1, \tfrac{1}{2}+c-b, 1-a)\, \frac{\Delta(c+m+1)}{\Delta(c+m+\tfrac{1}{2})}$$

$$\times\, {}_2H_2\begin{pmatrix} a, b \\ 1+m, c \end{pmatrix}; x\Big)\, {}_2H_2\begin{pmatrix} c-a+\tfrac{1}{2}+m, c-b+\tfrac{1}{2}+m \\ 1+m, c+1 \end{pmatrix}; x\Big)$$

$$= \Gamma_m(\tfrac{1}{2}+c-a, \tfrac{1}{2}+c, \tfrac{1}{2}+c, \tfrac{1}{2}-a, c-b)\, \frac{\Delta(c+\tfrac{1}{2}+m)}{\Delta(c+m)}$$

$$\times\, {}_2H_2\begin{pmatrix} b, a+\tfrac{1}{2} \\ 1+m, c+\tfrac{1}{2} \end{pmatrix}; x\Big)\, {}_2H_2\begin{pmatrix} c-a+\tfrac{1}{2}+m, c-b+m \\ 1+m, c+\tfrac{1}{2} \end{pmatrix}; x\Big).$$

Using (4.2) on the left of the above equation we find that it reduces to (4.3) with $a + \tfrac{1}{2}$, $c + \tfrac{1}{2}$ for a and c. Thus (4.3) is connected with (4.2).

5. Certain general hypergeometric expansions. In this section we obtain certain general expansions both for ordinary and bilateral hyper-

https://doi.org/10.4153/CJM-1960-003-4 Published online by Cambridge University Press

geometric series with the help of transformations deduced in previous sections. Before proceeding to the actual deduction of the general expansions we prove a lemma which is a bilateral generalization of a known transformation due to Chaundy (**4**, 42).

LEMMA. *If*

$$\Psi\binom{a,\ldots}{1+m, c,\ldots}; xq^{\lambda}\Big) \quad \text{and} \quad \Psi\binom{a',\ldots}{1+m, c',\ldots}; x\Big)$$

are two basic bilateral hypergeometric functions (of any order) and h, k, any two suitable constants then

(5.1)

$$\frac{(h+m;m)(k;m)\,\Delta_q(h+2m)}{(k+m;m)(h;m)\,\Delta_q(k+2m)}\Psi\begin{bmatrix}a,\ldots & \\ 1+m, c,\ldots & ; xq^{\lambda}\end{bmatrix}\Psi\begin{bmatrix}a',\ldots & \\ 1+m, c',\ldots & ; x\end{bmatrix}$$

$$= \sum_{r=0}^{\infty}\frac{(k-1;r)(k;2r)(k-h;r)[(h+m;r)]^2(a;r)\ldots(a';r)\ldots}{(1;r)(h;r)(k-1;2r)[(k+m;2r)]^2(c;r)\ldots(c';r)\ldots}$$

$$\times x^{2r}q^{r(r-1)+(\lambda+h)r}\,\Psi\begin{pmatrix}h+m+r, a+r,\ldots & \\ k+m+2r, 1+m, c+r,\ldots & ; xq^{\lambda}\end{pmatrix}$$

$$\Psi\begin{pmatrix}h+m+r, a'+r,\ldots & \\ k+m+2r, 1+m, c'+r,\ldots & ; x\end{pmatrix}.$$

Proof: It is easily seen that

$$\frac{\Delta_q(h)}{\Delta_q(k)}\Psi\begin{pmatrix}a,\ldots & \\ c,\ldots & ; x\end{pmatrix} = \Psi\begin{pmatrix}h, a,\ldots & \\ k, c\ \ldots & ; x\end{pmatrix}$$

and

$$q^{(\delta+m)r}\frac{(-\delta-m;r)\Delta_q(h+m)}{(\delta+m+k;r)\Delta_q(k+m)}\Psi\begin{pmatrix}a,\ldots & \\ 1+m, c\ \ldots & ; xq^{\lambda}\end{pmatrix}$$

$$= \frac{(a;r)\ldots(h+m;r)}{(c;r)\ldots(k+m;2r)}(-x)^r q^{\frac{1}{2}r(r-1)+\lambda r}$$

$$\Psi\begin{pmatrix}h+m+r, a+r,\ldots & \\ k+m+2r, 1+m, c+r,\ldots & ; xq^{\lambda}\end{pmatrix}.$$

