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DEFORMATIONS OF SECONDARY CLASSES FOR SUBFOLIATIONS

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ABSTRACT. The purpose of this paper is to study the rigidity and deformations of secondary characteristic classes for subfoliations.

1. **Introduction.** Let M be an n-dimensional manifold, TM its tangent bundle. A (q_1, q_2)-codimensional subfoliation on M is a couple (F_1, F_2) of integrable subbundles F_i of TM of dimension $n - q_i$, $i = 1, 2$, and such that $F_2 \subset F_1$. Feigin [9], Cordero-Masa [3], Carballés [1], Wolak [16] and the author ([4], [5], [6], [7], [8]) have studied the secondary characteristic classes for subfoliations.

In this paper, using the techniques of Cordero-Masa [3], we discuss the rigidity and deformations of secondary characteristic classes for subfoliations. This generalizes the result of Heitsch [11] on the rigidity of secondary characteristic classes of a foliation under one-parameter deformations.

In Section 2 we prove the rigidity theorem for subfoliations which generalizes Heitsch's rigidity theorem [11] for foliations. This result is then applied in Section 3 to see which classes of the Vey basis of $H^*(W0_I)$ (as defined in [4]) are rigid. It follows in particular that the Godbillon-Vey classes for subfoliations of codimension (q_1, q_2) are variable. A similar result holds for subfoliations with trivialized normal bundle (in the sense of [3]).

The variable classes are used in [8] to prove that the homology group $H_{2q_2+2}(B\Gamma_{(q_1,q_2)}; Z)$ admits an epimorphism onto Euclidean space, where $B\Gamma_{(q_1,q_2)}$ is the Haefliger classifying space for subfoliations of codimension (q_1, q_2) (as defined in [7]).

Throughout the paper all objects are of type C^∞.

2. **Deformations of secondary classes for subfoliations.** In this section, using the techniques of [3], [4] and [7], we discuss the rigidity and deformations of secondary characteristic classes for subfoliations.

For any manifold M, TM denotes the tangent bundle of M, and $A^*(M)$ the algebra of differential forms on M. If (F_1, F_2) is a subfoliation (q_1, q_2)-codimensional on M, then Q_i denotes the normal bundle $\nu F_i = TM/F_i$ of F_i, $i = 1, 2$, Q_0 the quotient bundle F_1/F_2, and $\nu(F_1, F_2)$ the normal bundle $Q_1 \oplus Q_0$ of (F_1, F_2). If ∇^1 and ∇^0 are two connections on a vector bundle E over M with structure group $\mathrm{GL}(q) = \mathrm{GL}(q; R)$, and if

$$\phi = y_{i_1} \wedge \cdots \wedge y_{i_s} \otimes c_1^{j_1} \cdots c_q^{j_q} \in \Lambda(y_1, \ldots, y_q) \otimes R[c_1, \ldots, c_q]$$

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is an element (here the y_{i_α} are the relative suspensions of the Chern polynomials $c_{i_\alpha} \in I\big(\mathrm{GL}(q)\big) = R[c_1, \ldots, c_q]$ with $i_1 < \cdots < i_s$), then $\phi(\nabla^1, \nabla^0)$ denotes the differential form

$$\Delta(\nabla^1, \nabla^0)(c_{i_1}) \wedge \cdots \wedge \Delta(\nabla^1, \nabla^0)(c_{i_s}) \wedge c_1(\Omega^1)^{j_1} \wedge \cdots \wedge c_q(\Omega^1)^{j_q} \in A^*(M),$$

where Ω^1 is the curvature of ∇^1 and $\Delta(\nabla^1, \nabla^0)(c_{i_\alpha}) = \pi_*\big(c_{i_\alpha}(\Omega)\big)$, where Ω is the curvature of the connection $\nabla = t\nabla^1 + (1-t)\nabla^0$ on the vector bundle $E \times [0,1]$ over $M \times [0,1]$ and $\pi_*: A^r(M \times [0,1]) \longrightarrow A^{r-1}(M)$ denotes integration over the fiber of the disc bundle $M \times [0,1]$ over M.

