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CONDITIONS FOR THE SEPARABILITY OF OBJECTS IN TWO-DIMENSIONAL VELOCITY FIELDS

STEPHAN FOLDES

ABSTRACT. We consider the directed graph representing the obstruction relation between objects moving along the streamlines of a two-dimensional velocity field. A collection of objects is sequentially separable if and only if the corresponding graph has no directed cycles. A sufficient condition for this is the permeability of closed Jordan curves.

1. **Introduction.** In this article we will be interested in the motions of certain deformable objects in the plane, and more precisely in the obstruction relations among such objects. The physical model we shall refer to is that of a steady fluid flow in some region D of the plane $\mathbb{R}^2$, described by a differential equation. The objects are thought of as released in the flow and moving along streamlines like a suspension after their release, being transported by convection in the fluid. The property we seek is sequential separability, *i.e.* the possibility of releasing the objects in some sequence so that they are "washed away" by the flow without coming into contact or mixing with one another. This will amount to the absence of directed cycles in a directed graph representing the possible obstruction relations. In this acyclic case, the partial order defined by the directed graph specifies all possible release sequences. Using the general framework of rectilinear motion planning for rigid objects rather than that of fluid flows and velocity fields, the partial orders arising from a uniform flow were characterized by Rival and Urrutia [7], and those arising from a central velocity field were also described [3]. In the former case, sequential separability is guaranteed, in the latter it is not. The purpose of this article is to shed some light on this difference and to show that under reasonable postulates in two dimensions, the permeability of Jordan curves in a flow is sufficient to guarantee the sequential separability of streamlined objects.

2. **Geometrical framework.** Our object of study will be a *two-dimensional velocity field*, *i.e.* a continuous function $V: D \longrightarrow \mathbb{R}^2$ from a subset $D \subseteq \mathbb{R}^2$ called the *domain* of V, to the vector space $\mathbb{R}^2$. For $x \in D$, the vector $V(x)$ is thought of as the *velocity* of a fluid flow in D at the point x. The flow is steady as $V(x)$ does not depend on time. A *streamline function* is then defined as a differentiable function $p: \mathbb{R} \longrightarrow D$, such that $p'(t) = V\big(p(t)\big)$ for all t. $p(t)$ is thought of as describing the changing position of a particle over time as it is carried by the fluid flow.

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A *streamline* is the image of some streamline function, endowed with the obvious *order structure* according to time.

For $a \in D$, the *streamline ray* from a, denoted $[a, \longrightarrow)$ is the maximal streamline interval with first point a. If $b \in [a, \longrightarrow)$, then the *streamline segment* from a to b, denoted $\overline{ab}$, is the streamline interval with first point a and last point b.

To ensure the well-behaved nature of the flow, the following conditions are postulated, as customary:

(1) EXISTENCE AND UNIQUENESS. For every $c \in D$, $r \in \mathbb{R}$, there exists a unique streamline function $p: \mathbb{R} \longrightarrow D$, satisfying for all $t \in \mathbb{R}$ the differential equation

$$p'(t) = V\big(p(t)\big) \tag{DE}$$

and the initial condition $p(r) = c$.

For $c \in D$, let $p: \mathbb{R} \longrightarrow D$ then be the unique streamline function satisfying (DE) and $p(0) = c$. Let $P(t, c) = p(t)$. Then P is a function from $\mathbb{R} \times D$ to D. We may call it the *transport function* of the velocity field V. The following property is also customarily part of the well-set requirements and we shall assume it henceforward:

(2) CONTINUITY. Let $c_1, c_2, c_3, \ldots$ be a sequence of points of D converging to some limit point $c \in D$. Let T be a bounded interval of $\mathbb{R}$ such that $P(t, c)$, as well as all the $P(t, c_k)$, $k = 1, 2, 3, \ldots$, are defined for all $t \in T$. Then the $P(t, c_k)$, defined as functions from T to D, converge uniformly to $P(t, c)$ as $k \longrightarrow \infty$.

Observe that the continuity postulate, as stated above, implies the continuity of the transport functions, in both variables. The theory of differential equations provides broad sufficient conditions for existence, uniqueness and continuity (see *e.g.* Birkhoff and Rota [1]). Here we shall be concerned with the topological and combinatorial implications of these postulates rather than with the origins of their validity.

We are interested in the combinatorial relationships among objects some of which may be brought into contact with some others by convection in the fluid flow. An *object* is formally defined as an arc-connected subset of D (an arc being a homeomorphic image of the real interval $[0, 1]$). Note that we require objects to be within the domain D of the flow. A subset A of $\mathbb{R}^2$ is *streamlined* if every streamline segment with both end-points in A is entirely contained in A.

