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Computing the effectively computable bound in Baker's inequality for linear forms in logarithms

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For certain number theoretical applications, it is useful to actually compute the effectively computable constant which appears in Baker's inequality for linear forms in logarithms. In this note, we carry out such a detailed computation, obtaining bounds which are the best known and, in some respects, the best possible. We show *inter alia* that if the algebraic numbers $\alpha_1, \ldots, \alpha_n$ all lie in an algebraic number field of degree D and satisfy a certain independence condition, then for some $n_0(D)$ which is explicitly computed, the inequalities (in the standard notation)

$$0 < |b_1 \log \alpha_1 + \ldots + b_n \log \alpha_n|$$
$$< \exp\left(-(n+1)^{\left(1+\log(n+1)^{-\frac{1}{2}}\right)(n+1)} \Omega \log \Omega' \log B\right)$$

have no solution in rational integers $b_1, \ldots, b_n$ $(b_n \neq 0)$ of absolute value at most B, whenever $n \geq n_0(D)$. The very favourable dependence on n is particularly useful.

1. Introduction

It is our purpose in the present work to describe the detailed computation of the lower bound in various versions of Baker's inequality

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for linear forms in logarithms. We have attempted to make no prejudgement as to the relative sizes of the quantities involved, so that our results remain quite general but, in effect, are best suited to certain applications in elementary number theory where one wants an optimal dependence on n, the number of logarithms. When n is small, there is now an alternative technique due to Cijsouw and Waldschmidt [8], which avoids the extrapolations necessary to the present argument and is almost surely sharper than is our present bound as regards the absolute constant appearing therein. Moreover, our work is incomplete to the extent that our results are stated subject to a certain independence condition on the logarithms; however, in the immediate applications we have in mind, the independence condition is automatically satisfied by the data. Subject then to the above qualifications, our results are nevertheless the best known and, for some of the variables, best possible in each of these variables. Of particular interest is the dependence on n and the quite good dependence on the heights of the algebraic numbers in each of our results. There is no suggestion that our absolute constants are in any way optimal, but they are nevertheless strikingly small.

In an attempt to make our work more transparent to the reader, we have indicated explicitly those points in our proof at which we become committed to a growth in the parameter k, and thence to a growth in the eventual constant C. Nevertheless, the work forces one to a number of pre-commitments which, even in hindsight, are difficult to coherently explain.

Throughout this work, we denote by $\alpha_1, \ldots, \alpha_n$ non-zero algebraic numbers of heights respectively not exceeding $A_1, \ldots, A_n$ (with $\log\log A_j \geq 1$). We suppose that we have $A_1 \leq A_2 \leq \ldots \leq A_n$ and sometimes write $A_1 \leq \ldots \leq A_{n-1} \leq A'$ and $A_n = A$. Throughout, we set

$$\Omega' = (\log A_1) \ldots (\log A_{n-1}) \quad \text{and} \quad \Omega = \Omega' \log A_n .$$

We require, of course, that $n \geq 2$. Furthermore, we denote by K the field $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ and we write $[K : \mathbb{Q}] = D$; the case $D = 1$ is admissible. Finally, we denote by B a rational integer with, say, $B \geq e^2$. (However, generally, our results are quite trivial unless B is substantially larger.)

In order to conveniently state our results it is useful to introduce real numbers μ, κ , and ε , satisfying

$$2/(n+1) \le \mu \le 2 , \quad 0 < \kappa \le \tfrac{1}{2}\mu , \quad \varepsilon = (\mu-\kappa)/(1+\mu)(1+\kappa)(n+1) .$$

Furthermore, we denote by k a constant satisfying

(1) $$k \ge \max\{(3^4D)^{(1+1/\kappa)(n+1)}, 13^{1/\varepsilon}, (10D/\varepsilon)^{(1+\mu)(n+1)}\} .$$

We prove the following results.

THEOREM 1. *Suppose there is a prime* q *satisfying* $13 \le q \le k\varepsilon$ *such that* $\left[K\left(\alpha_1^{1/q}, \ldots, \alpha_n^{1/q}\right) : K\right] = q^n$. *Then the inequalities*

$$0 < |b_1 \log \alpha_1 + \ldots + b_n \log \alpha_n| < \exp\left(-k^{1+\mu}\Omega' \log \Omega' \log A \log B\right)$$

have no solutions in rational integers $b_1, \ldots, b_n$ $(b_n \ne 0)$ *with absolute values at most* B .

By choosing $\mu = 1/3(\log(n+1))^{\frac{1}{2}}$ and $\kappa = \frac{1}{2}\mu$, one sees easily that for n sufficiently large relative to D , for example $\log(n+1) > (6 \log 3^4D)^2$, we have the corollary:

COROLLARY 1. *Suppose there is a prime* q *satisfying* $13 \le q \le \exp\left(\frac{1}{8}(\log(n+1))^{\frac{1}{2}}\right)$ *such that* $\left[K\left(\alpha_1^{1/q}, \ldots, \alpha_n^{1/q}\right) : K\right] = q^n$. *If* $n > n_0(D)$ *for some explicitly computable* $n_0(D)$ *, then the inequalities*

$$0 < |b_1 \log \alpha_1 + \ldots + b_n \log \alpha_n|$$
$$< \exp\left(-(n+1)^{\left(1+\log(n+1)^{-\frac{1}{2}}\right)(n+1)}\Omega' \log \Omega' \log A \log B\right)$$

have no solution in rational integers $b_1, \ldots, b_n$ $(b_n \ne 0)$ *with absolute values at most* B .

On the other hand, choosing $\mu = 2$ and $\kappa = \frac{1}{2}$, one can obtain the corollary:

COROLLARY 2. *Suppose there is a prime* q *satisfying* $13 \le q \le 2^5D(n+1)$ *such that* $\left[K\left(\alpha_1^{1/q}, \ldots, \alpha_n^{1/q}\right) : K\right] = q^n$. *Then the inequalities*

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$$0 < |b_1 \log \alpha_1 + \ldots + b_n \log \alpha_n|$$
$$< \exp\left(-\left(2^5 D(n+1)\right)^{9(n+1)} \Omega' \log \Omega' \log A \log B\right)$$

have no solution in rational integers $b_1, \ldots, b_n$ $(b_n \neq 0)$ *with absolute values at most* B .

The choice of parameters in the proof of Theorem 1 is made with a view to getting the best result of the form of Corollary 1. By making some minor changes to the argument, we can improve the constants in Corollary 2 and, in particular, we can lower the exponent $9(n+1)$ to $6n + 8$. By the argument of Baker [6] (note the minor corrections mentioned in [11]), one can get rid of the independence condition so as to immediately obtain the following result.

COROLLARY 3. *There is an effectively computable constant* $C = C(n, D) > 0$, *dependent only on* n *and* D , *such that the inequalities*

$$0 < |b_1 \log \alpha_1 + \ldots + b_n \log \alpha_n| < \exp(-C\Omega' \log \Omega' \log A \log B)$$

have no solution in rational integers $b_1, \ldots, b_n$ $(b_n \neq 0)$ *with absolute values at most* B .

