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ON POWERS OF HALF-TWISTS IN $M(0, 2n)$

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Abstract. We use elementary skein theory to prove a version of a result of Stylianakis (Stylianakis, The normal closure of a power of a half-twist has infinite index in the mapping class group of a punctured sphere, `arXiv:1511.02912`) who showed that under mild restrictions on m and n, the normal closure of the mth power of a half-twist has infinite index in the mapping class group of a sphere with $2n$ punctures.

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1. Introduction. Let $M(0, 2n)$ be the mapping class group of the 2-sphere S^2 fixing (setwise) a set of $2n$ points $p_1, \ldots, p_{2n} \in S^2$. It is well-known [**2**] that $M(0, 2n)$ is a quotient of the braid group B_{2n} on $2n$ strands, where the braid generator σ_i ($i = 1, \ldots, 2n-1$) maps to the mapping class $h_i \in M(0, 2n)$ which is a *half-twist* permuting p_i and p_{i+1} and fixing all other points p_j. Stylianakis recently showed the following:

Theorem 1.1 (Stylianakis [**10**]). *For $2n \geq 6$ and $m \geq 5$, the normal closure of h_i^m has infinite index in $M(0, 2n)$.*

(Note that the normal closure does not depend on i, as the h_i are all conjugate.)

For $2n = 6$, this result was known and is due to Humphries [**5**], as it is equivalent (by the Birman–Hilden Theorem) to Humphries' result [**5**, Theorem 4] that the normal closure of the mth power of a non-separating Dehn twist has infinite order in the genus 2 mapping class group for $m \geq 5$. Humphries' method was to employ the Jones representation [**4**] of the genus 2 mapping class group together with an explicit computation. Stylianakis' generalization proceeds by using certain Jones representations of $M(0, 2n)$, but his proof involves some non-trivial representation theory.

In this paper, we give an elementary skein-theoretic proof of the following:

Theorem 1.2. *For $2n \geq 4$ and $m \geq 6$, the normal closure of h_i^m has infinite index in $M(0, 2n)$.*

The key point in the proof of Theorem 1.2 is a simple 2×2 matrix calculation that I essentially did in [**8**]. Note that Theorem 1.2 implies Stylianakis' result for $m \geq 6$. Theorem 1.2 does not hold when $m = 5$ and $2n = 4$, as $M(0, 4)/(h_i^5 = 1)$ is a finite group (the alternating group A_5). I believe that the remaining case ($m = 5$, $2n \geq 6$) of Stylianakis' theorem can also be proved using the skein-theoretic method exposed below, but it would require a calculation with 5×5 matrices which I have not done (see Remark 3.5).

2. Strategy of the proof. The proof will be based on the representation of the braid group B_{2n} on the Kauffman bracket [**6**] skein module of the 3-ball relative to $2n$ marked points on the boundary. We will show that for an appropriate choice of Kauffman's skein variable A, this representation induces a projective-linear representation

$$\rho : M(0, 2n) \to \mathrm{PGL}_d(\mathbb{C})$$

(where d depends on n) so that
(i) $\rho(h_i^m) = 1$, and
(ii) the image $\rho(M(0, 2n))$ is an infinite group.
Clearly, this will imply that the normal closure of h_i^m has infinite index in $M(0, 2n)$.

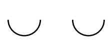
REMARK 2.1. Stylianakis used the same strategy applied to a certain Jones representation of $M(0, 2n)$. Actually, up to normalization and change of variables, the representation ρ is equivalent to the Jones representation for the rectangular Young diagram with two rows of length n. (We shall not make use of this fact in this paper.) For the purpose at hand, I find the skein-theoretic approach much easier.

REMARK 2.2. Funar [**3**] showed that the normal closure of the mth power of a Dehn twist has infinite index in the mapping class group of a genus g surface (with some restrictions on m and g) using the above strategy applied to TQFT-representations of mapping class groups. Our representation ρ can also be viewed as a TQFT representation of $M(0, 2n)$. But for us, TQFT is not actually needed. We shall only need Birman's presentation [**2**, Theorem 4.5] of $M(0, 2n)$ as a quotient of B_{2n} and elementary skein theory.

3. Proof of Theorem 1.2. We start with the representation of the braid group B_{2n} on the Kauffman bracket skein module of the 3-ball relative to $2n$ marked points on the boundary. Let us recall how this representation, which we denote by ρ, is defined. The skein module is a free $\mathbb{Z}[A, A^{-1}]$-module of dimension

$$d = \frac{1}{n+1}\binom{2n}{n}$$

(the Catalan number).[1] Its elements are represented by $\mathbb{Z}[A, A^{-1}]$-linear combinations of $(0, 2n)$-tangle diagrams, that is, tangle diagrams in a rectangle relative to $2n$ marked points at the top of the rectangle. The diagrams are considered modulo the Kauffman skein relations (which will be stated shortly). The skein module has a standard basis given by tangle diagrams without crossings and without closed circles. For example, if the number of points is $2n = 4$, the dimension is $d = 2$ and the basis is given by the two diagrams

$D_1 =$  $D_2 =$ 

Below we specialize A to a non-zero complex number, so that the skein module with this basis (ordered in some arbitrary fashion) is identified with $\mathbb{C}^d$.