Let (F_1, F_2) and (F, F_1) be subfoliations on M of codimension $(m + q_1, m + q_2)$ and $(m, m + q_1)$ respectively with $d = q_2 - q_1 \geqq 0$ and $m \geqq 1$. Let N be a leaf of F and $i_N: N \longrightarrow M$ the canonical immersion. Then the subfoliation (F_1, F_2) induces on N a (q_1, q_2)-codimensional subfoliation $(F_{1N}, F_{2N}) = (F_1 \mid_N, F_2 \mid_N)$. Analogously, the exact sequences of vector bundles

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_0 & \longrightarrow & Q_2 & \longrightarrow & Q_1 & \longrightarrow & 0 \\ 0 & \longrightarrow & F/F_1 & \longrightarrow & Q_1 & \longrightarrow & \nu F & \longrightarrow & 0 \end{array}$$

associated to (F_1, F_2) and (F, F_1) respectively with $\nu F = TM/F$, induce the following exact sequences of vector bundles over N:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_{0N} = F_{1N}/F_{2N} & \longrightarrow & Q_2 \mid_N & \longrightarrow & Q_1 \mid_N & \longrightarrow & 0 \\ 0 & \longrightarrow & Q_{1N} = TN/F_{1N} & \longrightarrow & Q_1 \mid_N & \longrightarrow & \nu N & \longrightarrow & 0 \end{array}$$

where $\nu N = \nu F \mid_N$ is the normal bundle of the leaf N of F. It is easy to verify that the vector bundle $(F/F_1) \oplus Q_0$ over M is canonically (F_1, F_2)-foliated and that the canonical (F_{1N}, F_{2N})-foliated bundle structure of the normal bundle $\nu(F_{1N}, F_{2N}) = Q_{1N} \oplus Q_{0N}$ of the subfoliation (F_{1N}, F_{2N}) on N is induced by the canonical (F_1, F_2)-foliated bundle structure of $(F/F_1) \oplus Q_0$.

LEMMA 2.1. *The following diagram is commutative*

$$\begin{array}{ccc} H^*(W0_{l'}) & \xrightarrow{\Delta_{*(F_1,F_2)}} & H^*_{DR}(M) \\ \big\downarrow W(d\rho)^* & & \big\downarrow i_N^* \\ H^*(W0_l) & \xrightarrow{\Delta_{*(F_{1N},F_{2N})}} & H^*_{DR}(N) \end{array}$$

where $\Delta_{*(F_1,F_2)}$ and $\Delta_{*(F_{1N},F_{2N})}$ are the characteristic homomorphisms of (F_1, F_2) and (F_{1N}, F_{2N}) respectively (as defined in [3]), $W0_{l'}$ (resp. $W0_l$) is the complex corresponding to the pair $(m + q_1, m + q_2)$ (resp. (q_1, q_2)), and $W(d\rho)^*$ denotes the homomorphism induced by the canonical inclusion

$$\rho: \mathrm{GL}(q_1) \times \mathrm{GL}(d) \longrightarrow \big(\mathrm{GL}(m) \times \mathrm{GL}(q_1)\big) \times \mathrm{GL}(d) \longrightarrow \mathrm{GL}(m + q_1) \times \mathrm{GL}(d).$$

PROOF. Let $z_{(i,i',j,j')} = y_{(i)} \wedge y'_{(i')} \otimes c_{(j)}c'_{(j')} \in W0_{l'}$ be a cocycle of the Vey basis (see [4]). Denote by ϕ (resp. by ϕ') the element $y_{(i)} \otimes c_{(j)} = y_{i_1} \wedge \cdots \wedge y_{i_s} \otimes$