This is Theorem 2 of [11] which is announced by analogy in [11]; the present argument thus confirms that assertion. Our result is sharper in A than that of Baker [6], which has $\Omega \log \Omega$ rather than our $\Omega' \log \Omega' \log A$, and is considerably sharper in A' than is that of Baker [2], which is indefinite in A' , or that of Tijdeman [16], Theorem 2, which has $(\log A')^{2n^2+7n}$ rather than our $\Omega' \log \Omega' < (\log A')^n$. The dependence on A separately is best possible; this dependence was of course achieved earlier by Fel'dman [10] and Baker [2]. It may be useful to remark in passing that our bound, which has $\Omega' \log \Omega' \log A \log B$, is strictly sharper than the bound announced by Čudnovskiĭ [9], namely with $\Omega(\log B)^2$, in that the latter bound is weaker than the trivial result (see [1], Lemma 6) if $B < \Omega'$ and is implied by our result if $B \geq \Omega'$.

The dependence on n and on D , for example in Corollary 2, is as good as has been achieved, whilst the bound in Corollary 1 is quite striking in applications when the α_j are distinct primes, so that

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$D = 1$, and n is large. However, these results are all qualified by the independence condition. Recently, Baker has announced a more general result [7], namely:

The inequalities

$$0 < |\beta_0+\beta_1 \log \alpha_1 + \dots + \beta_n \log \alpha_n|$$
$$< \exp\left(-(16nD)^{200n}\Omega' \log \Omega' \log A(\log B + \log \Omega)\right)$$

have no solutions in algebraic numbers $\beta_0, \dots, \beta_n$ $(\beta_n \neq 0)$ *belonging to the field* K *and of heights respectively not exceeding* B .

This last result sharpens and generalises a similar result of Shorey [14]. We are indebted to certain ideas of [14] in the present argument.

THEOREM 2. *Suppose there is a prime* q *satisfying* $13 \le q \le k^{\varepsilon}$ *such that* $\left[K\left(\alpha_1^{1/q}, \dots, \alpha_n^{1/q}\right) : K\right] = q^n$. *Let* $\delta > 0$ *and write* $C = k^{1+\mu}$ *and* $h = \left[\log\left(B'\delta^{-1}C\Omega' \log \Omega'\right)\right]$. *Then for any* δ *with* $0 < \delta < Ch\Omega' \log \Omega'$, *the inequalities*

$$0 < |b_1 \log \alpha_1 + \dots + b_n \log \alpha_n|$$
$$< \min\{\exp(-Ch\Omega' \log \Omega' \log A), \exp(-\delta B/B')\}$$

have no solution in rational integers $b_1, \dots, b_{n-1}$ *and* $b_n \neq 0$ *with absolute values at most* B *and* B' *respectively.*

There are a number of useful corollaries of Theorem 2. Firstly, we can transform the result to the shape of Theorem 1 of Baker [4].

COROLLARY 4. *Suppose there is a prime* q *satisfying* $13 \le q \le k^{\varepsilon}$ *such that* $\left[K\left(\alpha_1^{1/q}, \dots, \alpha_n^{1/q}\right) : K\right] = q^n$. *Then for any* δ *with* $0 < \delta \le \frac{1}{2}$, *say, the inequalities*

$$0 < |b_1 \log \alpha_1 + \dots + b_n \log \alpha_n| < (\delta/B')^{C(\log C)\Omega'(\log\Omega')^2\log A} \cdot e^{-\delta B/B'}$$

have no solution in rational integers $b_1, \dots, b_{n-1}$ *and* $b_n \neq 0$ *with absolute values at most* B *and* B' *respectively.*

Secondly, we readily obtain the following explicit form of Theorem 2

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of [4].

THEOREM 3. *Suppose there is a prime* q *satisfying* $13 \le q \le k^{\varepsilon}$ *such that* $\left[K\left(\alpha_1^{1/q}, \ldots, \alpha_n^{1/q}\right) : K\right] = q^n$. *Write* $C = k^{1+\mu}$. *If for some* $\delta > 0$, *there exist rational integers* $b_1, \ldots, b_{n-1}$ *with absolute values at most* B *such that*

$$0 < |b_1 \log \alpha_1 + \ldots + b_{n-1} \log \alpha_{n-1} - \log \alpha_n| < e^{-\delta B} ,$$

then $B < \left(\delta^{-1} C\Omega' \log \Omega'\right)\left(\log\left(\delta^{-1} C\Omega' \log \Omega'\right)\right) \log A$ *or* $B < \log A$ *according as* δ *is less than or equal to, or greater than,* $(C\Omega' \log \Omega')\left(\log\left(\delta^{-1} C\Omega' \log \Omega'\right)\right)$.

This assertion is immediate for $\delta \le Ch\Omega' \log \Omega'$, whilst the case $\delta > Ch\Omega' \log \Omega'$ is a weaker claim. One readily writes down analogues of Corollaries 1 and 2 above.

We obtain the above results by following the argument of Baker [4]. Incidentally, by taking $\delta = B'B^{-1}C\Omega' \log \Omega'$ and $B' \le B$, one sees that Theorem 2 implies Theorem 1. Subject to the independence condition, the above results make explicit the dependence on n, D , and Ω' in Baker's results in [4]. For comparison, we quote an earlier explicit result of Baker [1], in which there is also no undetermined constant.

Suppose $\deg \alpha_j \le d$ $(1 \le j \le n)$ *and take* $d \ge 4$. *If rational integers* $b_1, \ldots, b_n$ *exist with absolute values at most* B , *such that* $0 < \delta \le 1$ *and*

$$|b_1 \log \alpha_1 + \ldots + b_n \log \alpha_n| < e^{-\delta B} ,$$

then either all the b_j *vanish, or* $B < \left(4^{n^2} \delta^{-1} d^{2n} \log A\right)^{(2n+1)^2}$.

Whilst it would obviously be desirable to remove the independence conditions from the present theorems, doing so by the method of [6] has the preliminary requirement that one have certain weaker results which act as a base for an inductive argument. It is a tedious and lengthy process to construct such results, though there do not appear to be any essential difficulties involved in eventually retaining much the same constants as in the present results. On the other hand, the obtaining of any sharpening of the present results, for example the removing of the $\log \Omega'$ factor, or the improving of the dependence on n, appears to present intractable difficulties. It would seem that a significant breakthrough will be required to effect bounds essentially sharper than those of the present theorems. Of course, it is known that if $\alpha_1, \ldots, \alpha_n$ are all very close to 1, one can obtain very much sharper bounds (*cf.* Shorey [13]).

The first author has been able to prove the p-adic analogues of the present results and in so doing has obtained a particularly favourable dependence on the prime ideal involved ([12]). It may be useful to remark that some ideas culled from the p-adic situation have motivated aspects of the present argument.

Finally, we should remark on the significance of Tijdeman's Lemma (Lemma 1, below) which, perhaps surprisingly, is quite critical to the sharpness of the results we manage to obtain.

2. Proof of Theorem 1

Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers of heights respectively not exceeding $A_1, \ldots, A_n$ (with $\log\log A_j \geq 1$) and set $\Omega' = (\log A_1) \ldots (\log A_{n-1})$ and $\Omega = \Omega' \log A_n$. We denote by K the field $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ and write $D = [K : \mathbb{Q}]$. As in Section 1, we introduce parameters μ, κ, and ε satisfying

$$2/(n+1) \leq \mu \leq 2 , \quad 0 < \kappa \leq \tfrac{1}{2}\mu , \quad \varepsilon = (\mu-\kappa)/(1+\mu)(1+\kappa)(n+1) ,$$

and we choose further constants k and C such that

$$k \geq \max\left\{(3^4 D)^{(1+1/\kappa)(n+1)},\ 6^{1/\varepsilon},\ (10D/\varepsilon)^{(1+\mu)(n+1)}\right\}$$

and

$$C = k^{1+\mu} .$$

Finally, we write

$$\sigma = 1/(1+\kappa)(n+1) ,$$

$$L_{-1} + 1 = h = [\log B] ,$$

$$L_0 + 1 = [\tfrac{1}{8}k^{1-\sigma}\Omega] ,$$

and

$$L_j = \left[\frac{1}{8n} k^{1-\sigma}\Omega \log \Omega'/\log A_j\right] \quad (1 \le j \le n) .$$

We shall suppose that $b_1, \ldots, b_n$ $(b_n \ne 0)$ are rational integers with absolute values at most B (with $B \ge e^2$) such that

(2) $$0 < |b_1 \log \alpha_1 + \ldots + b_n \log \alpha_n| < \exp(-Ch\Omega \log \Omega') ,$$

and we proceed to show that if there is a prime q satisfying $13 \le q \le k^{\varepsilon}$ such that $\left[K\left(\alpha_1^{1/q}, \ldots, \alpha_n^{1/q}\right) : K\right] = q^n$, then our supposition leads to a contradiction.

