

GROUPS IN WHICH ALL LARGE SUBGROUPS HAVE BOUNDED NEAR DEFECT

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Abstract

If G is a group with subgroup H and m, k are two fixed nonnegative integers, H is called an (m, k) -subnormal subgroup of G if it has index at most m in a subnormal subgroup of G of defect less than or equal to k . We study the behaviour of uncountable groups of cardinality \aleph where all subgroups of cardinality \aleph are (m, k) -subnormal.

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1. Introduction

If G is a group, H is a subgroup of G and (m, k) is a pair of nonnegative integers, following [7], H is said to be (m, k) -subnormal in G if there is a subgroup H_0 containing H such that $|H_0 : H| \leq m$ and H_0 is subnormal in G with defect at most k . The pairs (m, k) are ordered lexicographically; with this ordering, if H is (m, k) -subnormal for some pair (m, k) , then the least such pair is called the *near defect* of H in G (see [9]). Lennox [9, 10] called a subgroup *almost subnormal* if it is (m, k) -subnormal for certain nonnegative integers m, k . Clearly, a subgroup H is $(m, 1)$ -subnormal in G if and only if it has index at most m in its normal closure H^G , while H is $(1, k)$ -subnormal if and only if it is subnormal of defect at most k .

It follows from theorems of Neumann [12] and Macdonald [11] that if a group admits only $(m, 1)$ -subnormal subgroups for a fixed positive integer m , then its commutator subgroup is finite of order bounded by a function of m . On the other hand, a well-known theorem of Roseblade [16] asserts that there exists a function f such that if G is a group in which every subgroup is $(1, k)$ -subnormal for a fixed positive integer k , then G is nilpotent of class bounded by $f(k)$.

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Lennox [9] generalised Roseblade's theorem, proving that there exists a function μ such that if every finitely generated subgroup of a group G is (m, k) -subnormal in G , then the term $\gamma_{\mu(m+k)}(G)$ of the lower central series of G is finite of order less than or equal to $m!$.

A group G is said to have finite rank if there is a positive integer r such that every finitely generated subgroup of G can be generated by at most r elements; if such an r does not exist, we say that the group G has infinite rank. In a long series of papers, it has been shown that the structure of a (soluble) group of infinite rank is strongly influenced by that of its proper subgroups of infinite rank; in particular, Evans and Kim [8] studied locally soluble groups of infinite rank in which all subgroups of infinite rank are subnormal with bounded defect, while in [7] locally soluble groups of infinite rank whose subgroups of infinite rank are (m, k) -subnormal were considered. The results obtained in that context suggest that the behaviour of small subgroups in a large group can be neglected, at least for the right choice of the definition of largeness (and within a suitable universe); for instance, in [3] the authors consider uncountable groups, of cardinality \aleph say, in which all subgroups of cardinality \aleph are normal or subnormal. In a series of subsequent papers (see [4–6]), similar problems were considered, by replacing normality with some of its relevant generalisations. The main obstacle in the study of groups of large cardinality \aleph is the existence of infinite groups in which all proper subgroups have cardinality strictly smaller than \aleph (the so-called *Jónsson groups*). Clearly, a countably infinite group is a Jónsson group if and only if all its proper subgroups are finite, so that Prüfer groups and Tarski groups (that is, infinite simple groups whose proper nontrivial subgroups have prime order) have the Jónsson property. On the other hand, it is known that if G is an uncountable Jónsson group, then $G/Z(G)$ is simple and $G' = G$ (see, for instance, [3, Corollary 2.6]). Relevant examples of Jónsson groups of cardinality \aleph_1 have been constructed by Shelah [17] and Obraztsov [13].

If G is an uncountable group, a subgroup H of G will be called *large* if it has the same cardinality as G and *small* otherwise. The aim of the paper is to make a further contribution to the study of the influence of the behaviour of large subgroups on the structure of a group, by considering uncountable groups whose large subgroups are (m, k) -subnormal for a fixed pair (m, k) .