The ith braid generator σ_i acts on a diagram D by gluing the usual braid diagram of σ_i^{-1} on top of D (that is, the braid diagram which has a crossing  at the ith and

[1]A proof of this formula can be found in [**7**, p. 661].

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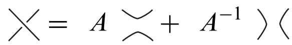
$(i + 1)$-st strand and all other strands are vertical). (We use inverses here so as to get a left action of B_{2n} on the skein module.) The Kauffman bracket skein relation



implies that

$$\rho(\sigma_i) = A\,\rho(E_i) + A^{-1}\,\mathrm{Id}\ ,$$

where E_i has  at the appropriate place and all other strands are vertical. The second Kauffman skein relation, which fixes the value of an unknot diagram to $-A^2 - A^{-2}$, implies that

$$\rho(E_i)^2 = (-A^2 - A^{-2})\,\rho(E_i)\ .$$

A simple recursion now establishes that

$$\rho(\sigma_i^m) = P_m(A)\,\rho(E_i) + A^{-m}\,\mathrm{Id},$$

where $P_m(A) = A^{2-m}(1 - A^4 + A^8 - \ldots + (-1)^{m-1}A^{4m-4})$. Thus, we have the following:

Proposition 3.1. *If $A \in \mathbb{C}$ satisfies $P_m(A) = 0$, then $\rho(\sigma_i^m) = A^{-m}\,\mathrm{Id}$ is the identity element in $\mathrm{PGL}_d(\mathbb{C})$.*

From now on, we assume that A is a zero of the polynomial $P_m(A)$. Note that all zeros of $P_m(A)$ are roots of unity. We shall make a precise choice of A later.

Proposition 3.2. *For any $A \in \mathbb{C}^*$, the homomorphism $\rho : B_{2n} \to \mathrm{PGL}_d(\mathbb{C})$ factors through $M(0, 2n)$.*

Proof. This is well-known but here is a proof. The group $M(0, 2n)$ is the quotient of B_{2n} by the relations $R_1 = R_2 = 1$, where

$$R_1 = \sigma_1\sigma_2 \cdots \sigma_{2n-1}\sigma_{2n-1}\sigma_{2n-2} \cdots \sigma_1$$

$$R_2 = (\sigma_1\sigma_2 \cdots \sigma_{2n-1})^{2n}$$

(see [**2**, Theorem 4.5]). Using the isotopy invariance of the Kauffman bracket, it is easy to check that

$$\rho(R_1) = (-A^3)^2\,\mathrm{Id},$$

$$\rho(R_2) = (-A^3)^{2n}\,\mathrm{Id}\,.$$

(see [**9**, Section 1.3]). This proves the proposition. □

Remark 3.3. For appropriate roots of unity A, the induced projective-linear representation of $M(0, 2n)$ is a TQFT representation, as follows from the skein-theoretic construction of Witten–Reshetikhin–Turaev TQFT in [**1**].

By abuse of notation, we denote the induced homomorphism $M(0, 2n) \to \mathrm{PGL}_d(\mathbb{C})$, which sends h_i to $\rho(\sigma_i)$, again by ρ. Thus, we have realized condition (i) of the strategy outlined in §2. To realize condition (ii), it suffices to find an element $\phi \in B_{2n}$ so that $\rho(\phi)$ has infinite order in $\mathrm{PGL}_d(\mathbb{C})$. We now show that $\phi = \sigma_1^2\sigma_2^{-2}$ works.

Recall the diagrams D_i $(i = 1, 2)$ depicted above. By taking disjoint union of D_i with some fixed $(0, 2n-4)$-tangle diagram $\widetilde{D}$ (so that the first four points are the boundary of D_i, and the remaining $2n-4$ points are the boundary of $\widetilde{D}$), we get two diagrams D'_1 and D'_2 which form part of a basis of our skein module. The two-dimensional subspace spanned by D'_1 and D'_2 is preserved by both $\rho(\sigma_1)$ and $\rho(\sigma_2)$. On this subspace, $\rho(\sigma_1)$ and $\rho(\sigma_2)$ act by the following matrices:

$$\rho(\sigma_1) = \begin{bmatrix} -A^3 & A \\ 0 & A^{-1} \end{bmatrix}, \qquad \rho(\sigma_2) = \begin{bmatrix} A^{-1} & 0 \\ A & -A^3 \end{bmatrix}.$$

(This follows immediately from the Kauffman relations.) A straightforward calculation now gives that the matrix of $\rho(\sigma_1^2\sigma_2^{-2})$ acting on this two-dimensional subspace is

$$M = \begin{bmatrix} 2 - A^4 - A^{-4} + A^8 & -A^{-2} + A^{-6} \\ A^{-2} - A^{-6} & A^{-8} \end{bmatrix}$$

Clearly, if M has infinite order in $\mathrm{PGL}_2(\mathbb{C})$, then $\rho(\sigma_1^2\sigma_2^{-2})$ has infinite order in $\mathrm{PGL}_d(\mathbb{C})$.

