ON P-INJECTIVE RINGS

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1. Definitions and preliminary results. Throughout this paper R will be an associative ring with unity and all R-modules are unitary. The right (resp. left) annihilator in R of a subset X of a module is denoted by $\mathbf{r}(X)$ (resp. $\mathbf{l}(X)$). The Jacobson radical of R is denoted by $J(R)$, the singular ideals are denoted by $Z(R_R)$ and $Z({}_RR)$ and the socles by $\mathrm{Soc}(R_R)$ and $\mathrm{Soc}({}_RR)$. For a module M, $E(M)$ and $\mathrm{PE}(M)$ denote the injective and pure-injective envelopes of M, respectively. For a submodule $A \subseteq M$, the notation $A \subseteq^{\oplus} M$ will mean that A is a direct summand of M.

A module M_R is called *p-injective* if for every $a \in R$, every R-linear map from aR to M can be extended to an R-linear map from R to M. R is called *right p-injective* if R_R is p-injective. Recall that a module M_R is called *uniserial* if its submodules are linearly ordered by inclusion and *serial* if it is a direct sum of uniserial submodules. A ring R is *right uniserial* (*serial*) if R_R is uniserial (serial).

We record some well-known results on serial and p-injective rings.

Lemma 1.1 [**5**, **6**]. *Let R be any ring.*

(1) *R is right p-injective if and only if $\mathbf{l}(\mathbf{r}(a)) = Ra$ for every $a \in R$.*

(2) *If R is right p-injective then $J(R) = Z(R_R)$.*

(3) *If R is left uniserial then R is right p-injective if and only if $J(R) = Z(R_R)$.*

(4) *If R is right p-injective and $A, B_1, \ldots, B_n$ are two-sided ideals of R then*

$$A \cap (B_1 \oplus \ldots \oplus B_n) = (A \cap B_1) \oplus \ldots \oplus (A \cap B_n).$$

Lemma 1.2 [**11**, p. 200, Theorem 3.3]. *Let R be a serial ring, P a finitely generated projective R-module, and M a finitely generated submodule of P. Then there is a decomposition $P = P_1 \oplus \ldots \oplus P_n$ with indecomposables P_i such that*

$$M = (M \cap P_1) \oplus \ldots \oplus (M \cap P_n).$$

The next two statements are proved using model theory for modules.

Lemma 1.3 [**3**]. *Let R be an arbitrary ring and M a finitely presented module over R. Then $\mathrm{PE}(M)$ is indecomposable if and only if M has a local endomorphism ring.*

Lemma 1.4 [**7**]. *Let R be a serial ring and M a pure-injective indecomposable module over R. Then either M is injective or, for every primitive idempotent $e \in R$ and every nonzero element $m \in Me$, there exists an element $r \in R$ such that $m \in E(M)re$ and $m \notin Mre$.*

Lemma 1.5 [**5**, Corollary 2.2, Theorem 2.3]. *Let R be a semiperfect right p-injective ring with $\mathrm{Soc}(R_R)$ essential as a right ideal in R. Then $\mathrm{Soc}(R_R) = \mathrm{Soc}({}_RR)$ is essential as a left ideal and $Z(R_R) = J(R) = Z({}_RR)$.*

Recall that a right R-module M is called *fp-injective* if every R-linear map from a finitely generated submodule of a free R-module F to M can be extended to an R-linear map from F to M. Evidently every fp-injective module is p-injective and the converse is true for some classes of rings including serial rings, see [**8**]. In the serial ring case we give a short proof of this fact using the above cited Warfield's result.

Lemma 1.6. *Every right p-injective module M over a serial ring R is fp-injective.*

Proof. Let N be a finitely generated submodule of a free module P of finite rank and f a homomorphism from N into M. In view of Lemma 1.2 we may assume that N is a finitely generated submodule of an indecomposable projective module eR for some primitive idempotent $e \in R$. Since eR is uniserial, it follows that N is cyclic. Now the existence of the desired extension follows from p-injectivity of M.

2. Serial p-injective rings. Now we formulate our criteria for serial rings to be right p-injective.

Theorem 2.1. *For a serial ring R with a complete set of primitive orthogonal idempotents $\{e_1, \ldots, e_n\}$ the following conditions are equivalent:*

(a) *R is right p-injective;*

(b) *R is right fp-injective;*

(c) $J(R) = Z(R_R)$;

(d) *for any pair of indices $i, j \leq n$ and any $r \in R$ with $0 \neq e_i r e_j \in J(Re_j)$ there exist $s \in R$ and $k \leq n$, such that $e_j s e_k \neq 0$ and $e_i r e_j s e_k = 0$.*

Proof. The equivalence between (a) and (b) follows from Lemma 1.6 and the implication (b)$\Rightarrow$(c) follows from Lemma 1.1.