We have :

(i) $\mathbb{R}, \emptyset$ and all singletons are streamlined,

(ii) the intersection of any family of streamlined sets is streamlined,

(iii) the union of any updirected family of streamlined sets is streamlined.
(A family is updirected if the union of any two members is always contained in some member of the family.)

Properties (i) to (iii) mean that the family of streamlined subsets of the plane forms a *convexity structure*, essentially in the sense of Duchet [2] and van de Vel [9, 10]. So does the family of streamlined subsets of D. Note that streamlined sets need not be convex in the standard sense, nor do standard convex sets need be streamlined. However, if all streamlines are rectilinear, as *e.g.* in the case of a uniform flow or a central velocity field, then all standard convex sets are streamlined.

Let A and B be disjoint objects. B is said to *obstruct* A if there is a streamline segment from A to B (*i.e.* from some point of A to some point of B). Let us observe that an object A is streamlined if and only if there is no $b \notin A$ such that A and $\{b\}$ mutually obstruct each other.

We are interested in the possibility of having objects released and carried "away" by the flow. "Away" means out of any bounded area of the plane. In addition to the usual well-set requirements, we shall therefore postulate a third property :

(3) NON-CONFINEMENT (OR "SINK AT INFINITY"). For every $c \in D$, the streamline ray from c is unbounded.

3. **Separability and obstruction digraphs.** By a *collection* of objects we shall always mean a finite, pairwise disjoint set of objects. A collection S of n objects is said to be *sequentially separable* if the objects can be ordered in a sequence $A_1, \ldots, A_n$ such that A_i is not obstructed by any A_j, $j = i+1, \ldots, n$. This means that the objects will be removed by the fluid from any specified bounded area if released one-by-one in the sequence $A_1, \ldots, A_n$, and no collision can occur in the bounded area if each object is released after the previous one clears the bounded area. Such a separation sequence is indeed equivalent to a linear (total) order $\leq$ defined on S in which

$$A_1 > A_2 > \cdots > A_{n-1} > A_n.$$

This linear order is obviously such that if $B > A$ then A does not obstruct B. Finding a linear order with this property, called a *separation order*, is thus essential if we wish to achieve a collision-free sequential separation of the objects. It should be noted that this concept of separability by fluid convection is distinct from, although in spirit akin to, those surveyed by Toussaint in a computational context [8], and in some special cases there are relationships between these distinct separability conditions.

In order to find a separation order, let us define the *obstruction digraph* as the directed graph whose vertices are the objects, and where there is a directed edge from A to B whenever B obstructs A. Obviously if the obstruction digraph contains a directed cycle, then the objects are not sequentially separable. If, on the other hand, this digraph is acyclic, then its transitive closure is a directed comparability graph, defining a partial order on the objects as follows: $A \leq B$ if and only if there is a directed path from A to B in the obstruction digraph. This partial order will be called the *obstruction order*. It is easily seen, by induction on the length of the shortest directed path from A to B, that if $A < B$ in the obstruction order, then $A < B$ in any separation order (*i.e.* every separation order is a linear extension of the obstruction order). Conversely, let us consider any linear extension of the obstruction order. Obviously, it satisfies the requirements for being a separation order. This reasoning is analogous to the one in [4] Proposition 1, although the approach to defining the objects' possible motions is entirely different here. The essential fact is the same :

PROPOSITION 1. *Let S be a collection of objects in a steady fluid flow in some domain of* $\mathbb{R}^2$. *Then the objects in S are sequentially separable if and only if the obstruction digraph of S is acyclic. In this case, the separation orders are precisely the linear extensions of the obstruction order.* ∎

4. **Acyclicity and permeable Jordan curves.** The idea of describing the obstruction digraph, and the obstruction order, as a function of the geometry of the objects and the motion rules, is inspired by the work of Rival and Urrutia [7] which deals, in the framework of theoretical robotics, with an everywhere-defined uniform flow (*i.e.* the velocity vector $V(x)$ is the same for all $x \in \mathbb{R}^2$). Standard convex objects, which are *a fortiori* streamlined, are always separable in this case, as was already observed by Guibas and Yao [5]. In the case of polygons, it was indeed shown by Nussbaum and Sack [6] that standard convexity could be somewhat relaxed and separability still guaranteed: the monotone polygons they consider are actually identical with the streamlined polygonal objects in a uniform flow, and are also directionally convex with respect to a single direction [4], which again confirms their guaranteed separability.