Most of our notation is standard and can be found in [15].

2. Statements and proofs

Throughout this paper, \aleph will denote a fixed uncountable regular cardinal number and, for nonnegative integers (m, k) , we define the group class $\mathfrak{X}(m, k)$ to be the class of all groups of cardinality \aleph (and the trivial groups) whose large subgroups are (m, k) -subnormal in G . We begin this section by showing that the Baer radical of an $\mathfrak{X}(m, k)$ -group has all large subgroups subnormal with bounded defect.

LEMMA 2.1. *Let G be a locally nilpotent group. Then every (m, k) -subnormal subgroup of G is subnormal in G of defect at most $m + k$.*

PROOF. Let H be an (m, k) -subnormal subgroup of G , so that there exists a subnormal subgroup X of G of defect at most k such that $|X : H| \leq m$. If $N = H_X$ is the normal core of H in X , then X/N is a finite nilpotent group, so H/N is subnormal in X/N with defect at most m . Hence, H is a subnormal subgroup of G of defect at most $m + k$. \square

COROLLARY 2.2. *Let G be a locally nilpotent $\aleph(m, k)$ -group. If G contains a large abelian subgroup, then G is nilpotent of class at most $f(m + k)$.*

PROOF. The assertion is a consequence of Lemma 2.1 and [3, Theorem 2.5]. \square

COROLLARY 2.3. *Let G be an $\aleph(m, k)$ -group containing a large abelian subgroup. Then the Baer radical of G is large and nilpotent of class at most $f(m + k)$.*

PROOF. Let A be a large abelian subgroup; then A is (m, k) -subnormal in G , so that there exists a subnormal subgroup C of defect at most k such that $|C : A| \leq m$. It follows that A_C , the normal core of A in C , is a large abelian subnormal subgroup of G . Thus, B is a locally nilpotent $\aleph(m, k)$ -group containing a large abelian subgroup. The statement is now a consequence of Corollary 2.2. \square

PROPOSITION 2.4. *Let G be a group of cardinality \aleph . If there exists a positive integer r such that $\gamma_r(G)$ is small, then for every finitely generated subgroup F of G , the centraliser $C_G(F)$ is a large subgroup of G .*

PROOF. Let s be the least positive integer such that $\gamma_{s+1}(G)$ is small and let g be any element of G ; if we put

$$g^{\gamma_s(G)} = \{g^h \mid h \in \gamma_s(G)\},$$

the function

$$g^h \in g^{\gamma_s(G)} \longrightarrow [g, h] \in [g, \gamma_s(G)] \leq \gamma_{s+1}(G)$$

is a bijection and, therefore, the set $g^{\gamma_s(G)}$ has cardinality strictly smaller than \aleph . However,

$$|g^{\gamma_s(G)}| = |\gamma_s(G) : C_{\gamma_s(G)}(g)|$$

where $C_{\gamma_s(G)}(g)$ is the centraliser in $\gamma_s(G)$ of g .

Put $F = \langle g_1, \dots, g_t \rangle$; then $|\gamma_s(G) : C_{\gamma_s(G)}(g_i)|$ is strictly smaller than \aleph , for each $i \in \{1, \dots, t\}$. Moreover,

$$C_G(F) = \bigcap_{i=1}^t C_G(g_i),$$

so that $|\gamma_s(G) : C_{\gamma_s(G)}(F)|$ is strictly smaller than \aleph , and hence $C_G(F)$ is a large subgroup of G . \square

Phillips [14] introduced the following generalisation of subnormality for infinite groups: a subgroup H of a group G is called f -subnormal in G if there exists an f -series

from H to G , that is, a finite series

$$H = H_0 \leq H_1 \leq \dots \leq H_n = G$$

such that either H_i is normal in H_{i+1} or $|H_{i+1} : H_i| < \infty$ for every $i \in \{0, \dots, n - 1\}$.