For any integer $h \ge 1$, we denote by $\nu(h)$ the least common multiple of $1, 2, \ldots, h$. We write

$$\Delta(x; h) = (x+1)(x+2) \ldots (x+h)/h! , \quad \Delta(x; 0) = 1 ,$$

and further, for any integers $\lambda \ge 0$, $m \ge 0$, we denote by $\Delta(x; h, \lambda, m)$ the m-th derivative with respect to x of $(\Delta(x; h))^{\lambda}/m!$.

LEMMA 1. *Let q and qx be positive integers. Then*

$$q^{2h\lambda}(\nu(h))^m \Delta(x; h, \lambda, m)$$

is a positive integer and we have

$$\Delta(x; h, \lambda, m) \le 4^{\lambda(x+h)} \quad \textit{and} \quad \nu(h) \le 4^h .$$

Proof. This is Lemma T1 of Tijdeman [*16*].

LEMMA 2. *There are integers $p(\lambda) = p(\lambda_{-1}, \ldots, \lambda_n)$, not all 0, with absolute values at most $\exp\left(3^{-1}hk\Omega \log \Omega'\right)$ such that, for all integers l with $1 \le l \le 16Dh$ and all non-negative integers $m_0, \ldots, m_{n-1}$ satisfying $m_0 + \ldots + m_{n-1} \le k\Omega \log \Omega'$, we have*

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(3) $g(l;\, m) = g(l;\, m_0,\, \ldots,\, m_{n-1})$

$$= \sum_{\lambda_{-1}=0}^{L_{-1}} \sum_{\lambda_0=0}^{L_0} \cdots \sum_{\lambda_n=0}^{L_n} p(\lambda)\Lambda(l;\, m)\alpha_1^{\lambda_1 l} \cdots \alpha_n^{\lambda_n l} = 0 ,$$

where

$$\Lambda(z;\, m) = \Delta(z+\lambda_{-1};\, h,\, \lambda_0+1,\, m_0) \prod_{r=1}^{n-1} \Delta(b_n\lambda_r - b_r\lambda_n;\, m_r) .$$

Proof. Let $a_1, \ldots, a_n$ denote the leading coefficients (supposed positive) in the minimal defining polynomials of $\alpha_1, \ldots, \alpha_n$ respectively. For any non-negative integer j, we have

$$(a_r\alpha_r)^j = \sum_{s=0}^{D-1} a_{rs}^{(j)}\alpha_r^s ,$$

where the $a_{rs}^{(j)}$ denote rational integers with absolute values at most $(2A_r)^j$ $(1 \le r \le n)$. Hence, on multiplying (3) by $\left(a_1^{L_1} \cdots a_n^{L_n}\right)^l$, we obtain

$$\sum_{s_1=0}^{D-1} \cdots \sum_{s_n=0}^{D-1} V(s)\alpha_1^{s_1} \cdots \alpha_n^{s_n} = 0 ,$$

where

$$V(s) = \sum_{\lambda_{-1}=0}^{L_{-1}} \cdots \sum_{\lambda_n=0}^{L_n} p(\lambda)\Lambda(l;\, m) \prod_{r=1}^{n} \left\{a_r^{(L_r-\lambda_r)l}\, a_{r,s_r}^{(\lambda_r l)}\right\} .$$

Hence the conditions (3) are satisfied if the D^n equations $V(s) = 0$ hold for all choices of l and m. These represent

$$M \le 16hD^{n+1}\binom{k\Omega\log\Omega'+n}{n} < 16e^nD^{n+1}hk^n(\Omega \log \Omega')^n n^{-n}$$

linear equations in the

$$N = (L_{-1}+1) \cdots (L_n+1) \ge \tfrac{1}{8}hk^{n+\kappa/(\kappa+1)}(\Omega \log \Omega')^n(8n)^{-n}$$

unknowns $p(\lambda)$. Further, Lemma 1 shows that after multiplying by

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$(\nu(h))^{m_0}$, the coefficients in these equations will be rational integers. We have

$$(\nu(h))^{m_0}\Delta(l+\lambda_{-1};\ h,\ \lambda_0+1,\ m_0) \le 4^{hm_0+(L_0+1)(l+h+h)} ,$$

$$\prod_{r=1}^{n-1} |\Delta(b_n\lambda_r-b_r\lambda_n;\ m_r)| \le \prod_{r=1}^{n-1} B^{m_r}\Delta(\lambda_r+\lambda_n;\ m_r)$$

$$\le \prod_{r=1}^{n-1} (2B)^{m_r}4^{L_r} \le e^{h(m_1+\ldots+m_{n-1})}2^{m_1+\ldots+m_{n-1}}4^{L_1+\ldots+L_{n-1}} ,$$

$$\prod_{r=1}^{n} \left|a_r^{(L_r-\lambda_r)l}a_{r,s_r}^{(\lambda_r l)}\right| \le \prod_{r=1}^{n} (2A_r)^{L_r l} \le \exp\left(\tfrac{1}{8}lk^{1-\sigma}\Omega \log \Omega'\right)2^{l(L_1+\ldots+L_n)} .$$

We can therefore conclude that these rational integer coefficients have absolute values at most

$$U \le 5^{hk\Omega\log\Omega'}\exp\left(\tfrac{1}{3}lk^{1-\sigma}\Omega \log \Omega'\right) .$$

Provided that, say,

$$k \ge \left(3^4D\right)^{(1+1/\kappa)(n+1)} ,$$

we have $N \ge (1+7)M$, whence by the box principle (see, for example, [5], p. 13), the system of equations $V(s) = 0$ can be solved non-trivially and the integers $p(\lambda)$ can be chosen to have absolute values at most

$$(2NU)^{1/7} \le \exp(\tfrac{1}{3}hk\Omega \log \Omega') .$$

Our efforts will be directed at proving the following inductive step:

Let q *denote a positive integer satisfying* $13 \le q \le k^{\varepsilon}$. *Then for each integer* $J = 0, 1, 2, \ldots$ *with* $q^J < (8n)^{-1}k^{1-(\sigma-\varepsilon)}\Omega' \log \Omega'$, *there exist integers* $p^{(J)}(\lambda_{-1}, \ldots, \lambda_n)$ *not all* 0 *, with absolute values at most* $\exp\left(3^{-1}hk\Omega \log \Omega'\right)$ *, such that*

$$g_{(J)}(l;\ m_0,\ \ldots,\ m_{n-1}) =$$

$$\sum_{\lambda_{-1}=0}^{L_{-1}^{(J)}} \cdots \sum_{\lambda_n=0}^{L_n^{(J)}} p^{(J)}(\lambda)\Lambda\left(l/q^J;\ m_0,\ \ldots,\ m_{n-1}\right)\alpha_1^{\lambda_1 l} \cdots \alpha_n^{\lambda_n l} = 0$$

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for all integers l *with* $1 \le l \le 16q^J Dh$ *and all non-negative integers* $m_0, \ldots, m_{n-1}$ *with* $m_0 + \ldots + m_{n-1} \le q^{-J} k\Omega \log \Omega'$ *, where* $L_{-1}^{(J)} = L_{-1}$ *,* $L_0^{(J)} = L_0$ *, and* $L_j^{(J)} \le q^{-J} L_j$ *for* $1 \le j \le n$ *.*

Of course, Lemma 2 is the case $J = 0$ and accordingly we shall assume the above proposition to have been proved for $J = 0, 1, \ldots, N$. It is a matter of notational convenience that we suppress the affixes and suffixes (N) that should appear in our subsequent discussion.