Separability is no longer guaranteed in the case of the central velocity field defined by $V(x) = x$ on all $x \neq (0,0)$. The arising obstruction orders were fully described in [3], but directed cycles may exist in the obstruction digraph. Let us now point out an easily verifiable difference between the uniform flow and the flow corresponding to the above central velocity field. Call a Jordan curve C (a homeomorph of the unit circle) *permeable* if there is a streamline ray from C that is not disjoint from its inner region. All Jordan curves are permeable in a uniform flow, and this is obviously not the case in a central velocity field. This observation is the basis of the following Theorem, establishing the permeability of Jordan curves as a sufficient condition for separability.

THEOREM. *If all Jordan curves in the domain of a steady fluid flow are permeable, then every collection of streamlined objects is separable in the flow.*

The following simple but important fact will be used in the proof of the Theorem:

LEMMA. *If* $\overline{a_i b_{i+1}}$ *is any streamline segment from* A_i *to* A_{i+1} *and* $\overline{a_j b_{j+1}}$ *any streamline segment from* A_j *to* A_{j+1}, *such that the two streamline segments intersect, then* $i = j$.

PROOF OF THE LEMMA. The uniqueness postulate implies, without loss of generality, that $a_j \in \overline{a_i b_{i+1}}$. Thus there is a streamline segment of the form $\overline{a_i b_{j+1}}$, and $j + 1 = i + 1$, *i.e.* $i = j$. ∎

PROOF OF THE THEOREM. Assume that some collection of objects is not separable. We shall show the existence of a non-permeable Jordan curve. Let $A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_n \longrightarrow A_1$ be a directed cycle of minimal length in the obstruction digraph. (The indices $1, \ldots, n$ are to be interpreted modulo n.) Intuitively it should seem plausible that streamline segments responsible for these obstructions can somehow be concatenated and completed to form the desirable Jordan curve. The proof of the Theorem is a precise verification of this.

Since n is minimal, there can be no streamline segments from A_i to A_j unless $j = i+1$. For the same reason, a streamline segment from A_i to A_{i+1} does not meet any other A_k, $k \neq i, i+1$.

A streamline segment $\overline{ab}$ from A_i to A_{i+1} is called a *boundary streamline segment* if $\overline{ab} \cap (A_i \cup A_{i+1}) = \{a, b\}$.

As the objects are topologically closed, boundary streamline segments must always exist from A_i to A_{i+1}. For each $i = 1, \ldots, n$ let $\overline{a_i b_{i+1}}$ be one of the boundary streamline segments from A_i to A_{i+1}. We claim that $b_i \neq a_i$ for all i. If $b_i = a_i$ for some i, then we may assume $a_{i-1} \neq b_{i+1}$, because otherwise $\overline{a_{i-1} b_i} \cup \overline{a_i b_{i+1}}$ would be a non-permeable Jordan curve and the proof would be complete. (Non-permeability would follow from the uniqueness postulate.) There is a streamline segment $\overline{a_{i-1} b_{i+1}}$ from A_{i-1} to A_{i+1}, but as A_{i+1} cannot obstruct A_{i-1}, we must have $A_{i-1} = A_{i+1}$. The point $a_i = b_i$ belongs to A_i, and does not belong to $A_{i-1} = A_{i+1}$. As it also belongs to the streamline segment $\overline{a_{i-1} b_{i+1}}$, $A_{i-1} = A_{i+1}$ cannot be streamlined: a contradiction. Therefore $b_i \neq a_i$ as claimed, for all i.

For each i, let $[b_i a_i]$ be an arc in A_i from b_i to a_i (the endpoint and the starting point, respectively, of boundary streamline segments). By juxtaposition of

$$\overline{a_1 b_2}, [b_2 a_2], \overline{a_2 b_3}, [b_3 a_3], \ldots, \overline{a_n b_1}, [b_1 a_1]$$

we obtain a Jordan curve C.

Call a streamline segment $\overline{ab}$ *internal* if it is disjoint from the open exterior of C, *strongly internal* if it is internal and $\overline{ab} \cap C = \{a, b\}$.

For each i, the points of the arc $[b_i, a_i]$ are totally ordered from b_i to a_i. Using the Lemma it can be seen that if $\overline{xb}$ is a strongly internal streamline segment from A_{i-1} to A_i and $\overline{ay}$ a strongly internal streamline segment from A_i to A_{i+1}, then a and b lie on $[b_i a_i]$, and $b < a$ in the order on $[b_i a_i]$. Therefore, for the least upper bound β_i of all such b and the greatest lower bound α_i of all such a, we have $b_i \leq \beta_i \leq \alpha_i \leq a_i$ in the arc order.