LEMMA 3.4. *M has infinite order in* $\mathrm{PGL}_2(\mathbb{C})$ *provided the order r of the root of unity $q = A^4$ satisfies $r \geq 3$ and $r \notin \{4, 6, 10\}$.*

Proof. For $r \geq 5$ and $r \notin \{6, 10\}$, this is shown in **[8]**, as one can check that the matrix M is conjugate to the one computed in **[8]**. We can also apply the argument of **[8]** directly to our matrix, as follows. Note that M has determinant 1 and trace

$$t = 2 - q - q^{-1} + q^2 + q^{-2}$$

where $q = A^4$. If M has finite order in $\mathrm{PGL}_2(\mathbb{C})$, then its eigenvalues λ and λ^{-1} must satisfy $\lambda^N = \lambda^{-N}$ for some N, so λ is a root of unity. But this is impossible, as we can find a primitive rth root $q \in \mathbb{C}$ such that $|t| = |\lambda + \lambda^{-1}| > 2$ (see **[8]**). Thus, M has infinite order in $\mathrm{PGL}_2(\mathbb{C})$.

In the remaining case $r = 3$, it suffices to observe that in this case we have $t = 2$, so M is conjugate to

$$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$$

with $c \neq 0$ (since M is not the identity matrix). □

The proof of Theorem 1.2 is now completed as follows. For $m \geq 6$, we choose A to be a primitive Nth root of unity, as follows:

- For $m = 6$, we take $N = 12$.
- For $m = 10$, we take $N = 20$.
- For odd $m \geq 7$, we take $N = 8m$.
- For even $m \geq 8$, $m \neq 10$, we take $N = 4m$.

Then, $P_m(A) = 0$, so Proposition 3.1 applies. Also, $q = A^4$ has order $r \geq 3$, $r \notin \{4, 6, 10\}$, so Lemma 3.4 applies. Thus, ρ satisfies condition (i) because of

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Proposition 3.1, and ρ satisfies condition (ii) because the matrix $\rho(\sigma_1^2\sigma_2^{-2})$ has infinite order in $\mathrm{PGL}_d(\mathbb{C})$. This completes the proof.

REMARK 3.5. I expect that the remaining case ($m = 5$, $2n \geq 6$) of Stylianakis' theorem (see Theorem 1.1) can also be proved using the skein-theoretic representation ρ evaluated at a root of unity A so that $P_5(A) = 0$. It suffices to find $\phi \in B_6$ so that the 5×5 matrix $\rho(\phi)$ has infinite order in $\mathrm{PGL}_5(\mathbb{C})$. This will imply the result for $M(0, 2n)$ with $2n \geq 6$ for the same reason as above. Stylianakis describes such an element ϕ and shows that it has infinite order in the Jones representation he uses. Actually ϕ is closely related to the element originally used by Humphries [**5**]. Note that *modulo* identifying our skein-theoretic representation of $M(0, 6)$ with the Jones representation used by Humphries, the fact that $\rho(\phi)$ has infinite order is already shown by Humphries. There seems to be no advantage in redoing the relevant 5×5 matrix computation directly from the skein-theoretic approach, and I have not attempted to do so.

REMARK 3.6. The proof of Proposition 3.2 shows that one can rescale ρ to get a representation $\hat{\rho}$ of B_{2n} which descends to $M(0, 2n)$ as a *linear* representation: put

$$\hat{\rho}(\sigma_i) = \theta^{-1}\rho(\sigma_i),$$

where $\theta^{4n-2} = (-A^3)^2 = A^6$; then $\hat{\rho}(R_1) = \hat{\rho}(R_2) = \mathrm{Id}$. Note that

$$\hat{\rho}(\sigma_i^m) = (\theta A)^{-m}\,\mathrm{Id}\,.$$

One may wonder whether θ can be chosen so that $(\theta A)^{-m} = 1$. In general, the answer is no. For example, if m is odd, then $P_m(A) = 0$ implies $A^{4m} = -1$, and one computes (using $\theta^{4n-2} = A^6$) that

$$((\theta A)^{-m})^{4n-2} = A^{-4m(n+1)} = (-1)^{n+1}\,.$$

Thus, $(\theta A)^{-m} \neq 1$ if m is odd and n is even.

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