LEMMA 3. *For all non-negative integers* $m_0, \ldots, m_{n-1}$ *with* $m_0 + \ldots + m_{n-1} \le q^{-N} k\Omega \log \Omega'$ *, let*

$$f(z; m) = f(z; m_0, \ldots, m_{n-1}) = \sum_{\lambda_{-1}=0}^{L_{-1}} \ldots \sum_{\lambda_n=0}^{L_n} p(\lambda)\Lambda(z/q^N; m)\alpha_1^{\gamma_1 z} \ldots \alpha_{n-1}^{\gamma_{n-1} z} ,$$

where $\gamma_r = \lambda_r - b_r\lambda_n/b_n$ $(1 \le r \le n)$ *. Then, for any integer* l *with* $1 \le l \le 16q^N Dhk^{\sigma+\frac{1}{2}\mu}$ *, either* $g(l; m) = 0$ *or*

$$(4) \qquad |f(l; m)| \ge \tfrac{1}{2} \exp\left(-D\left(2hk+\tfrac{1}{3}lq^{-N}k^{1-\sigma}\right)\Omega \log \Omega'\right) .$$

Similarly, for any integer q *such that* $1 \le q \le k^{\varepsilon}$ *and any integer* l *such that* $1 \le l \le 16q^{N+1}Dh$ *, either* $g(l/q; m) = 0$ *or*

$$(5) \qquad |f(l/q; m)| \ge \tfrac{1}{2} \exp\left(-q^n D\left(2hk+\tfrac{1}{3}lq^{-N}k^{1-\sigma}\right)\Omega \log \Omega'\right) .$$

Furthermore, for any complex number z *with* $|z| \le \frac{1}{3}q^N hk^{\sigma+\mu}$ *, we have*

$$|f(z; m)| \le \exp\left(\left(2hk+\tfrac{1}{3}|z|q^{-N}k^{1-\sigma}\right)\Omega \log \Omega'\right) .$$

Proof. We have

$$g(z) - f(z) = \sum_{\lambda_{-1}=0}^{L_{-1}} \ldots \sum_{\lambda_n=0}^{L_n} p(\lambda)\Lambda(z/q^N)\alpha_1^{\lambda_1 z} \ldots \alpha_n^{\lambda_n z}\left(1-\left(\alpha_1^{b_1} \ldots \alpha_n^{b_n}\right)^{-\lambda_n z/b_n}\right) .$$

On noting that $|e^{\zeta}-1| \le 2|\zeta|$ for $|\zeta| < 1$, we see that (2) implies that,

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for $|z| \le q^N hk^{\sigma+\mu}$,

$$\left|\left(1-\left(\alpha_1^{b_1} \cdots \alpha_n^{b_n}\right)^{-\lambda_n z/b_n}\right)\right| \le 2|\lambda_n z/b_n| . |b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n| \le \exp(-\tfrac{7}{8}Ch\Omega \log \Omega') .$$

The estimates of Lemma 2 readily reveal that for $|z| \le \frac{1}{3}q^N hk^{\sigma+\mu}$,

$$(6) \qquad |g(z)-f(z)| \le \exp\left(\left(\tfrac{7}{4}hk+\tfrac{1}{3}|z|q^{-N}k^{1-\sigma}\right)\Omega \log \Omega'\right).\exp(-\tfrac{7}{8}Ch\Omega \log \Omega') \le \exp(-\tfrac{3}{4}Ch\Omega \log \Omega') .$$

Hence, as asserted,

$$|f(z)| \le |g(z)| + \exp(-\tfrac{3}{4}Ch\Omega \log \Omega') \le \exp\left(\left(2hk+\tfrac{1}{3}|z|q^{-N}k^{1-\sigma}\right)\Omega \log \Omega'\right) .$$

We now recall that by Lemma 1,

$$q^{2h(N+1)(L_0+1)}(\nu(h))^{m_0}\left(a_1^{L_1} \cdots a_n^{L_n}\right)^l g(l/q;\, m)$$

is an algebraic integer of degree at most $q^n D$, and on recalling that $|\alpha_j| \le DA_j$ with the same inequality holding for each of the conjugates of each of the α_j , and noting that $q^{N+1} < k\Omega' \log \Omega'$, we see that each of the conjugates of the algebraic integer is bounded above by

$$\exp\left(\left(2hk+\tfrac{1}{3}lq^{-N}k^{1-\sigma}\right)\Omega \log \Omega'\right) ,$$

and by virtue of (6), the assertions (4) and (5) follow immediately.

LEMMA 4. *Let* J *be an integer with* $0 \le J \le \mu/2\varepsilon$ *and set*

$$S_0 = [q^{-N}k\Omega \log \Omega] , \quad S_1 = [\tfrac{1}{2}S_0] , \quad S_{J+1} = [(1-\varepsilon_J)S_J] \quad (J \ge 1)$$

where $\varepsilon_J = \max\{\varepsilon, 3/k^{\varepsilon J}\}$. *Then* $g(l;\, m_0, \ldots, m_{n-1}) = 0$ *for all integers* l *with* $1 \le l \le 16q^N Dhk^{\varepsilon J}$ *and all non-negative integers* $m_0, \ldots, m_{n-1}$ *with* $m_0 + \cdots + m_{n-1} \le S_J$.

Proof. The assertion is valid for $J = 0$, since this is the inductive assumption announced subsequent to Lemma 2. We suppose the lemma to have been verified for $J = 0, 1, \ldots, T$, where T is an integer

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satisfying $0 \le T < \mu/2\varepsilon$. We shall write

$$R_J = \left[16q^N Dhk^{\varepsilon J}\right] \quad (J \ge 0) .$$

By virtue of the present inductive hypothesis, we claim that

$$(7) \qquad \left|f_m\left(r;\, m_0,\, \ldots,\, m_{n-1}\right)\right| \le \exp\left(-\tfrac{2}{3}Ch\Omega \log \Omega'\right) ,$$

for all integers r, m satisfying $1 \le r \le R_T$, $0 \le m \le S_T - S_{T+1}$ and all non-negative integers $m_0, \ldots, m_{n-1}$ satisfying $m_0 + \ldots + m_{n-1} \le S_{T+1}$. Here, and henceforth, f_m is defined by

$$f_m = (m!)^{-1} f^{(m)} .$$

To prove the assertion (7), we firstly recall that $\Delta\left(b_n\lambda_j - b_j\lambda_n;\, m_j\right)$ is a polynomial in γ_j with coefficients independent of the λ's and of degree m_j . We claim now that our present inductive assumption implies that

