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POSITIVE LINEAR OPERATORS AND THE APPROXIMATION OF CONTINUOUS FUNCTIONS ON LOCALLY COMPACT ABELIAN GROUPS

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Abstract

In 1953 P. P. Korovkin proved that if (T_n) is a sequence of positive linear operators defined on the space C of continuous real 2π-periodic functions and $\lim_{n\to\infty} T_n f = f$ uniformly for $f = 1$, cos and sin, then $\lim_{n\to\infty} T_n f = f$ uniformly for all $f \in C$. We extend this result to spaces of continuous functions defined on a locally compact abelian group G, with the test family $\{1, \cos, \sin\}$ replaced by a set of generators of the character group of G.

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Throughout X will denote a nonempty set, $B(X)$ the space of bounded complex-valued functions on X, and $A(X)$ a subalgebra of $B(X)$ containing the constant functions and closed under complex conjugation.

A linear operator T from $A(X)$ into $B(X)$ will be called positive if $Tf \geqslant 0$ whenever $f \geqslant 0$. It is easy to show that such an operator takes real functions into real functions, and $|Tf| \leqslant Tg$ whenever $f, g \in A(X)$ and $|f| \leqslant g$. In particular T is continuous with $\|T\| = \|T1\|$, where 1 denotes the constant function with value 1 and $B(X)$ and its subspaces are given the uniform norm $\|\cdot\|$.

It is of considerable interest in approximation theory to examine the convergence properties of a sequence (T_n) of positive linear operators. Fundamental to this are the results that deduce from the convergence of $T_n g$ to g ($g \in S(X)$) that of $T_n f$ to f for all $f \in A(X)$, where $S(X)$ is a (small) subset of $A(X)$. We refer to $S(X)$ as a test family for $A(X)$ (and (T_n)); see Freud (1964). One of the earliest results of this type was that of P. P. Korovkin who showed that $\{1, \cos, \sin\}$ is a test family for the space of continuous 2π-periodic real functions; see Korovkin (1960), Theorem 4, p. 17.

In this paper we extend Korovkin's result to where the underlying space is an arbitrary Hausdorff locally compact abelian group G. The subalgebras of $B(G)$ that will be of interest are $T(G)$ and $C(G)$, the spaces of respectively trigonometric polynomials and bounded uniformly continuous functions. The former is just the algebra generated by the character group Γ of G. Subsets of Γ generating the group will serve as test families for $T(G)$ and $C(G)$.

Our first result is quite general, giving conditions under which the convergence of a net (T_ρ) of positive linear operators on two functions can be extended to their product. Here and elsewhere in the paper, unless otherwise stated, convergence will be understood to be in the uniform topology.

THEOREM 1. *Let (T_ρ) be a net of positive linear operators from $A(X)$ into $B(X)$, $f, g \in A(X)$, and suppose that $T_\rho h \to h$ for $h = 1, f, g, |f|^2$ and $|g|^2$. Then $T_\rho \bar{f} \to \bar{f}$ and $T_\rho(fg) \to fg$.*

PROOF. The positivity of T_ρ gives immediately that $T_\rho \bar{f} = (T_\rho f)^-$, and this proves the first assertion.

For each $t \in X$ define $\phi_t \in A(X)$ by

$$\phi_t = |f - f(t)|^2 + |g - g(t)|^2.$$

Then

$$\begin{aligned} T_\rho \phi_t = T_\rho |f|^2 + |f(t)|^2 T_\rho 1 + T_\rho |g|^2 + |g(t)|^2 T_\rho 1 \\ - f(t) T_\rho \bar{f} - \bar{f}(t) T_\rho f - g(t) T_\rho \bar{g} - \bar{g}(t) T_\rho g \end{aligned}$$

and the assumptions on (T_ρ) show that $\lim T_\rho \phi_t(t) = 0$, the convergence being uniform in t. Choose $\varepsilon > 0$ and write $\delta = \varepsilon^2(\|f\| + \|g\|)^{-2}$, where we assume that f, g are not identically zero. Then, for $t \in X$,

$$\begin{aligned} |fg - (fg)(t)| &\leqslant \|f\| \, |g - g(t)| + \|g\| \, |f - f(t)| \\ &\leqslant (\|f\| + \|g\|) \phi_t^{\frac{1}{2}} \\ &= \varepsilon(\delta^{-1} \phi_t)^{\frac{1}{2}} \\ &\leqslant \varepsilon(1 + \delta^{-1} \phi_t). \end{aligned}$$

Hence, appealing to the remarks in the Introduction,

$$|T_\rho(fg) - (fg)(t) T_\rho 1| \leqslant \varepsilon(T_\rho 1 + \delta^{-1} T_\rho \phi_t),$$

and the result follows.