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$c_1^{j_1} \cdots c_{m+q_1}^{j_{m+q_1}} \in \Lambda(y_1, y_3, \ldots) \otimes R[c_1, c_2, \ldots, c_{m+q_1}]$ (resp. $y'_{(i')} \otimes c'_{(j')} = y'_{i'_1} \wedge \cdots \wedge y'_{i'_{s'}} \otimes c_1'^{j'_1} \cdots c_d'^{j'_d} \in \Lambda(y'_1, y'_3, \ldots) \otimes R[c'_1, c'_2, \ldots, c'_d]$) with $i_1 < \cdots < i_s$ and $i'_1 < \cdots < i'_{s'}$, where the y_i (resp. the y'_i) are the relative suspensions of the odd Chern polynomials $c_i \in I\big(\mathrm{GL}(m+q_1)\big) = R[c_1, \ldots, c_{m+q_1}]$ (resp. $c'_i \in I\big(\mathrm{GL}(d)\big) = R[c'_1, \ldots, c'_d]\big)$, $\deg c_i = \deg c'_i = 2i$ and $\deg y_i = \deg y'_i = 2i - 1$. Then $z_{(i,i',j,j')} = \phi \cdot \phi' \in W0_{l'}$. Consider now the exact sequence of vector bundles

$$0 \longrightarrow F/F_1 \xrightarrow{i} Q_1 \xrightarrow{\pi} \nu F \longrightarrow 0$$

associated to (F, F_1). By Theorem 3.3 in [3], we can choose basic connections ∇'_1, ∇_1, ∇_F (analogously, Riemannian connections $\nabla_1'^r$, ∇_1^r, ∇_F^r) on F/F_1, Q_1 and νF respectively, and such that they are compatible with the homomorphisms i and π (in the sense of [3]). Therefore, $\nabla' = \nabla_F \oplus \nabla'_1$ is a basic connection and $\nabla'^r = \nabla_F^r \oplus \nabla_1'^r$ is a Riemannian connection on $\nu F \oplus (F/F_1)$ (in the sense of [3]).

Let $\sigma_1, \ldots, \sigma_{m+q_1}$ be a local framing of Q_1 such that $\pi(\sigma_1), \ldots, \pi(\sigma_m)$ is a local framing of νF and $\sigma_{m+1}, \ldots, \sigma_{m+q_1}$ is a local framing of F/F_1. An easy computation shows that with respect to the local framing $\sigma_1, \ldots, \sigma_{m+q_1}$, the local connection forms θ_1 and θ_1^r of ∇_1 and ∇_1^r are given by

$$\theta_1 = \begin{bmatrix} \theta_F & 0 \\ * & \theta'_1 \end{bmatrix},$$

$$\theta_1^r = \begin{bmatrix} \theta_F^r & 0 \\ * & \theta_1'^r \end{bmatrix},$$

respectively, where θ_F and θ_F^r (resp. θ'_1 and $\theta_1'^r$ are the local connection forms of ∇_F and ∇_F^r (resp. of ∇'_1 and $\nabla_1'^r$) with respect to the local framing $\pi(\sigma_1), \ldots, \pi(\sigma_m)$ (resp. $\sigma_{m+1}, \ldots, \sigma_{m+q_1}$). Hence we have

$$\phi(\nabla_1, \nabla_1^r) = \phi(\nabla', \nabla'^r) \in A^*(M). \tag{2.2}$$

Now, let ∇_0 be a basic connection and ∇_0^r a Riemannian connection on Q_0. Then $\nabla^b = \nabla_1 \oplus \nabla_0$ (resp. $\nabla^r = \nabla_1^r \oplus \nabla_0^r$) is a basic connection (resp. a Riemannian connection) on $\nu(F_1, F_2) = Q_1 \oplus Q_0$, and we can use ∇^b and ∇^r to compute the characteristic homomorphism $\Delta_{*(F_1,F_2)}$ of (F_1, F_2) (see [3]). From (2.2) it follows that the cohomology class $\Delta_{*(F_1,F_2)}[z_{(i,i',j,j')}] \in H^*_{DR}(M)$ is represented by the closed form

$$\phi(\nabla_1, \nabla_1^r) \wedge \phi'(\nabla_0, \nabla_0^r) = \phi(\nabla', \nabla'^r) \wedge \phi'(\nabla_0, \nabla_0^r) \in A^*(M).$$