Using the above transformation on the right-hand side of (5.1), it becomes

(5.2)

$$\frac{\Gamma_q(\theta+h+m)\Gamma_q(\phi+h+m)}{\Gamma_q(\theta+k+m)\Gamma_q(\phi+k+m)}\left[\frac{\Gamma_q(k+m)}{\Gamma_q(h+m)}\right]^2$$

$$\times \sum_{r=0}^{\infty}\frac{(k-1;r)(k;2r)(k-h;r)(-\theta-m;r)(-\phi-m;r)}{(1;r)(k-1;2r)(h;r)(\theta+k+m;r)(\phi+k+m;r)}q^{(\theta+\phi+2m+h)r}$$

$$\times \Psi\begin{pmatrix}a,\ldots & \\ 1+m, c,\ldots & ; xq^{\lambda}\end{pmatrix}\Psi\begin{pmatrix}a',\ldots & \\ 1+m, c',\ldots & ; x\end{pmatrix},$$

where $\theta \equiv x\ \partial/\partial x$ and $\phi \equiv x\ \partial/\partial x$ operate on the first and second series alone respectively and hence $q^{\theta+\phi} = q^{\delta}$. Summing up the above well-poised ${}_6\Phi_5$ series we get the required result (5.1). If we take the limit $q \to 1$ we get the corresponding transformation for ordinary bilateral hypergeometric series, viz.:

$$(5.3)\quad \frac{(h+m)_m(k)_m}{(k+m)_m(h)_m}\frac{\Delta(h+2m)}{\Delta(k+2m)} H\left[\begin{matrix} a, \ldots \\ 1+m, c, \ldots \end{matrix}; x\right] H\left[\begin{matrix} a', \ldots \\ 1+m, c', \ldots \end{matrix}; x\right]$$

$$= \sum_{r=0}^{\infty} \frac{(k-1)_r(k)_{2r}(k-h)_r((h+m)_r)^2(a)_r \ldots (a')_r \ldots}{r!(k-1)_{2r}((k+m)_{2r})^2(h)_r(c)_r \ldots (c')_r \ldots} x^{2r}$$

$$\times H\left(\begin{matrix} h+m+r, a+r, \ldots \\ 1+m, k+m+2r, c+r, \ldots \end{matrix}; x\right) H\left(\begin{matrix} h+m+r, a'+r, \ldots \\ 1+m, k+m+2r, c'+r, \ldots \end{matrix}; x\right).$$

Applications. Applying the transformation (5.1) to the identity (2.1) we get

$$(5.4)\quad {}_1\Psi_1\left(\begin{matrix} d-b+m \\ 1+m \end{matrix}; x\right) {}_2\Psi_2\left(\begin{matrix} a+c-m, b \\ 1+m, c \end{matrix}; x\, q^{-a-b+2m}\right)$$

$$= \frac{(c-b;m)(a+d-2m;m)(d-m;m)}{(d-b;m)(a+c-2m;m)(c-m;m)}$$

$$\times \sum_{r=0}^{\infty} \frac{(c-m-1;r)(c-m;2r)(c-d;r)(d;r)(c-b+m;r)(d+a-m;r)}{(1;r)(c-m-1;2r)(c;2r)(c;2r)(d-m;r)}$$

$$\times (b;r) x^{2r} q^{r(r-1)+(d-a-b+m)r}\, {}_2\Psi_2\left(\begin{matrix} d+r, c-b+m+r \\ 1+m, c+2r \end{matrix}; x\right)$$

$$\times {}_3\Psi_3\left(\begin{matrix} d+r, a+d-m+r, b+r \\ 1+m, d+r, c+2r \end{matrix}; x\, q^{-a-b+2m}\right).$$

Similarly, applying the transformation (5.1) to (2.2) and (5.3) to (4.2) and (4.3) one can obtain three other expansions of similar type.

Next, applying the transformation (5.3) for $m = 0$ (**4**, 42) to the right-hand side of the identities (3.1), (3.2), and (3.3) respectively, we get the following three expansions

$$(5.5)\quad {}_3F_2\left(\begin{matrix} a, b, c \\ 1+a-b, 1+a-c \end{matrix}; x\right)$$

$$= \sum_{r=0}^{\infty} \frac{(2b)_r(1+a-2c)_r(a-2b)_r(\tfrac{1}{2}a-b+1)_r(-b)_r(c)_r}{r!(a-c+r)_r(1+a-c)_{2r}(\tfrac{1}{2}a-b)_r(1+a-b)_r} x^{2r}$$

$$\times {}_2F_1\left(\begin{matrix} c+r, 2b+r \\ 1+a-c+2r \end{matrix}; x\right)$$

$${}_4F_3\left(\begin{matrix} a-2b+r, \tfrac{1}{2}a-b+1+r, -b+r, c+r \\ \tfrac{1}{2}a-b+r, 1+a-b+r, 1+a-c+2r \end{matrix}; x\right)$$

https://doi.org/10.4153/CJM-1960-003-4 Published online by Cambridge University Press