Since characters have modulus 1, Theorem 1 gives immediately:

COROLLARY 1. *Let (T_ρ) be a net of positive linear operators from $T(G)$ into $B(G)$ and suppose that $S = \{\gamma \in \Gamma : T_\rho \gamma \to \gamma\}$ contains the identity character. Then S is a subgroup of Γ.*

Our analogue of Korovkin's theorem for compact abelian groups is:

THEOREM 2. *Suppose G is compact and let (T_ρ) be a net of positive linear operators from $C(G)$ into $B(G)$. Then any set S of generators of Γ containing the identity is a test family for $C(G)$.*

PROOF. Assume $T_\rho \gamma \to \gamma$ for all $\gamma \in S$. Since S generates Γ and contains the identity, Corollary 1 gives that the convergence holds for all $\gamma \in \Gamma$. Now $\|T_\rho\| = \|T_\rho 1\| \to \|1\| = 1$, so that the T_ρ are eventually uniformly bounded. As $T(G)$ is dense in $C(G)$ it follows that $T_\rho f \to f$ for all $f \in C(G)$.

For noncompact groups we restrict attention to positive convolution operators T given by $Tf = \mu * f \ (f \in C(G))$, where μ is a positive bounded Radon measure and $\mu * f(x) = \int f(xy^{-1})\, d\mu(y)$ for all $x \in G$.

THEOREM 3. *Let (T_n) be a sequence of positive convolution operators from $C(G)$ into $B(G)$. Then any set S of generators of Γ containing the identity is a test family for $C(G)$.*

PROOF. As in the compact case we have that $\mu_n * \gamma = T_n \gamma \to \gamma$, which implies that $\hat{\mu}_n(\gamma) \to 1$, for all $\gamma \in \Gamma$. It follows from Lévy's continuity theorem (Heyer (1977), Theorem 1.4.2) that the sequence (μ_n) converges weakly to the Dirac measure δ_1 placed at the identity, that is $\int f\, d\mu_n \to f(1)$ for all $f \in C(G)$. We now appeal to Heyer (1977), 1.2.20(2), to deduce that (μ_n) is a tight sequence of measures; in fact for our case the proof shows even more, namely given any $\varepsilon > 0$ and compact neighbourhood V of the identity in G there exists a positive integer n_0 such that $\|\mu_n\| - \mu_n(V) \leqslant \varepsilon$ for all $n \geqslant n_0$.

We now show that for $f \in C(G)$, $\mu_n * f \to f$. Let $\varepsilon \in (0, 1)$ be given and, without loss of generality, assume that $\|f\| = 1$. Since f is uniformly continuous there exists a compact neighbourhood V of the identity in G such that

$$|f(xy^{-1}) - f(x)| < \varepsilon/6$$

for all $x \in G$, $y \in V$. By the results of the previous paragraph we have the existence of n_0 such that $\|\mu_n\| - \mu_n(V) \leqslant \varepsilon/6$ for all $n \geqslant n_0$. We may also assume that $|\|\mu_n\| - 1| < \varepsilon/3$ for the same range of n, using the properties that $\hat{\mu}_n(1) \to 1$ and $\mu_n \geqslant 0$. Then, for $x \in G$, $n \geqslant n_0$,

$$\begin{aligned}
|\mu_n * f(x) - f(x)| &= \left| \int \{f(xy^{-1}) - f(x)\}\, d\mu_n(y) + f(x)\{\|\mu_n\| - 1\} \right| \\
&< \int |f(xy^{-1}) - f(x)|\, d\mu_n(y) + \varepsilon/3 \\
&\leqslant \int_V |f(xy^{-1}) - f(x)|\, d\mu_n(y) \\
&\qquad\qquad + 2\|f\|\{\|\mu_n\| - \mu_n(V)\} + \varepsilon/3 \\
&\leqslant \|\mu_n\|\varepsilon/6 + \varepsilon/3 + \varepsilon/3 < \varepsilon,
\end{aligned}$$

giving the uniform convergence of $\mu_n * f$ to f, and the theorem follows.