The Jordan curve C may well be permeable. Parts of the Jordan region bounded by C will now be carved away along the strongly internal streamline segments joining the objects A_i, leaving a Jordan region with a non-permeable boundary. For this purpose, we shall prove that there is an internal streamline segment from $\alpha_i \in A_i$ to $\beta_{i+1} \in A_{i+1}$. Consider therefore the streamline ray $[\alpha_i \longrightarrow)$ and see how it is approximated by strongly internal streamline segments from A_i to A_{i+1}.

We can show that $[\alpha_i \longrightarrow)$ meets A_{i+1}. For any $i = 1, \ldots, n$, if $\overline{ab}$ and $\overline{a'b'}$ are strongly internal streamline segments from A_i to A_{i+1} such that say $a < a'$ in the arc order on $[b_i a_i]$, then it is easy to verify, using the Lemma again, that $b' < b$ on $[b_{i+1}, a_{i+1}]$. By the definition of α_i, there is an infinite sequence of points $a^1, a^2, a^3, \ldots$ of the sub-arc $[\alpha_i a_i]$ of $[b_i a_i]$ that converges to α_i, and a corresponding sequence of points $b^1, b^2, b^3, \ldots$ of the sub-arc $[b_{i+1} \beta_{i+1}]$ of $[b_{i+1} a_{i+1}]$, such that each $\overline{a^k b^k}$ is a strongly internal streamline segment from A_i to A_{i+1}. Indeed the points b^k must converge to β_{i+1}, and we may also suppose that the points a^k form a decreasing sequence in the arc order of $[b_i a_i]$, while the points b^k from an increasing sequence on $[b_{i+1} a_{i+1}]$. Let $T > 0$ be any positive time bound. By the continuity postulate, the functions $t \longrightarrow P(a^k, t)$ converge uniformly to the function $t \longrightarrow P(\alpha_i, t)$ in the time interval $[0, T]$. Suppose now that $[\alpha_i \longrightarrow)$ does not meet A_{i+1}. Choose T such that $P(\alpha_i, T)$ is in the open exterior of C. Such T exists because of non-confinement. By uniform convergence, there is a k such that

(i) $P(a^k, T)$ is in the open exterior of C,

(ii) the streamline segment $\overline{a^k, P(a^k, T)}$ does not meet A_{i+1}.

For this k then, $b^k > P(a^k, T)$ on the streamline ray originating at a^k. But then the streamline segment $\overline{a^k b^k}$ cannot be internal, a contradiction. Therefore $[\alpha_i \longrightarrow) \cap A_{i+1} \neq \emptyset$.

Now let us show that $[\alpha_i \longrightarrow)$ contains an internal streamline segment from α_i to A_{i+1}. Let t_0 be the first positive time such that $P(\alpha_i, t_0) \in A_{i+1}$. (The existence of t_0 is guaranteed by the compactness of A_{i+1}.) Suppose that for some $0 < T < t_0$, the point $P(\alpha_i, T)$ is in the open exterior of C. By uniform convergence, as above, there would be some k such that (i) and (ii) hold as above for the T as just re-defined. This would again contradict the internality of the streamline segment $\overline{a^k b^k}$. Therefore no $P(\alpha_i, T)$ is in the open exterior of C for $0 < T < t_0$. If follows that the streamline segment $\overline{\alpha_i, P(\alpha_i, t_0)}$ is internal.

We now define a sub-segment $\overline{\delta_i \gamma_{i+1}}$ of $\overline{\alpha_i, P(\alpha_i, t_0)}$ that links $[b_i, a_i]$ directly to $[b_{i+1}, a_{i+1}]$. It is not difficult to verify that there must be a smallest $t_1 \geq t_0$ such that $P(\alpha_i, t_1) \in [b_{i+1}, a_{i+1}]$. Let us denote $P(\alpha_i, t_1)$ by γ_{i+1}. The streamline segment $\overline{\alpha_i \gamma_{i+1}}$ is still internal, and it is a streamline segment from A_i to A_{i+1}. Let δ_i be the last point of $\overline{\alpha_i \gamma_{i+1}}$ that lies on $[b_i a_i]$.

Let us show that we must have indeed $\delta_i = \alpha_i$ and $\gamma_{i+1} = \beta_{i+1}$, completing the proof that there is an internal streamline segment from α_i to β_{i+1}.

Either $\alpha_i = a_i$ or $\alpha_i \neq a_i$. In the former case obviously $\alpha_i = \delta_i$ and $\gamma_{i+1} = \beta_{i+1} = b_{i+1}$. The streamline segment $\overline{a_i b_{i+1}}$ is identical with a streamline segment $\overline{\alpha_i \beta_{i+1}}$.