Clearly, every (m, k) -subnormal subgroup is f -subnormal.

COROLLARY 2.5. *Let G be an $\mathfrak{X}(m, k)$ -group. If there exists a positive integer r such that $\gamma_r(G)$ is small, then every finitely generated subgroup F of G is f -subnormal in G .*

PROOF. It follows from Proposition 2.4 that $C_G(F)F$ is an (m, k) -subnormal subgroup of G in which F is normal. □

LEMMA 2.6. *Let G be an $\mathfrak{X}(m, k)$ -group. If G is abelian-by-cyclic, then it is finite-by-nilpotent and $|\gamma_{\mu(m+k)}(G)| \leq (m!)^2$.*

PROOF. We can write $G = A\langle x \rangle$, where A is a normal abelian subgroup of G and x is an element of G . Of course, A is large and it can be regarded as a module over the countable ring $R = \mathbb{Z}\langle x \rangle$. Moreover, the R -submodules of A are all its G -invariant subgroups. By [3, Lemma 2.3], A contains the direct product

$$\text{Dr}_{i \in I} M_i,$$

where I is a set of cardinality \aleph and, for each $i \in I$, M_i is a G -invariant subgroup of A . It follows that there exist two subsets I_1 and I_2 of I , each of cardinality \aleph , such that

$$I_1 \cap I_2 = \emptyset.$$

For $j = 1, 2$, put

$$B_j = \text{Dr}_{i \in I_j} M_i,$$

then B_1 and B_2 are G -invariant subgroups of A of cardinality \aleph whose intersection is trivial. Thus, [9, Theorem A] ensures that, for $j = 1, 2$,

$$|\gamma_{\mu(m+k)}(G/B_j)| \leq m!.$$

Therefore, $|\gamma_{\mu(m+k)}(G)| \leq (m!)^2$ and G is finite-by-nilpotent. □

LEMMA 2.7. *Let G be an $\mathfrak{X}(m, k)$ -group containing a large abelian normal subgroup. Then G is locally finite-by-nilpotent, and if B is the Baer radical of G , then G/B has finite exponent.*

PROOF. Let x be any element of G and let A be a large abelian normal subgroup of G ; the subgroup $\langle A, x \rangle = A\langle x \rangle$ is an abelian-by-cyclic $\mathfrak{X}(m, k)$ -group so, by Lemma 2.6, $|\gamma_{\mu(m+k)}(A\langle x \rangle)| \leq (m!)^2$. Therefore, $\langle x \rangle$ has index at most $(m!)^2$ in $\langle x \rangle \gamma_{\mu(m+k)}(A\langle x \rangle)$. Since $A\langle x \rangle / \gamma_{\mu(m+k)}(A\langle x \rangle)$ is nilpotent of class at most $\mu(m+k) - 1$, the subgroup $\langle x \rangle \gamma_{\mu(m+k)}(A\langle x \rangle)$ is subnormal in $A\langle x \rangle$ with defect at most $\mu(m+k) - 1$. Finally, $A\langle x \rangle$ is (m, k) -subnormal in G and so it has index at most m in $(A\langle x \rangle)^{G,k}$ which is subnormal in G with defect at most k .

By an application of [9, Theorem B], G is locally finite-by-nilpotent.

Let X be a cyclic subgroup of G . By the above argument, there exists a chain

$$X = X_n \leq \dots \leq X_0 = G$$

such that either X_{i+1} is subnormal in X_i with defect at most $\mu(m+k) - 1$ or X_{i+1} has index at most $(m!)^2$ in X_i ; we shall prove by induction on the length of the chain that for every cyclic subgroup X of G , there is a positive integer s dividing $\beta(m) = ((m!)^2)!$ such that X^s is subnormal in G .