$$(8) \qquad \sum_{\lambda_{-1}=0}^{L_{-1}} \cdots \sum_{\lambda_n=0}^{L_n} p(\lambda)\Lambda\left\{r/q^N;\, m_0+\mu_0,\, m_1,\, \ldots,\, m_{n-1}\right\} \,.$$

$$. \gamma_1^{\mu_1} \cdots \gamma_{n-1}^{\mu_{n-1}} \alpha_1^{\lambda_1 r} \cdots \alpha_n^{\lambda_n r} = 0$$

for all non-negative integers $\mu_0, \ldots, \mu_{n-1}$ and $m_0, \ldots, m_{n-1}$ respectively satisfying $\mu_0 + \ldots + \mu_{n-1} \le m$ and $m_0 + \ldots + m_{n-1} \le S_{T+1}$. To prove this claim, we argue by induction on $\mu = \mu_0 + \ldots + \mu_{n-1}$, observing that the case $\mu = 0$ is the inductive assumption of the lemma, since $S_{T+1} < S_T$. Indeed, this assumption states that

$$\sum_{\lambda_{-1}=0}^{L_{-1}} \cdots \sum_{\lambda_n=0}^{L_n} p(\lambda)\Lambda\left\{r/q^N;\, m_0+\mu_0,\, m_1+\mu_1,\, \ldots,\, m_{n-1}+\mu_{n-1}\right\}\alpha_1^{\lambda_1 r} \cdots \alpha_n^{\lambda_n r} = 0 ,$$

which we can rewrite as

$$\sum_{j_1=0}^{\mu_1} \cdots \sum_{j_{n-1}=0}^{\mu_{n-1}} d_{1,j_1} \cdots d_{n-1,j_{n-1}} \Xi\left(j_1,\, \ldots,\, j_{n-1}\right) = 0 ,$$

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where

$$\Xi(j_1, \ldots, j_{n-1}) = \sum_{\lambda_{-1}=0}^{L_{-1}} \cdots \sum_{\lambda_n=0}^{L_n} p(\lambda)\Lambda\left(r/q^N; m_0+\mu_0, m_1, \ldots, m_{n-1}\right) \cdot \\ \cdot \gamma_1^{j_1} \cdots \gamma_{n-1}^{j_{n-1}} \alpha_1^{\lambda_1 r} \cdots \alpha_n^{\lambda_n r} .$$

Since $d_{i,\mu_i} \neq 0$ for each i, we can conclude from the inductive assumption that $\Xi(j_1, \ldots, j_{n-1}) = 0$ for all non-negative integers $j_1, \ldots, j_{n-1}$ satisfying $j_1 + \ldots + j_{n-1} < \mu$, that also $\Xi(\mu_1, \ldots, \mu_{n-1}) = 0$, which confirms our claim (8).

Now write

$$\Phi(z_0, \ldots, z_{n-1}; m_0, \ldots, m_{n-1}) \\ = \sum_{\lambda_{-1}=0}^{L_{-1}} \cdots \sum_{\lambda_n=0}^{L_n} p(\lambda)\Lambda\left(z_0/q^N; m_0, \ldots, m_{n-1}\right)\alpha_1^{\gamma_1 z_1} \cdots \alpha_{n-1}^{\gamma_{n-1} z_{n-1}}$$

and observe that

$$f_m(r) = (m!)^{-1}\left(\partial/\partial z_0 + \ldots + \partial/\partial z_{n-1}\right)^m \Phi(z_0, \ldots, z_{n-1})\Big|_{z_0=\ldots=z_{n-1}=r}$$

$$= \sum_{\mu_0+\ldots+\mu_{n-1}=m} (\mu_0! \ldots \mu_{n-1}!)^{-1}\left(\frac{\partial}{\partial z_0}\right)^{\mu_0} \cdots \left(\frac{\partial}{\partial z_{n-1}}\right)^{\mu_{n-1}} \cdot \\ \cdot \Phi(z_0, \ldots, z_{n-1})\Big|_{z_0=\ldots=r}$$

$$= \sum_{\mu_0+\ldots+\mu_{n-1}=m} (\mu_1! \ldots \mu_{n-1}!)^{-1}\binom{m_0+\mu_0}{\mu_0}(\log \alpha_1)^{\mu_1} \cdots (\log \alpha_{n-1})^{\mu_{n-1}} \cdot \\ \cdot \sum_{\lambda_{-1}=0}^{L_{-1}} \cdots \sum_{\lambda_n=0}^{L_n} p(\lambda)\Lambda\left(r/q^N; m_0+\mu_0, m_1, \ldots, m_{n-1}\right) \cdot \\ \cdot \gamma_1^{\mu_1} \cdots \gamma_{n-1}^{\mu_{n-1}} \alpha_1^{\lambda_1 r} \cdots \alpha_{n-1}^{\lambda_{n-1} r} .$$

In view of (2), (8), and the argument opening the proof of Lemma 3, we have

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(7) as asserted. It is relevant to remark that the quantities $\left(\gamma_j \log \alpha_j\right)^{\mu_j}/\mu_j!$ are, relatively, not large.

We write

$$F(z) = \left\{(z-1) \ldots \left(z-R_T\right)\right\}^{S_T+1-S_{T+1}} .$$

By the integral form of the Hermite interpolation formula, we have, for each integer l satisfying $R_T < l \le R_{T+1}$,

$$(9) \quad \frac{1}{2\pi i} \int_\Gamma \frac{F(l)}{F(z)} \frac{f(z)}{z-l}\, dz$$

$$= f(l) + \frac{1}{2\pi i} \sum_{r=1}^{R_T} \sum_{m=0}^{S_T-S_{T+1}} f_m(r) \int_{\Gamma_r} \frac{F(l)}{F(z)} \frac{(z-r)^m}{z-l}\, dz ,$$

where, for each r , Γ_r is a circle with centre r and radius $\frac{1}{2}$ and Γ is a circle about the origin of radius $3R_{T+1}$. For z on Γ_r , we have

$$|F(l)/F(z)| \le \left\{\left(R_T k^\varepsilon\right)!/\left(R_T\left(k^\varepsilon-1\right)\right)!\left(\tfrac{1}{2}R_T\right)!^2\right\}^{S_T+1-S_{T+1}}$$

$$< \left(2ek^\varepsilon\right)^{R_T\left(S_T+1-S_{T+1}\right)} < \exp\left(16Dhk^{\varepsilon T+1}(2+\varepsilon \log k)\Omega \log \Omega'\right) .$$

Since $T < \mu/2\varepsilon$, we may conclude in view of the size of C and on recalling (2), that the double sum on the right in (9) is bounded above by $\exp(-\frac{1}{2}Ch\Omega \log \Omega')$.

For z on Γ , we have, in the case $T = 0$,

$$|F(l)/F(z)| \le 3^{-R_0\left(S_0-S_1\right)} < \exp(-8Dhk\Omega \log \Omega') .$$

But by Lemma 3, for z on Γ , we have

$$|f(z)| < \exp\left(\left(2hk+\tfrac{16}{3}Dhk^{1-(\sigma-\varepsilon)}\right)\Omega \log \Omega'\right) .$$

Thus, already if

$$k \ge (16/3)^{(1+\mu)(n+1)} ,$$

we see that the integral on the left in (9) is bounded above by $\exp(-5Dhk\Omega \log \Omega)$. Hence the interpolation formula implies that

$$|f(l;\ m)| \le \exp(-5Dhk\Omega \log \Omega') + \exp(-\tfrac{1}{2}Ch\Omega \log \Omega') ,$$

whilst, by Lemma 3, we have $g(l;\ m) = 0$ or

$$|f(l;\ m)| > \tfrac{1}{2} \exp\left(-D\left(2hk+\tfrac{16}{3}Dhk^{1-(\sigma-\varepsilon)}\right)\Omega \log \Omega'\right) .$$

Evidently, the inequalities for $|f(l;\ m)|$ contradict one another, so we obtain $g(l;\ m) = 0$ for $1 \le l \le R_1$ and $m_0 + \ldots + m_{n-1} \le S_1$, as required.