The case $\alpha_i \neq a_i$ requires more attention. We shall in this case show the existence of a strongly internal streamline segment from α_i to β_{i+1}.

First, observe that γ_{i+1} is indeed the first and only point of $\overline{\alpha_i \gamma_{i+1}}$ that meets $[b_{i+1} a_{i+1}]$, by the definition of t_0 and t_1 above.

Second, we claim that $\alpha_i = \delta_i$. Otherwise δ_i either precedes or follows α_i in the arc order on $[b_i a_i]$. If δ_i precedes α_i, then $\overline{\delta_i \gamma_{i+1}}$ is a strongly internal streamline segment from A_i to A_{i+1}, contradicting the definition of α_i. If δ_i follows α_i, then take any a^k, as defined above, such that $\alpha_i \leq a^k < \delta_i$ in the arc order. We may indeed suppose that $\alpha_i < a^k$, for otherwise $\alpha_i = a^k$, $\beta_{i+1} = b^k$, and $\overline{a^k b^k}$ would be a streamline segment from α_i to β_{i+1}, the existence of which we intend to show. Consider now the two Jordan curves C_1 and C_2 formed by $\overline{a^k b^k}$ and one or the other sub-arc of C joining a^k to b^k. Let $\overline{C_j}$ be the union of C_j and its inner region, $j = 1, 2$. Let $B_j = \overline{C_j} \setminus \overline{a^k b^k}$. B_1 and B_2 are disjoint, and each of them is open relative to $B_1 \cup B_2$. If $\overline{\alpha_i \delta_i} \cap \overline{a^k b^k} = \emptyset$, then $\overline{\alpha_i \delta_i} \subseteq B_1 \cup B_2$. As α_i and δ_i are in different components B_j, $j = 1, 2$, this is impossible. Thus $\overline{\alpha_i \delta_i}$ and $\overline{a^k b^k}$ intersect at some internal point c of $\overline{\alpha_i \delta_i}$. But this would violate the uniqueness postulate at c. Thus $\alpha_i = \delta_i$ as claimed.

To show that $\gamma_{i+1} = \beta_{i+1}$, let us first observe that the streamline segment $\overline{\alpha_i\gamma_{i+1}}$ must then be strongly internal, implying that $\gamma_{i+1} \leq \beta_{i+1}$ in the arc order on $[b_{i+1}a_{i+1}]$. If $\gamma_{i+1} < \beta_{i+1}$, then for some b^k such that $\gamma_{i+1} < b^k \leq \beta_{i+1}$, the strongly internal streamline segments $\overline{a^kb^k}$ and $\overline{\alpha_i\gamma_{i+1}}$ would intersect, violating the uniqueness postulate. Thus $\gamma_{i+1} = \beta_{i+1}$ and $\overline{\alpha_i\beta_{i+1}}$ is a strongly internal streamline segment as claimed.

For each $i = 1, \ldots, n$ consider the sub-arc $[\beta_i\alpha_i]$ of $[b_ia_i]$. Take the juxtaposition of

$$\overline{\alpha_1\beta_2},\ [\beta_2\alpha_2],\ \overline{\alpha_2\beta_3},\ [\beta_3\alpha_3],\ \ldots,\ \overline{\alpha_n\beta_1},\ [\beta_1\alpha_1].$$

It is a non-permeable Jordan curve. ∎

It is easy to see that a non-permeable Jordan curve must have a point c in its inner region where the velocity field is not defined, $c \notin D$. (If d is any point in the inner region of the curve where V is defined, then a required c would be obtained by $c = \lim_{t \to -\infty} P(t, d)$.) Hence, we obtain a corollary that generalizes the results of Guibas and Yao [5], and of Nussbaum and Sack [6]:

COROLLARY 1. *Every collection of streamlined objects is separable in a steady flow defined over the entire plane* $\mathbb{R}^2$. ∎

Uniform flows are the simplest examples of such flows. They are a particular case of the flows of velocity fields of the form $V(x, y) = (r, qye^{qx})$, $r > 0$, the streamlines of which are generally non-rectilinear for $q \neq 0$. An even more general corollary is the following:

COROLLARY 2. *Every collection of streamlined objects is separable in a steady flow defined in a simply connected region of the plane* $\mathbb{R}^2$. ∎

Examples are restrictions of a central velocity field (flow proceeding along straight streamlines from a single source) to points in the plane in a given angular sector (bounded by two straight line rays emanating from the source). The angular sector may be wider than 180°. In this sense, the motion rules under which sequential separability can be guaranteed are more relaxed here than in [4].

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