Next, consider the canonical immersion $i_N : N \longrightarrow M$. Then $\nabla_N = i_N^*(\nabla_F)$ (resp. $\nabla_N^r = i_N^*(\nabla_F^r)$) is the natural flat connection (resp. a Riemannian connection) on νN, and $\nabla_N^b = \nabla_{1N} \oplus \nabla_{0N}$ (resp. $\overline{\nabla}^r_N = \nabla_{1N}^r \oplus \nabla_{0N}^r$)is a basic connection (resp. a Riemannian connection) on $\nu(F_{1N}, F_{2N}) = Q_{1N} \oplus Q_{0N}$, where $\nabla_{1N} = i_N^*(\nabla'_1)$, $\nabla_{0N} = i_N^*(\nabla_0)$, $\nabla_{1N}^r = i_N^*(\nabla_1'^r)$ and $\nabla_{0N}^r = i_N^*(\nabla_0^r)$. Whence, we can use ∇_N^b and $\overline{\nabla}^r_N$ to compute the characteristic homomorphism $\Delta_{*(F_{1N},F_{2N})}$ of (F_{1N}, F_{2N}). Denote by ∇'_N (resp. by $\nabla_N'^r$) the

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connection $\nabla_N \oplus \nabla_{1N}$ (resp. the Riemannian connection $\nabla^r_N \oplus \nabla^r_{1N}$) on $\nu N \oplus Q_{1N}$. By (2.2) we have then

$$(2.3)\qquad i^*_N\big(\phi(\nabla_1, \nabla^r_1) \wedge \phi'(\nabla_0, \nabla^r_0)\big) = \phi(\nabla'_N, \nabla'^r_N) \wedge \phi'(\nabla_{0N}, \nabla^r_{0N}) \in A^*(N).$$

In order to compute the differential form $\phi(\nabla'_N, \nabla'^r_N)$ we consider the restriction homomorphism

$$\begin{array}{ccc} I\big(\mathrm{GL}(m+q_1)\big) & \xrightarrow{\rho_1^*} & I\big(\mathrm{GL}(m)\big) \otimes I\big(\mathrm{GL}(q_1)\big) \\ \| & & \| \\ R[c_1, \ldots, c_{m+q_1}] & \longrightarrow & R[c'_1, \ldots, c'_m] \otimes R[c_1, \ldots, c_{q_1}] \end{array}$$

given by

$$(2.4)\qquad \rho_1^* c_i = \sum_{j=0}^{i} c'_j c_{i-j}, \quad i = 1, \ldots, m + q_1$$

with $c'_0 = 1$, $c_0 = 1$, $c'_i = 0$ for $i > m$, and $c_i = 0 \in I\big(\mathrm{GL}(q_1)\big)$ for $i > q_1$, where the c_i and c'_i denote the Chern polynomials. By (2.4) we have

$$(2.5)\qquad \rho_1^* c_{2i-1} = \sum_{k=1}^{i} (c'_{2k-1} c_{2(i-k)} + c'_{2(i-k)} c_{2k-1}),$$

$i = 1, \ldots, [(m + q_1 + 1)/2]$. Now, denote by Ω'_N, Ω_N and Ω_{1N} the curvatures of ∇'_N, ∇_N and ∇_{1N} respectively. Since $\Omega_N = 0$, it follows from (2.4) that

$$(2.6)\qquad c_{(j)}(\Omega'_N) = c_{(j)}(\Omega_{1N}),$$

where $c_{(j)} \in I\big(\mathrm{GL}(m + q_1)\big)$ on the left, and $c_{(j)} \in I\big(\mathrm{GL}(q_1)\big)$ on the right. On the other hand, using (2.5), we obtain by an easy computation the formula

$$(2.7)\qquad \begin{aligned} \Delta(\nabla'_N, \nabla'^r_N)(c_{2i-1}) = {} & \sum_{k=1}^{i} \Delta(\nabla_N, \nabla^r_N)(c'_{2k-1}) \wedge c_{2(i-k)}(\Omega_{1N}) \\ & + \Delta(\nabla_{1N}, \nabla^r_{1N})(c_{2i-1}) + \text{ exact}. \end{aligned}$$