Generally, for $1 \le T < \mu/2\varepsilon$, we have for z on Γ ,

$$|F(l)/F(z)| < 3^{-R_T\left(S_T-S_{T+1}\right)} < \exp\left(-\tfrac{16}{6}\varepsilon_T Dhk^{\varepsilon T+1}\Omega \log \Omega'\right)$$

and

$$|f(z)| < \exp\left(\left(2hk+\tfrac{16}{3}Dhk^{\varepsilon T+1-(\sigma-\varepsilon)}\right)\Omega \log \Omega'\right) .$$

Thus, if say

$$k \ge (10D/\varepsilon)^{(1+\mu)(n+1)} ,$$

the interpolation formula implies that

$$|f(l;\ m)| \le \exp\left(-\tfrac{8}{6}\varepsilon_T Dhk^{\varepsilon T+1}\Omega \log \Omega'\right) ,$$

whilst for $1 \le l \le 16q^N Dhk^{\varepsilon T+\varepsilon}$, we have by Lemma 3 that $g(l;\ m) = 0$ or

$$|f(l;\ m)| > \tfrac{1}{2} \exp\left(-D\left(2hk+\tfrac{16}{3}Dhk^{\varepsilon T+1-(\sigma-\varepsilon)}\right)\Omega \log \Omega'\right) .$$

For $1 \le T < \mu/2\varepsilon$, the estimates for $|f(l;\ m)|$ contradict one another, whence we have $g(l;\ m) = 0$ for $1 \le l \le R_{T+1}$ and $m_0 + \ldots + m_{n-1} \le S_{T+1}$, as required. The lemma now follows by induction.

LEMMA 5. *For all integers* l, q *with* $1 \le l \le 16q^{N+1}Dh$, $7 \le q \le k^{\varepsilon}$ *and all non-negative integers* $m_0, \ldots, m_{n-1}$ *with* $m_0 + \ldots + m_{n-1} \le 9^{-1}q^{-N}k\Omega \log \Omega'$, *we have* $g\left(l/q;\ m_0, \ldots, m_{n-1}\right) = 0$.

Proof. By Lemma 4, we see that $g\left(l;\ m_0, \ldots, m_{n-1}\right) = 0$ for all integers l satisfying $1 \le l \le R$ and all non-negative integers $m_0, \ldots, m_{n-1}$ with $m_0 + \ldots + m_{n-1} \le S$, where

$$R = \left[16q^N Dhk^{\frac{1}{2}\mu}\right] \quad \text{and} \quad S = \left[6^{-1}q^{-N}k\Omega \log \Omega'\right] .$$

As shown in the proof of Lemma 4, this implies that

$$(10) \qquad |f_m(r;\, m_0,\, \ldots,\, m_{n-1})| \le \exp(-\tfrac{2}{3}Ch\Omega \log \Omega')$$

for all integers r, m satisfying $1 \le r \le R$, $0 \le m \le S/3$ and all non-negative integers $m_0, \ldots, m_{n-1}$ satisfying $m_0 + \ldots + m_{n-1} < 2S/3$.

Now suppose that l/q is not an integer and set

$$F(z) = \{(z-1) \ldots (z-R)\}^{[S/3]+1} .$$

As in the proof of Lemma 4, we have

$$(11) \quad \frac{1}{2\pi i}\int_\Gamma \frac{F(l/q)}{F(z)}\frac{f(z)}{z-l/q}\,dz$$
$$= f(l/q) + \frac{1}{2\pi i}\sum_{r=1}^{R}\sum_{m=0}^{[S/3]} f_m(r)\int_{\Gamma_r}\frac{F(l/q)}{F(z)}\frac{(z-r)^m}{z-l/q}\,dz$$

where, for each r , Γ_r is a circle with centre r and radius $\frac{1}{2}\min\{1, |r-l/q|\}$, and Γ is a circle about the origin of radius $3R$. For z on Γ_r , we have

$$|F(l/q)/F(z)| \le \left(R!/(\tfrac{1}{2}R)!^2\right)^{1+S/3} < \exp\left(2Dhk^{1+\frac{1}{2}\mu}\Omega \log \Omega'\right) ,$$

whence (10) suffices to show that the double sum on the right in (11) is bounded above by $\exp(-\frac{1}{2}Ch\Omega \log \Omega')$. For z on Γ , we have

$$|F(l/q)/F(z)| \le \left(3^{-R}\right)^{S/3} < \exp\left(-\tfrac{8}{9}Dhk^{1+\frac{1}{2}\mu}\Omega \log \Omega'\right) ,$$

whilst by Lemma 3,

$$|f(z)| \le \exp\left(\left(2hk+16Dhk^{1+\frac{1}{2}\mu-\sigma}\right)\Omega \log \Omega'\right) ,$$

whence the interpolation formula implies that

$$|f(l/q)| \le \exp\left(-\tfrac{2}{3}Dhk^{1+\frac{1}{2}\mu}\Omega \log \Omega'\right) .$$

On the other hand, by Lemma 3, we have for $1 \le l \le 16q^{N+1}Dh$ that either $g(l/q;\, m) = 0$ or

$$|f(l/q;\, m)| \ge \tfrac{1}{2}\exp\left(-q^n D\left(2hk+\tfrac{16}{3}qDhk^{1-\sigma}\right)\Omega \log \Omega'\right) .$$

Plainly, the two inequalities for $|f(l/q;\, m)|$ contradict one another if,

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say,

$$7 \le q \le k^{\mu/2(n+1)} ,$$

and recalling that the data includes the cases l/q an integer, we obtain the assertion of the lemma.

We are now in a position to state formally the inductive argument introduced prior to Lemma 3.

LEMMA 6. *Let* q *be a prime satisfying* $13 \le q \le k^{\varepsilon}$ *and suppose that the field* $K\left(\alpha_1^{1/q}, \ldots, \alpha_n^{1/q}\right)$ *is an extension of* $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ *of degree* q^n. *Then, for each non-negative integer* J *with* $q^J < (8n)^{-1}k^{1-(\sigma-\varepsilon)}\Omega' \log \Omega'$, *there exist integers* $p^{(J)}(\lambda_{-1}, \ldots, \lambda_n)$ *not all* 0 *and with absolute values at most* $\exp\left(3^{-1}hk\Omega \log \Omega'\right)$, *such that*

$$g_{(J)}(l; m_0, \ldots, m_{n-1}) = \sum_{\lambda_{-1}=0}^{L_{-1}^{(J)}} \cdots \sum_{\lambda_n=0}^{L_n^{(J)}} p^{(J)}(\lambda)\Lambda\left(l/q^J; m_0, \ldots, m_{n-1}\right)\alpha_1^{\lambda_1 l} \cdots \alpha_n^{\lambda_n l} = 0$$

for all integers l *with* $1 \le l \le 16q^J Dh$ *and all non-negative integers* $m_0, \ldots, m_{n-1}$ *with* $m_0 + \ldots + m_{n-1} \le q^{-J}k\Omega \log \Omega'$, *where* $L_{-1}^{(J)} = L_{-1}$, $L_0^{(J)} = L_0$, *and* $L_j^{(J)} \le q^{-J}L_j$ *for* $1 \le j \le n$.