We can assume that X^s is subnormal in X_1 , for some positive integer s dividing $\beta(m)$. If X_1 is subnormal in G , then X^s is subnormal in G . So we can assume that the index $|G : X_1| \leq (m!)^2$; then the core Y of X_1 in G has index at most $((m!)^2)! = \beta(m)$ in G . Hence, $X^{\beta(m)}$ is a normal subgroup of $Y \cap X^s$, which is subnormal in Y . It follows that $X^{\beta(m)}$ is subnormal in G , and hence $X^{\beta(m)} \leq B$.

Therefore, G/B has finite exponent. □

The following result depends on [1, Theorem 1’]. Recalling that an abelian normal subgroup A of a group G can be regarded as a module over the group ring $\mathbb{Z}(G/C_G(A))$, we obtain the following interesting lemma.

LEMMA 2.8. *Let G be a group containing an uncountable abelian normal subgroup A . If $G/C_G(A)$ is finite, then A contains a subgroup of the form*

$$D_I \langle a_i \rangle^G,$$

where the set I has the same cardinality as A .

LEMMA 2.9. *Let G be an $\mathfrak{X}(m, k)$ -group. If G contains a large nilpotent normal subgroup N such that G/N is locally finite, then $\gamma_{\mu(m^2+k)}(G)$ is small.*

PROOF. Since N is nilpotent of cardinality \aleph , there exists a nonnegative integer r such that $|Z_r(N)| < \aleph$ and $|Z_{r+1}(N)| = \aleph$. So, the section $\bar{A} = Z_{r+1}(N)/Z_r(N)$ is an abelian normal subgroup of $\bar{G} = G/Z_r(N)$. Let \bar{E} be any finitely generated subgroup of \bar{G} and put $\bar{L} = \bar{E}\bar{A}$. Since $\bar{N} \cap \bar{L} \leq C_{\bar{L}}(\bar{A})$, then $\bar{L}/C_{\bar{L}}(\bar{A})$ is finite, so that Lemma 2.8 ensures that \bar{A} contains a subgroup \bar{H} of the form

$$\bar{H} = D_I \bar{A}_i$$

where I is a set of cardinality \aleph and, for each $i \in I$, \bar{A}_i is a nontrivial normal subgroup of \bar{L} . So we can find two \bar{L} -invariant subgroups \bar{H}_1 and \bar{H}_2 of \bar{H} , both of cardinality \aleph and such that

$$\bar{H}_1 \cap \bar{H}_2 = \bar{E} \cap \bar{H}_1 \bar{H}_2 = \{1\}.$$

Therefore,

$$\bar{E} = \overline{E\bar{H}_1} \cap \overline{E\bar{H}_2}$$

is the intersection of two (m, k) -subnormal subgroups of \bar{G} and, by [7, Lemma 2.5], \bar{E} is (m^2, k) -subnormal. An application of [9, Theorem A] completes the proof. □

We are now able to state and prove our first main theorem.

THEOREM 2.10. *Let G be an $\mathfrak{X}(m, k)$ -group. If G contains a large abelian normal subgroup, then G is finite-by-nilpotent and $|\gamma_{\mu(m^2+k)}(G)| \leq (m^2)!$.*

PROOF. Let B be the Baer radical of G ; it follows from Corollary 2.3 and Lemma 2.7 that B is nilpotent and G/B is a locally finite group, so that $\gamma_{\mu(m+k)}(G)$ is small by Lemma 2.9. Let F be any finitely generated subgroup of G . Applying Proposition 2.4 to the group AF , we can say that $C_{AF}(F)$ is large, so that also $C_{AF}(F) \cap A$ is a large subgroup. It follows that $Z(AF)$ is large, and hence it contains two large subgroups U_1 and U_2 such that

$$U_1 \cap U_2 = F \cap U_1 U_2 = \{1\}.$$

Therefore,

$$F = F U_1 \cap F U_2$$

is the intersection of two (m, k) -subnormal subgroups of G and, by [7, Lemma 2.5], F is (m^2, k) -subnormal. The statement follows from [9, Theorem A]. \square

The last part of this section is devoted to our second main theorem concerning $\mathfrak{X}(m, k)$ -groups which contain a large abelian subgroup not necessarily normal.