There is no possibility of extending Theorem 3 to nets, as we now show using an example of Kendall and Lamperti (1970). Let G be any noncompact group and take a compact symmetric neighbourhood V of the identity. For each finite subset F of Γ and $\varepsilon \in (0, 1)$ we shall define a probability measure $\mu_{F,\varepsilon}$. The function $f = \sum_{\gamma \in F} \gamma$ is bounded, continuous and almost periodic, and satisfies $f(1) = n$, where n is the cardinality of F. It follows from Hewitt and Ross (1963), (18.1), that there exists $y \notin V$ such that $|n - f(y)| < \varepsilon^2/2$, and hence $|1 - \gamma(y)| < \varepsilon$ for all $\gamma \in F$. Now put $\mu_{F,\varepsilon} = \delta_y$, the Dirac measure placed at y. The above construction shows that $|1 - \hat{\mu}_{F,\varepsilon}(\gamma)| < \varepsilon$ for all $\gamma \in F$, so that $(\mu_{F,\varepsilon})$ is a net of probability measures with Fourier transforms converging pointwise to 1. However, it is clear that if $h \in C(G)$ is supported inside V and $h(1) \neq 0$ then

$$\mu_{F,\varepsilon} * h(1) = \int h(x^{-1})\, d\mu_{F,\varepsilon}(x) = 0,$$

so that the corresponding convolution operators $(T_{F,\varepsilon})$ cannot converge on $C(G)$ to the identity.

If the requirement that S generate Γ be dropped then we can no longer deduce that S is a test family for $C(G)$. In fact if G is compact and S is any subgroup of Γ then there exists a net (T_ρ) of positive linear operators from $C(G)$ into $B(G)$ such that $T_\rho \gamma \to \gamma$ *for all* $\gamma \in S$ and $T_\rho \gamma \to 0$ for $\gamma \notin S$. To see this consider the compact abelian group $G/A(G, S)$, which has character group isomorphic to S (Hewitt and Ross (1963), (23.25), (24.10)); here $A(G, S)$ denotes the annihilator of S in G. By Hewitt and Ross (1970), (28.53), $L^1\{G/A(G, S)\}$ admits a bounded approximate unit (k'_ρ) of trigonometric polynomials. Now define k_ρ on G by $k_\rho = k'_\rho \circ \pi$, where π is the natural homomorphism of G onto $G/A(G, S)$. Then k_ρ is a trigonometric polynomial on G, $\operatorname{supp} \hat{k}_\rho \subset S$ and $\hat{k}_\rho$, $\hat{k}'_\rho$ agree on S. It follows that (k_ρ) is a net of trigonometric polynomials on G satisfying $\hat{k}_\rho(\gamma) \to 1$ for all $\gamma \in S$ and $\hat{k}_\rho(\gamma) = 0$ for $\gamma \notin S$, and the corresponding convolution operators satisfy the assertion above.

For noncompact G we may not even have the convergence of (T_ρ) on the closure of S (Γ has the topology of uniform convergence on compact subsets of G). Indeed consider the sequence (γ_n) of characters of the real line $\mathbf{R}$ given by $\gamma_n(x) = \exp(ia_n x)$, where $x \in \mathbf{R}$ and $a_n = 2^n(2^n - 1)^{-1}$. Clearly $\gamma_n \to \gamma$ in $\Gamma_{\mathbf{R}}$, where $\gamma(x) = \exp(ix)$. Define (k_n) by

$$k_n(x) = \begin{cases} n/2, & x \in [\alpha_n - 1/n, \alpha_n + 1/n], \\ 0, & \text{otherwise}, \end{cases}$$

where $\alpha_n = \pi \prod_{i=1}^{n} (2^i - 1)$. Note that $\gamma(\alpha_n) = -1$ for $n = 1, 2, \ldots$. Furthermore, for $n \geqslant m$,

$$\gamma_m(\alpha_n) = \exp\left[i2^m(2^m - 1)^{-1} \pi \prod_{i=1}^{n} (2^i - 1) \right] = 1.$$

Thus $\hat{k}_n(\gamma_m) \to 1$ for all m but $\hat{k}_n(\gamma) \to -1$.