Proof. The case $J = 0$ is just Lemma 2 and, on supposing the lemma to have been verified for $J = 0, 1, \ldots, N$, we have established that for all integers l with $1 \le l \le 16q^{N+1}Dh$ and all non-negative integers $m_0, \ldots, m_{n-1}$ with $m_0 + \ldots + m_{n-1} \le 9^{-1}q^{-N}k\Omega \log \Omega'$ we have

$$g_{(N)}(l/q; m_0, \ldots, m_{n-1}) = 0 .$$

In view of the supposition $\left[K\left(\alpha_1^{1/q}, \ldots, \alpha_n^{1/q}\right) : K\right] = q^n$, we may now conclude that for some n-tuple $(\lambda_1', \ldots, \lambda_n')$ of integers satisfying $0 \le \lambda_j' < q$ $(1 \le j \le n)$, we have

$$(12) \quad \sum_{\lambda_{-1}=0}^{L'_{-1}} \cdots \sum_{\lambda_n=0}^{L'_n} p^{(N)}(\lambda_{-1}, \lambda_0, \mu_1, \ldots, \mu_n) \cdot$$

$$\cdot \Lambda'\left(l/q^{N+1};\, m_0, \ldots, m_{n-1}\right)\alpha_1^{\lambda_1 l} \cdots \alpha_n^{\lambda_n l} = 0$$

for all integers l with $1 \le l \le 16q^{N+1}Dh$ and $(l, q) = 1$ and all non-negative integers $m_0, \ldots, m_{n-1}$ satisfying

$$m_0 + \ldots + m_{n-1} \le \frac{1}{q}\, q^{-N} k\Omega \log \Omega' ,$$

and where $L'_{-1} = L_{-1}$, $L'_0 = L_0$, $L'_j = \left[\left[L_j^{(N)} - \lambda'_j\right]/q\right]$ $(1 \le j \le n)$, $\mu_j = \lambda'_j + q\lambda_j$ $(1 \le j \le n)$, and the integers $\mu_1, \ldots, \mu_n$ are so chosen that not all the integers $p^{(N)}(\lambda_{-1}, \lambda_0, \mu_1, \ldots, \mu_n)$ vanish. Here, Λ' is defined as is Λ (see Lemma 2), but with $\lambda_1, \ldots, \lambda_n$ replaced by $\mu_1, \ldots, \mu_n$. On recalling that $\Delta\left(b_n\mu_r - b_r\mu_n;\, m_r\right)$ is a polynomial in $\gamma_r = \mu_r - b_r\mu_n/b_n$ with coefficients independent of the μ's and with degree m_r , we can argue by induction with respect to $m_1 + \ldots + m_{n-1}$ as in Lemma 4 (one of the "inner" inductions) and infer that (12) remains valid when the product over r in the definition of Λ' is replaced by $\gamma_1^{m_1} \cdots \gamma_{n-1}^{m_{n-1}}$. Then, by taking linear combinations, we can conclude that in (12), Λ' can be replaced by Λ . So we obtain that

$$\sum_{\lambda_{-1}=0}^{L'_{-1}} \cdots \sum_{\lambda_n=0}^{L'_n} p'(\lambda_{-1}, \lambda_0, \ldots, \lambda_n)\Lambda\left(l/q^{N+1};\, m_0, \ldots, m_{n-1}\right) \cdot$$

$$\cdot\, \alpha_1^{\lambda_1 l} \cdots \alpha_n^{\lambda_n l} = 0 \,;$$

where the $p'(\lambda)$ are integers not all 0 , with absolute values at most $\exp\left(3^{-1}hk\Omega \log \Omega'\right)$, and this is the assertion of the lemma for $J = N + 1$. Hence the lemma is proved by induction, provided we cope with the cases where $(l, q) \ne 1$.

To see this, set $R = \left[16q^{N+1}Dh\right]$ and $S = \left[9^{-1}q^{-N}k\Omega \log \Omega'\right]$. We

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have shown that $g(l;\ m)$ $\left(= g_{(N+1)}(l;\ m)\right) = 0$ for all integers l with $(l,\ q) = 1$ and $1 \le l \le R$ and all non-negative integers $m_0,\ \ldots,\ m_{n-1}$ with $m_0 + \ldots + m_{n-1} \le S$. Just as in the proof of Lemma 4, this implies that

$$\left|f_m\left(l;\ m_0,\ \ldots,\ m_{n-1}\right)\right| \le \exp\left(-\tfrac{2}{3}Ch\Omega \log \Omega'\right)$$

for all integers $l,\ m$ with $(l,\ q) = 1$, $1 \le l \le R$, and $0 \le m \le \frac{1}{4}S$ and all non-negative integers $m_0,\ \ldots,\ m_{n-1}$ with $m_0 + \ldots + m_{n-1} < \frac{3}{4}S$. We write

$$F(z) = \prod_{\substack{r=1\\(r,q)=1}}^{R} (z-r)^{\left[\frac{1}{4}S\right]+1} .$$

As before, for each integer l with $(l,\ q) > 1$ and $1 \le l \le R$, we have the interpolation formula

$$\frac{1}{2\pi i}\int_\Gamma \frac{F(l)}{F(z)}\,\frac{f(z)}{z-l}\,dz = f(l) + \frac{1}{2\pi i}\sum_{\substack{r=1\\(r,q)=1}}^{R}\ \sum_{m=0}^{\left[\frac{1}{4}S\right]} f_m(r)\int_{\Gamma_r} \frac{F(l)}{F(z)}\,\frac{(z-r)^m}{z-l}\,dz ,$$

where, for each r , Γ_r is a circle with centre r and radius $\frac{1}{2}$ and Γ is a circle about the origin with radius $3R$. Proceeding as in Lemma 4, we see that the double sum on the right is bounded above by $\exp\left(-\frac{1}{2}Ch\Omega \log \Omega'\right)$ and, for z on Γ ,

$$|F(l)/F(z)| \le \left(3^{-R(q-1)/q}\right)^{\frac{1}{4}S} < \exp\left(-\tfrac{16}{3}Dhk\Omega \log \Omega'\right) ,$$

whilst

$$|f(z)| \le \exp\left(\left(2hk+16Dkh^{1-\sigma}\right)\Omega \log \Omega'\right) .$$

So the interpolation formula gives

$$|f(l;\ m)| \le \exp\left(-3Dhk\Omega \log \Omega'\right) .$$

Finally, by invoking Lemma 3, we find that $g(l;\ m) = 0$ for all integers l with $(l,\ q) > 1$ and $1 \le l \le R$ and all non-negative integers $m_0,\ \ldots,\ m_{n-1}$ with $m_0 + \ldots + m_{n-1} \le q^{-N-1}k\Omega \log \Omega'$, as required.

To complete this part of the proof, we now require only an elementary

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lemma on polynomials.

LEMMA 7. *If* $P(x)$ *is a polynomial of degree* $m > 0$ *and with coefficients in a field* K *, then, for any integer* t *with* $0 \le t \le m$ *, the polynomials* $P(x), P(x+1), \ldots, P(x+t)$ *and* $1, x, \ldots, x^{m-t-1}$ *are linearly independent over* K *.*

Proof. This is Lemma 2 of Baker [2].