LEMMA 2.11. *Let G be an $\mathfrak{X}(m, k)$ -group. If G contains a large abelian subgroup, then the Baer radical of G is a finite-index subgroup.*

PROOF. Let B be the Baer radical of G . By Corollary 2.3, B is a large nilpotent subgroup of G , so that the section B/B' has cardinality \aleph . Hence, G/B' is an $\mathfrak{X}(m, k)$ -group and contains the large abelian normal subgroup B/B' , so that, as a consequence of Theorem 2.10,

$$|\gamma_{\mu(m^2+k)}(G/B')| \leq (m^2)!$$

By Hall's theorem, $|G/B' : Z_{2\mu(m^2+k)}(G/B')|$ is finite so, with $Z/B' = Z_{2\mu(m^2+k)}(G/B')$, also $|G : ZB|$ is finite. Clearly, ZB/B' is nilpotent by Fitting's theorem, and so the nilpotency of the normal subgroup B of ZB ensures that ZB is a nilpotent normal subgroup of G . It follows that $Z \leq B$ and the statement is proved. \square

THEOREM 2.12. *Let G be an $\mathfrak{X}(m, k)$ -group. If G contains a large abelian subgroup A , then G is finite-by-nilpotent and $\gamma_{\mu(m^2+k)}(G)$ is small.*

PROOF. If B is the Baer radical of G , then B is a large and nilpotent subgroup of G by Corollary 2.3. Since G/B is finite by Lemma 2.11, as a consequence of Lemma 2.9, $\gamma_{\mu(m^2+k)}(G)$ is small.

In addition, Lemma 2.11 ensures the existence of a finitely generated subgroup F such that $G = FB$. By Corollary 2.5, F is f -subnormal in G , so that there exists an f -series of the form

$$F = F_0 < F_1 < \dots < F_n = G,$$

where either $|F_{i+1} : F_i|$ is finite or F_i is normal in F_{i+1} , for each $i \in \{0, \dots, n - 1\}$.

We will prove that G is finite-by-nilpotent by induction on the length n of the f -series above. If $n = 1$, since G is an uncountable group, F is normal in G . On the other hand, [2, Theorem 1.1] tells us that F is also finite-by-nilpotent so we can find a positive integer r such that the G -invariant subgroup $\gamma_r(F)$ is finite, and

$$G/\gamma_r(F) = BF/\gamma_r(F) = (F/\gamma_r(F)) \cdot (B\gamma_r(F)/\gamma_r(F))$$

is nilpotent by Fitting's theorem.

Assume now that $n > 1$. Then $|F_{n-1} : F_{n-1} \cap B|$ is finite and $B \cap F_{n-1}$ is contained in the Baer radical of F_{n-1} , so we can apply the inductive hypothesis to F_{n-1} which has to be a finite-by-nilpotent group. It follows that there exists a finite characteristic subgroup M of F_{n-1} such that F_{n-1}/M is nilpotent (see [2, Lemma 2.9]). If F_{n-1} is normal in G , also M is G -invariant and

$$G/M = (F_{n-1}/M) \cdot (BM/M)$$

is nilpotent by Fitting's theorem. Assume that $|G : F_{n-1}|$ is finite; then M has finitely many conjugates in G , so that M^G is finite by Dietzmann's lemma. On the other hand,

$$G/M^G = (F_{n-1}M^G/M^G) \cdot (BM^G/M^G)$$

is finite-by-(locally nilpotent) (see [2, Lemma 2.7]), so that G itself is finite-by-(locally nilpotent), and hence actually finite-by-nilpotent by Corollary 2.2. \square

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