with the condition $1 + a - c - e = 2b$,

$$(5.6)\quad {}_4F_3\left(\begin{matrix} a, \tfrac{1}{2}a+1, b, c \\ \tfrac{1}{2}a, 1+a-b, 1+a-c \end{matrix}; x\right)$$

$$= \sum_{r=0}^{\infty} \frac{(2b)_r(1+a-2c)_r(a-2b)_r(-b)_r(c)_r}{r!(a-c+r)_r(1+a-c)_{2r}(1+a-b)_r} x^{2r}$$

$$\times\ {}_2F_1\left(\begin{matrix} c+r, 2b+r \\ 1+a-c+2r \end{matrix}; x\right) {}_3F_2\left(\begin{matrix} a-2b+r, -b+r, c+r \\ 1+a-b+r, 1+a-c+2r \end{matrix}; x\right)$$

with $1 + a - c - e = 2b$,

(5.7)

$$= \sum_{r=0}^{\infty} \frac{(2b-1)_r(1+a-2c)_r(a-2b-1)_r(\tfrac{1}{2}a+\tfrac{1}{2}-b)_r(-b-1)_r(c)_r}{r!(a-c+r)_r(1+a-c)_{2r}(\tfrac{1}{2}a-\tfrac{1}{2}-b)_r(1+a-b)_r} x^{2r}$$

$$\times\ {}_2F_1\left(\begin{matrix} c+r, 1+2b+r \\ 1+a-c+2r \end{matrix}; x\right)$$

$${}_4F_3\left(\begin{matrix} a-2b-1+r, \tfrac{1}{2}a+\tfrac{1}{2}-b+r, -b-1+r, c+r \\ \tfrac{1}{2}a-\tfrac{1}{2}-b+r, 1+a-c+2r, 1+a-b+r \end{matrix}; x\right)$$

provided $a - c - e = 2b$.

The basic analogue of (5.7) may be written as below (by using (5.1) for $m = 0$ in (3.5))

$$(5.8)\quad {}_5\Phi_4\left(\begin{matrix} q^a, q^{\frac{1}{2}a+1}, -q^{\frac{1}{2}a+1}, q^b, q^c \\ q^{\frac{1}{2}a}, -q^{\frac{1}{2}a}, q^{1+a-b}, q^{1+a-c} \end{matrix}; x\, q^{a-2b-2c}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(q^{2b-1}; r)(q^{1+a-2c}; r)(q^{a-2b-1}; r)(q^{\frac{1}{2}a+\frac{1}{2}-b}; r)}{(q; r)(q^{a-c+r}; r)(q^{1+a-c}; 2r)(q^{\frac{1}{2}a-\frac{1}{2}-b}; r)}$$

$$\times \frac{(-q^{\frac{1}{2}a+\frac{1}{2}-b}; r)(q^{-b-1}; r)(q^c; r)}{(-q^{\frac{1}{2}a-\frac{1}{2}-b}; r)(q^{1+a-b}; r)} x^{2r} q^{r^2+(e+a-2c)r}$$

$$\times\ {}_2\Phi_1\left(\begin{matrix} q^{c+r}, q^{2b+1+r} \\ q^{1+a-c+2r} \end{matrix}; x\, q^{e-c}\right)$$

$$\times\ {}_5\Phi_4\left(\begin{matrix} q^{a-2b-1+r}, q^{\frac{1}{2}a+\frac{1}{2}-b+r}, -q^{\frac{1}{2}a-b+r+\frac{1}{2}}, q^{-b-1+r}, q^{c+r} \\ q^{\frac{1}{2}a-\frac{1}{2}-b+r}, -q^{\frac{1}{2}a-\frac{1}{2}-b+r}, q^{1+a-b+r}, q^{1+a-c+2r} \end{matrix}; xq^{1+a-2c}\right)$$

with $a - c - e = 2b$.

Similar expansions could also be obtained from results due to Shukla (**8**, vii) and Henrici (**5**, $a \sim b$, $a \sim c$).

I am grateful to Dr. R. P. Agarwal for his kind help and guidance during the preparation of this paper.

References

1. N. Agarwal, J. Lond. Math. Soc. *34* (1959), 37–46.
2. W. N. Bailey, *Generalized hypergeometric series* (Cambridge Tract, 1935).
3. ——— J. Lond. Math. Soc., *22* (1947), 237–40.
4. T. W. Chaundy, *Proc. Lond. Math. Soc.* (*2*), *50* (1949), 56–74.
5. P. Henrici, Pacific J. Math., *5* (1955), 923–31.
6. M. Jackson, J. Lond. Math. Soc., *27* (1952), 116–23.
7. F. H. Jackson, Amer. J. Math., *32* (1910), 305–14.
8. H. S. Shukla, Quart. J. Math. (Oxford), *10* (1959), 48–59.

Lucknow University