Now, if $2k-1 \leqq m$, and if $2i-1+p_1 > m+q_1$ or $2i-1+p > m+q_2$, then $2(i-k)+p_1 > q_1$ or $2(i-k)+p > q_2$, where $2p_1 = \deg c_{(j)}$, $2p_2 = \deg c'_{(j')}$ and $p = p_1 + p_2$. Hence, by (2.3), (2.6) and (2.7) it follows that

$$\begin{aligned} i^*_N(\Delta_{*(F_1,F_2)}[z_{(i,i',j,j')}]) &= [\phi(\nabla'_N, \nabla'^r_N) \wedge \phi'(\nabla_{0N}, \nabla^r_{0N})] \\ &= [(\bigwedge_{\alpha=1}^{s} \Delta(\nabla_{1N}, \nabla^r_{1N})(c_{i_\alpha})\big) \wedge c_{(j)}(\Omega_{1N}) \wedge \phi'(\nabla_{0N}, \nabla^r_{0N})] \\ &= \Delta_{*(F_{1N},F_{2N})}\big(W(d\rho)^*[z_{(i,i',j,j')}]\big). \qquad \blacksquare \end{aligned}$$

REMARKS. 1) In the previous results, the leaf N of F can be replaced by any $(n-m)$-dimensional integral manifold of F, where n is the dimension of the manifold M.

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2) A similar result holds for subfoliations with trivialized normal bundle (in the sense of [3]).

Let $f: M \longrightarrow X$ be a submersion, where X is a manifold of dimension $m \geqq 1$. Consider now the case where F is the tangent bundle $T(f)$ along the fibers of f. Then the $(m+q_1, m+q_2)$-codimensional subfoliation (F_1, F_2) on M can be considered as a deformation of the subfoliations $(F_{1x}, F_{2x}) = (F_{1N_x}, F_{2N_x})$ of codimension (q_1, q_2) on the fibers $N_x = f^{-1}(x)$, $x \in f(M) \subset X$. Then, from Lemma 2.1 we obtain the following result.

THEOREM 2.8. *For every $x \in f(M) \subset X$, the following diagram is commutative*

$$\begin{array}{ccc} H^*(W0_{I'}) & \xrightarrow{\Delta_{*(F_1,F_2)}} & H^*_{DR}(M) \\ \Big\downarrow W(d\rho)^* & & \Big\downarrow i^*_x \\ H^*(W0_I) & \xrightarrow{\Delta_{*(F_{1x},F_{2x})}} & H^*_{DR}(N_x) \end{array}$$

where $W(d\rho)^*$ is as in Lemma 2.1 and $i_x: N_x = f^{-1}(x) \longrightarrow M$ denotes the canonical inclusion.

Let N be a manifold and X an m-dimensional connected manifold with $m \geqq 1$. Assume now that $M = N \times X$ and that $f: M \longrightarrow X$ is the canonical projection. Then the homomorphism $i^*_x: H^*_{DR}(M) \longrightarrow H^*_{DR}(N)$ induced by the canonical inclusion $i_x: N \cong N \times \{x\} = f^{-1}(x) \longrightarrow M = N \times X$ does not depend on the choice of $x \in X$. From Theorem 2.8 it follows then that the classes

$$\Delta_{*(F_{1x},F_{2x})}(u) \in H^*_{DR}(N) \text{ for } u \in \operatorname{Im} W(d\rho)^* \subset H^*(W0_I)$$

do not depend on the choice of $x \in X$. Hence, we have

COROLLARY 2.9. *The classes $\Delta_{*(F_{1x},F_{2x})}(u)$, $u \in \operatorname{Im} W(d\rho)^*$, are rigid for $m \geqq 1$.*

REMARK. This generalizes the result of Heitsch [11] on the rigidity of secondary characteristic classes of a foliation under one-parameter deformations. That is the case where $F_1 = F_2$, $q_1 = q_2$, $m = 1$ and $f: M = N \times R \longrightarrow R$ is the canonical projection.

Let $\Delta_*: H^*(W0_I) \longrightarrow H^*(B\Gamma; R)$ and $\Delta'_*: H^*(W0_{I'}) \longrightarrow H^*(B\Gamma'; R)$ be the universal characteristic homomorphisms for subfoliations of codimension (q_1, q_2) and $(m+q_1, m+q_2)$ respectively (as defined in [7]), where $B\Gamma$ (resp. $B\Gamma'$) denotes the Haefliger classifying space for subfoliations of codimension (q_1, q_2) (resp. $(m+q_1, m+q_2)$). Then the following is easily verified.