Proof of Theorem 1. Choose the prime q such that $13 \le q \le k^{\varepsilon}$ and $\left[K\left(\alpha_1^{1/q}, \ldots, \alpha_n^{1/q}\right) : K\right] = q^n$. We may suppose that we have established Lemma 6 for $J = N$ with

$$\left[(8n)^{-1}k^{1-\sigma}\Omega \log \Omega'/\log A_n\right] < q^N \le (8n)^{-1}k^{1-(\sigma-\varepsilon)}\Omega' \log \Omega' ,$$

whence we have

$$(13) \quad \sum_{\lambda_{-1}=0}^{L_{-1}^{(N)}} \cdots \sum_{\lambda_{n-1}=0}^{L_{n-1}^{(N)}} p^{(N)}(\lambda)\Lambda\left(l/q^N; m_0, \ldots, m_{n-1}\right)\alpha_1^{\lambda_1 l} \cdots \alpha_{n-1}^{\lambda_{n-1} l} = 0$$

for all integers l with $(l, q) = 1$ and $1 \le l \le 16q^N Dh$ and all non-negative integers $m_0, \ldots, m_{n-1}$ with $m_0 + \ldots + m_{n-1} \le q^{-N}k\Omega \log \Omega'$, and so, *a fortiori*, for $0 \le m_0 \le \frac{7}{8}q^{-N}k\Omega \log \Omega'$ and $0 \le m_j \le L_j^{(N)}$ $(1 \le j \le n-1)$, because $L_j^{(N)} \le q^{-N}L_j$ $(1 \le j \le n)$.

We claim that the equations (13) are impossible under the given conditions. Indeed, the equations imply that for $0 \le m_{n-1} \le L_{n-1}^{(N)}$,

$$(14) \quad \sum_{\lambda_{n-1}=0}^{L_{n-1}^{(N)}} \left\{\sum_{\lambda_{-1}=0}^{L_{-1}^{(N)}} \cdots \sum_{\lambda_{n-2}=0}^{L_{n-2}^{(N)}} p^{(N)}(\lambda)\Lambda\left(l/q^N; m_0, \ldots, m_{n-2}\right) \cdot \alpha_1^{\lambda_1 l} \cdots \alpha_{n-1}^{\lambda_{n-1} l}\right\}\Delta_{n-1} = 0 ,$$

where $\Delta_{n-1} = \Delta\left(b_n\lambda_{n-1}-b_{n-1}\lambda_n; m_{n-1}\right)$ and

$$\Lambda\left(l/q^N;\ m_0,\ \ldots,\ m_{n-2}\right) = \Lambda\left(l/q^N;\ m\right)/\Delta_{n-1} .$$

By Lemma 7, the polynomials $\Delta\left(x;\ m_{n-1}\right)$ for $0 \le m_{n-1} \le L_{n-1}^{(N)}$ are linearly independent, so it follows that the $\left(L_{n-1}^{(N)}+1\right) \times \left(L_{n-1}^{(N)}+1\right)$ determinant with Δ_{n-1} as its typical entry (in the $\left(\lambda_{n-1}+1\right)$-th row and $\left(m_{n-1}+1\right)$-th column) does not vanish. Hence the sums in parentheses in (14) all vanish and, after $n - 1$ applications of this argument, we obtain

$$(15) \qquad P_{m_0}\left(l/q^N\right) = \sum_{\lambda_{-1}=0}^{L_{-1}} \sum_{\lambda_0=0}^{L_0} p^{(N)}(\lambda)\Lambda\left(l/q^N+\lambda_{-1},\ h,\ \lambda_0+1,\ m_0\right) = 0$$

for all integers l with $(l,\ q) = 1$ and $1 \le l \le 16q^N Dh$ and all integers m_0 with $0 \le m_0 \le \frac{7}{8}q^{-N}k\Omega \log \Omega'$. But (15) asserts that for each admissible choice of $\lambda_1,\ \ldots,\ \lambda_n$, $P(z)$ is a polynomial of degree at most $h\left(L_0+1\right)$ with at least

$$(16) \qquad 16q^N Dh(1-1/q) \ . \ \tfrac{7}{8}q^{-N}k\Omega \log \Omega'$$

zeros, counted according to multiplicity. The quantity (16) is at least as large as $12Dhk\Omega \log \Omega'$, which exceeds $h\left(L_0+1\right)$, so that it follows that each $P(z)$ vanishes identically. Consequently, by Lemma 7, the $p^{(N)}(\lambda)$ all vanish, which is contrary to the construction of Lemma 6. This contradiction establishes the theorem.

3. Proofs of Theorems 2 and 3

The proofs of Theorems 2 and 3 involve only minor modifications to our proof of Theorem 1. We follow the argument of Baker [4]. We choose the parameters μ, κ, ε, k, C , and σ as in Section 2 and, for $\delta > 0$, we write

$$L_{-1} + 1 = h = \left[\log\left(B'\delta^{-1}C\Omega' \log \Omega'\right)\right] ,$$

$$L_0 + 1 = \left[\tfrac{1}{8}k^{1-\sigma}\Omega'\Theta\right] ,$$

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$$L_j = \left[(8n)^{-1}k^{1-\sigma}\Omega' \log \Omega'\Theta/\log A_j\right] \quad (1 \le j \le n-1) ,$$

$$L_n = [(8n)^{-1}k^{1-\sigma}\Omega' \log \Omega'] ,$$

and

$$\Theta = \max^{\prime}\{\log A, \delta B/(B'Ch\Omega' \log \Omega')\} .$$

Other than the new definition of h and, in effect, the replacement of $\log A$, wherever it implicitly appears, by Θ (so, for example, we commence Lemma 2 with $m_0 + \ldots + m_{n-1} \le k\Omega' \log \Omega'\Theta$) there is no change from the proof of Theorem 1. However, our primary supposition (2) is here replaced by the supposition that $b_1, \ldots, b_{n-1}$ and $b_n \ne 0$ are rational integers with absolute values at most B and B' respectively such that

(17) $\qquad 0 < |b_1 \log \alpha_1 + \ldots + b_n \log \alpha_n| < \exp(-Ch\Omega' \log \Omega'\Theta) .$

One easily confirms that there is only one point in the proof that is sufficiently different so as to require verification. Namely, in the proof of Lemma 2, one needs to notice that indeed

$$(18) \quad \prod_{r=1}^{n-1} |\Delta(b_n\lambda_r - b_r\lambda_n;\, m_r)| \le \prod_{r=1}^{n-1} \left(B'\delta^{-1}C\Omega' \log \Omega'\right)^{m_r} \Delta\left(\frac{k^{1-\sigma}}{8n}\left(\frac{\delta\Theta}{C\log A_r} + \frac{\delta B}{B'C}\right);\, m_r\right) \le e^{h(m_1+\ldots+m_{n-1})} \cdot 2^{m_1+\ldots+m_{n-1}} \cdot 2^{\frac{1}{4}k^{1-\sigma}h\Omega'\log\Omega'\Theta} ,$$

the estimate being valid for $\delta \le Ch\Omega' \log \Omega'$. The proof of Theorem 1 now allows one *mutatis mutandis* to conclude that, if there is a prime q satisfying $13 \le q \le k^{\varepsilon}$ such that $\left[K\left(\alpha_1^{1/q}, \ldots, \alpha_n^{1/q}\right) : K\right] = q^n$, then

$$|b_1 \log \alpha_1 + \ldots + b_n \log \alpha_n| \ge \exp(-Ch\Omega' \log \Omega'\Theta) .$$

To obtain Theorem 3, we observe that in view of the remark following the inequality (18), the assertion is immediate for $\delta \le Ch\Omega' \log \Omega'$, whilst the case $\delta > Ch\Omega' \log \Omega'$ is a weaker claim.

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