However we do have the following result.

THEOREM 4. *Let (T_ρ) be a net of uniformly bounded positive linear operators from $C(G)$ into $B(G)$ with the following property. For each $x \in G$ there exists a compact set $K_x \subset G$ such that*

(i) *for all ρ, $T_\rho f(x) = 0$ whenever $K_x \cap \operatorname{supp} f$ is empty.*

(ii) *$\cup \{K_x \colon x \in J\}$ is compact whenever J is.*

Suppose that $S = \{\gamma \in \Gamma \colon T_\rho \gamma \xrightarrow{c} \gamma\}$, where c denotes uniform convergence on compact subsets of G. Then S is closed.

PROOF. Let $H \subset G$ be any compact set, V a compact neighbourhood of the identity, and put

$$F = H \cup \bigcup \{K_x V \colon x \in H\}.$$

Using (ii) above we have that F is compact. Assume that S is nonempty and consider χ in the closure of S. Take $\varepsilon > 0$ and choose $\gamma \in S$ satisfying

$$\sup \{ |\chi(x) - \gamma(x)| \colon x \in F\} < \varepsilon/3t,$$

where $t = \sup\{\|T_\rho\|, 1\}$, and then choose ρ_0 such that for $\rho \geqslant \rho_0$,

$$\sup \{ |T_\rho \gamma(x) - \gamma(x)| \colon x \in F\} < \varepsilon/3.$$

We shall estimate the term $\sup\{|T_\rho \chi(x) - T_\rho \gamma(x)| \colon x \in H\}$, and the result will follow from the inequality

$$|T_\rho \chi(x) - \chi(x)| \leqslant |T_\rho \chi(x) - T_\rho \gamma(x)| + |T_\rho \gamma(x) - \gamma(x)| + |\gamma(x) - \chi(x)|.$$

First we show that for any continuous function f,

$$|T_\rho f(x)| \leqslant \|T_\rho\| \sup\{|f(x)| \colon x \in F\}$$

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holds for all $x \in H$. Let g denote the constant function taking the value $\sup\{|f(x)|: x \in F\}$. Then $|f| \leqslant g$ on F, so that the negative part $(g-|f|)^-$ is supported inside the complement of K_x for each $x \in H$. It follows from (i) above that

$$T_\rho(g-|f|)(x) = T_\rho(g-|f|)^+(x) \geqslant 0,$$

and

$$|T_\rho f(x)| \leqslant T_\rho|f|(x) \leqslant T_\rho g(x) \leqslant \|T_\rho\| \sup\{|f(x)|: x \in F\},$$

the inequality holding for all $x \in H$.

Applying this to the continuous function $\chi - \gamma$ gives

$$\sup\{|T_\rho \chi(x) - T_\rho \gamma(x)|: x \in H\} \leqslant \|T_\rho\| \sup\{|\chi(x)-\gamma(x)|: x \in F\} < \varepsilon/3,$$

as required.

COROLLARY 2. *Let (μ_ρ) be a net of probability measures on G and suppose there is a compact set $K \subset G$ such that* $\operatorname{supp} \mu_\rho \subset K$ *for all ρ. Then $S = \{\gamma \in \Gamma: \hat{\mu}_\rho(\gamma) \to 1\}$ is a closed subgroup of Γ.*

PROOF. Consider the positive linear operator T_ρ on $C(G)$ given by $T_\rho f = \mu_\rho * f$. Then $T_\rho \gamma = \hat{\mu}_\rho(\gamma)\gamma$ and Corollary 1 gives that S is a subgroup of Γ. To show that S is closed, observe that (T_ρ) satisfies the conditions of Theorem 4. Indeed uniform boundedness is clear, and (i) and (ii) are satisfied with $K_x = xK^{-1}$ since $\cup\{xK^{-1}: x \in J\} = JK^{-1}$ is compact whenever J is. Thus $S = \{\gamma \in \Gamma: \mu_\rho * \gamma \xrightarrow{c} \gamma\}$ is closed.

REMARK. The requirement on the supports of the μ_ρ can be weakened by demanding only that (μ_ρ) be a tight net of probability measures. This would involve a corresponding change in Theorem 4.

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