(c)$\Rightarrow$(d). If $0 \neq e_i r e_j \in J(Re_j)$ then $e_i r e_j \in J(R) = Z(R_R)$, hence $\mathbf{r}(e_i r e_j)$ is essential in R_R and $\mathbf{r}(e_i r e_j) \cap e_j R \neq 0$. It follows that $e_i r e_j s = 0$ for some nonzero $e_j s \in e_j R$. Since $e_j R$ is uniserial and $e_j s R = e_j s e_1 R + \ldots + e_j s e_n R$ we obtain $e_j s R = e_j s e_k R$ for some k and $e_j s e_k$ is the desired element.

(d)$\Rightarrow$(a). Suppose that R_R is not p-injective. Then $e_j R$ is not p-injective as a right R-module for some j. Let M be the pure-injective envelope of $e_j R$. Since $e_j R$ has a local (in fact uniserial) endomorphism ring it follows from Lemma 1.3 that M is an indecomposable pure-injective module. Now if M is injective, it will follow that $e_j R$ is *fp*-injective since it is a pure submodule of M, a contradiction. By Lemma 1.4, applied to the element $e_j \in Me_j$, we can find an element $r \in R$ such that $e_j \in E(M) r e_j$ and $e_j \notin Mre_j$. If $re_j \notin J(Re_j)$ then $tre_j = e_j$ for some $t \in R$. Now, $e_j t \in e_j R \subseteq M$ implies $e_j = e_j t \,.\, re_j \in Mre_j$, a contradiction. Hence we may assume $re_j \in J(Re_j)$. Since $Re_i re_j = Rre_j$, for some i, it follows $e_i r e_j \in J(Re_j)$ and hence by assumption $e_i r e_j s e_k = 0$ for some k and some $s \in R$.

Since $e_j \in E(M) r e_j$ we obtain $e_j = mre_j$ for some $m \in E(M)$. Multiplying this equality by $e_j s e_k$ from the right side we obtain $e_j s e_k = mre_j \,.\, e_j s e_k = 0$, a contradiction.

Corollary 2.2. *Let R be a serial right p-injective ring with essential right socle. Then R is left p-injective with essential left socle.*

Proof. From Lemma 1.5 we obtain $Z(R_R) = J(R) = Z({}_R R)$ and the socle of R is essential in ${}_R R$. From Theorem 2.1 it follows that R is left p-injective.

Example 2.3. *Let F be an arbitrary field and consider the ring*

$$R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}.$$

Then R is a (*two-sided*) *serial artinian ring which is neither left nor right p-injective.*

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Proof. We check this for the right side only. We have $e_{12} \in J(Re_2) \cap e_1Re_2$ and $e_{12}s \neq 0$ for every nonzero element $s \in e_2R$ which contradicts (d) of Theorem 2.1.

Next we provide an example of a ring R which is a right uniserial right artinian right duo left p-injective ring which is neither right p-injective nor left uniform. Also every non-invertible element of R has an essential left and right annihilator. Recall that a ring R is *right duo* if every right ideal of R is two-sided.

EXAMPLE 2.4. *Let K be a field and $K(x)$ the field of rational functions over K. Let α be an endomorphism of $K(x)$ which sends x to x^2. Clearly the image of α is $K(x^2)$. Let R be a matrix ring of the form*

$$\left\{\begin{bmatrix} \alpha(a) & b \\ 0 & a \end{bmatrix} : a, b \in K(x)\right\}.$$

Clearly

$$\begin{bmatrix} 0 & K(x) \\ 0 & 0 \end{bmatrix}$$

is the unique non-trivial right ideal of R. If we view $K(x)$ as a vector space over $K(x^2)$ then every proper left ideal of R has the form

$$\begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix},$$

where V is a subspace of $K(x)$. It is easy to check that for every $a \in J$, the Jacobson radical of R, $\mathbf{r}(a) = \mathbf{l}(a) = J$. Clearly R is right artinian right uniserial right duo and not left uniserial. It follows from Lemma 1.1 *that R is left p-injective and not right p-injective.*

3. Semiperfect p-injective rings. In this section we show that semiperfect right p-injective right duo rings are right continuous. Recall that a module M_R is called *continuous* if it satisfies the following two conditions: (C1) every submodule of M is essential in a direct summand, and (C2) if A and B are submodules of M with $A \cong B$ and $B \subseteq^{\oplus} M$ then $A \subseteq^{\oplus} M$.