THEOREM 2.10. *There is a commutative diagram*

$$\begin{array}{ccc} H^*(W0_{I'}) & \xrightarrow{\Delta'_*} & H^*(B\Gamma'; R) \\ \Big\downarrow W(d\rho)^* & & \Big\downarrow i^* \\ H^*(W0_I) & \xrightarrow{\Delta_*} & H^*(B\Gamma; R) \end{array}$$

with canonical vertical homomorphisms.

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3. **Results on $H^*(W0_l)$.** *In order to see which classes of the Vey basis of $H^*(W0_l)$ are rigid, we consider the homomorphism $W(d\rho)^*: H^*(W0_{l'}) \longrightarrow H^*(W0_l)$ induced by the DG-algebra homomorphism $W(d\rho): W0_{l'} \longrightarrow W0_l$ given by*

$$W(d\rho)(c_i) = \begin{cases} c_i & \text{for } 1 \leqq i \leqq q_1, \\ 0 & \text{for } q_1 + 1 \leqq i \leqq m + q_1, \end{cases}$$

$$W(d\rho)(y_i) = \begin{cases} y_i & \text{for } 1 \leqq i \leqq q_1,\ i \text{ odd}, \\ 0 & \text{for } q_1 + 1 \leqq i \leqq m + q_1,\ i \text{ odd} \end{cases}$$

$$W(d\rho)(c'_i) = c'_i \text{ for } 1 \leqq i \leqq d,$$

$$W(d\rho)(y'_i) = y'_i \text{ for } 1 \leqq i \leqq d,\ i \text{ odd}.$$

Then, from Theorem 2.8 and Corollary 2.9 we obtain for $m = 1$ the following result.

THEOREM 3.1. *Let the notation be as in [4]. Consider in $H^*(W0_l)$ the cohomology classes $[z_{(i,i',j,j')}]$ of the cocycles $z_{(i,i',j,j')} = y_{(i)} \wedge y'_{(i')} \otimes c_{(j')} c'_{(j')} \in W0_l$ of the Vey basis with* $\deg c_{(j)} = 2p_1$, $\deg c'_{(j')} = 2p_2$ *and $p = p_1 + p_2$. Then we have*

(i) An R-basis of the rigid classes of $H^*(W0_l)$ is given by the elements $[z_{(i,i',j,j')}]$ of the Vey basis of $H^*(W0_l)$ satisfying

$$i_0 + p_1 \geqq q_1 + 2 \text{ or } i_0 + p \geqq q_2 + 2, \text{ and } i'_0 + p \geqq q_2 + 2.$$

(ii) The elements $[z_{(i,i',j,j')}]$ of the Vey basis of $H^*(W0_l)$ satisfying at least one of the following conditions:

(a) $i_0 + p_1 = q_1 + 1,\ i_0 + p \leqq q_2 + 1,\ i'_0 + p \geqq q_2 + 1;$
(b) $i_0 + p_1 \leqq q_1 + 1,\ i_0 + p = q_2 + 1,\ i'_0 + p \geqq q_2 + 1;$
(c) $i_0 + p_1 \geqq q_1 + 1,\ i'_0 + p = q_2 + 1;$
(d) $i_0 + p \geqq q_2 + 1,\ i'_0 + p = q_2 + 1$

are the only elements of the Vey basis of $H^*(W0_l)$ which do not belong to $\operatorname{Im} W(d\rho)^* \subset H^*(W0_l)$. *Thus these secondary classes are variable.*

COROLLARY 3.2. *The Godbillon-Vey classes $[y_1 \otimes c_1^{q_1}] \in H^{2q_1+1}(W0_l)$, $[y'_1 \otimes c_1^j c_1'^{q_2-j}] \in H^{2q_2+1}(W0_l)$ and $[y_1 \wedge y'_1 \otimes c_1^j c_1'^{q_2-j}] \in H^{2q_2+2}(W0_l)$, $0 \leqq j \leqq q_1$, for subfoliations of codimension (q_1, q_2) are variable.*

REMARKS. 1) Similar results hold for subfoliations with trivialized normal bundle.

2) For $q_1 = q_2 = q$, we have the result of Heitsch [11].

3) The computations for some examples of subfoliations with variable classes are given in [8].

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