In [**5**, Theorem 1.2], it was shown that if R_R is right p-injective then R_R satisfies the C2-condition. In particular, if A and B are right ideals of R with $A \subseteq^{\oplus} R_R$, $B \subseteq^{\oplus} R_R$ and $A \cap B = 0$ then $A \oplus B \subseteq^{\oplus} R_R$. If R is right duo we have the following more general result which is of independent interest.

THEOREM 3.1. *Let R be a right p-injective right duo ring. If A and B are right ideals of R with $A \subseteq^{\oplus} R_R$ and $B \subseteq^{\oplus} R_R$ then $(A \cap B) \subseteq^{\oplus} R_R$ and $(A + B) \subseteq^{\oplus} R_R$.*

Proof. Write $R = A \oplus A_1 = B \oplus B_1$ for some right ideals A_1 and B_1 of R. By Lemma 1.1, $B = B \cap (A \oplus A_1) = (B \cap A) \oplus (B \cap A_1)$. Hence

$$R = (B \cap A) \oplus (B \cap A_1) \oplus B_1$$

and so $(A \cap B) \subseteq^{\oplus} R_R$. Also

$$A + B = A + ((B \cap A) \oplus (B \cap A_1)) = (A + (B \cap A)) \oplus (B \cap A_1) = A \oplus (B \cap A_1).$$

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Since both A and $(B \cap A_1)$ are summands of R_R, it follows from the remark preceding the theorem that $A \oplus (B \cap A_1)$ is a summand of R_R and so $A + B$ is also a summand of R_R.

LEMMA 3.2. *Let R be a local right p-injective ring. Then for any non-zero (two-sided) ideals I and J of R, $I \cap J \neq 0$.*

Proof. Suppose that $I \cap J = 0$ and let $0 \neq u \in I$, $0 \neq v \in J$. Define the map

$$\varphi : (u+v)R \to R, \qquad (u+v)r \mapsto ur.$$

Clearly φ is a well defined R-homomorphism. By right p-injectivity, φ is given by left multiplication by an element $t \in R$. Hence $t(u+v) = u$, and so $(1-t)u = tv = 0$. Since R is a local ring it follows that $u = 0$ or $v = 0$, a contradiction.

COROLLARY 3.3. *Suppose R is a local right p-injective right duo ring. Then R is right uniform.*

REMARK 3.4. Note that without the condition "right duo" the above result is not true. The ring R given in Example 2.4 is a local left p-injective ring which is not left uniform.

THEOREM 3.5. *Suppose R is a semiperfect right duo right p-injective ring. Then R is right continuous.*

Proof. By Corollary 3.3, clearly R is a direct sum of local right uniform rings R_i. By [**5**, Theorem 1.2], any right p-injective ring satisfies the C2-condition. We only need to show that R_R satisfies the C1-condition. Let A be a non-zero right ideal of R and write $R = R_1 \oplus \ldots \oplus R_n$. By Lemma 1.1, without loss of generality we may write $A = (A \cap R_1) \oplus \ldots \oplus (A \cap R_k)$, for some $k \leq n$ with $A \cap R_i \neq 0$, $1 \leq i \leq k$. Since each $A \cap R_i$ is essential as a right ideal in R_i, $1 \leq i \leq k$, it follows that A_R is essential in $R_1 \oplus \ldots \oplus R_k \subseteq^{\oplus} R_R$.

REMARK 3.6. Note that the ring R given in Example 2.4 is a left p-injective right artinian ring which is not left finite dimensional. Hence R can not be left continuous.

4. Completely p-injective rings. A ring R is called *completely right p-injective* (*right cp-injective*) if every ring homomorphic image of R is right p-injective. R is called *cp-injective* if it is both left and right cp-injective. In this section, for right duo rings, we give a characterization for serial rings with nil Jacobson radical in terms of cp-injectivity. Recall that a module M is said to be *distributive* if its lattice of submodules is distributive: for all $A, B, C \subset M$, $A \cap (B + C) = A \cap B + A \cap C$.

THEOREM 4.1. *Let R be a right cp-injective ring. Then the lattice of two-sided ideals of R is distributive.*

Proof. Suppose the lattice of two-sided ideals of R is a non-distributive (modular) lattice. It follows from [**2**, Theorem 2] that it contains a minimal non-distributive modular sublattice consisting of five elements. Hence we can find three noncomparable two-sided ideals I, J and K in R such that $I \cap J = I \cap K = J \cap K$ and $I + J = I + K = J + K$. Then factorizing by the common intersection we may suppose that all these sums are direct and all these intersections are zero. Now by Lemma 1.1 it follows that $0 \neq I = I \cap (J \oplus K) = (I \cap J) \oplus (I \cap K) = 0$, a contradiction.

COROLLARY 4.2. *Every right duo right cp-injective ring is right and left distributive.*

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Proof. The right distributivity follows from the above theorem and we can apply the following result from [**9**, Corollary 2.10]: every right distributive right p-injective ring is left distributive.

Recall that a ring R is *strongly regular* if for every $a \in R$ there exists $b \in R$ such that $a = ba^2$.

LEMMA 4.3. *For a ring R the following are equivalent*:

(a) *R is strongly regular*;

(b) *R is right p-injective with no non-zero nilpotent elements*;

(c) *R is a semiprime right p-injective right duo ring.*

Proof. (a)$\Rightarrow$(b), (c) is standard.

(c)$\Rightarrow$(a). We adopt the argument given in Example 6 of [**5**]. Let $a \in R$ and set $T = aR \cap \mathbf{r}(a)$. Then clearly T is a two-sided ideal of R with $T^2 = 0$. Since R is semiprime, $T = 0$ and hence $\mathbf{r}(a^2) = \mathbf{r}(a)$. By Lemma 1.1 we get $Ra = Ra^2$ and hence R is (strongly) regular.

(b)$\Rightarrow$(a). Note that in rings without non-zero nilpotent elements for every $a \in R$, $\mathbf{r}(a) = \mathbf{l}(a)$. Now the same argument as before applies

REMARK 4.4. More results of the type given in Lemma 4.3 may be found in some of Yue Chi Ming's work on p-injectivity (e.g. [**14**]).

A ring R is *π-regular* if every descending chain of the form $aR \supseteq a^2R \supseteq \ldots$ becomes stationary.

LEMMA 4.5. *Let R be right duo and right cp-injective. Then R is π-regular.*

Proof. Let $a \in R$ and consider the following ascending chain of right annihilators $\mathbf{r}(a) \subseteq \mathbf{r}(a^2) \subseteq \ldots$. Let $I = \bigcup_{i=1}^{\infty} \mathbf{r}(a^i)$ and consider the ring $\bar{R} = R/I$. Clearly $\mathbf{r}_{\bar{R}}(\bar{a}) = \bar{0}$ and hence it follows from Lemma 1.1 that $\bar{R}\bar{a} = \bar{R}$. So $1 - sa \in \mathbf{r}(a^m)$ for some $s \in R$ and $m > 0$. Since R is right duo there exists $t \in R$ such that $sa = at$ and hence $a^m = a^{m+1}t$ from which we infer that R is π-regular.

THEOREM 4.6. *For a right duo ring R the following conditions are equivalent*:

(a) *R is right cp-injective with no infinite set of orthogonal idempotents*;

(b) *R is cp-injective with no infinite set of orthogonal idempotents*;

(c) *R is a finite direct sum of (two-sided) uniserial rings with nil Jacobson radical.*

Proof. (a)$\Rightarrow$(c). By Lemma 4.5, R is π-regular and hence $J(R)$ is a nil ideal and so idempotents can be lifted modulo $J(R)$. By assumption and Lemma 4.3, it follows that $R/J(R)$ is semisimple artinian and hence R is semiperfect. Hence $R = R_1 \oplus \ldots \oplus R_n$ where each R_i is a local ring which is left and right distributive by Corollary 4.2. Since local right distributive rings are right uniserial we are done.

(c)$\Rightarrow$(b). We may assume that R is uniserial with nil radical J. Let I be any (two-sided) ideal of R and consider the ring $\bar{R} = R/I$. Clearly, every element of $J(\bar{R})$ has a nonzero left and right annihilator. Hence by [**6**, Lemma 1], $\bar{R}$ is right and left p-injective.

(b)$\Rightarrow$(a) is trivial.

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Notice that any von Neumann regular ring which is not right noetherian is cp-injective with an infinite set of orthogonal idempotents.

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