ON ZEROS OF DERIVATIVES OF POLYNOMIALS

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1. In an earlier paper [2], we raised the question of determining the minimum span of the k^{th} derivative of a polynomial with real zeros having a given span. More precisely let $\pi_{n,s}$ denote the class of polynomials $P(x) = C \prod_{i=1}^{n} (x - x_i)$, with $x_1 \le x_2 \le \dots \le x_n$, and the span $\sigma(P) \equiv x_n - x_1 = 2s$ (fixed). The problem is to determine

$$(1) \qquad \min_{P \in \pi_{n,s}} \sigma(P^{(k)}) , \quad k = 1, 2, \dots, n - 2.$$

We showed that if $k = n - 2$, the above minimum is attained only for the polynomials of the form $C\{(x - a)^2 - s^2\} (x - a)^{n-2}$, a real. In a recent paper, Ahmad [1] has obtained an upper estimate for (1) when $k = 1$.

Here we obtain the exact value of (1) for all k and determine the polynomials for which this is attained.

2. THEOREM. In the notation of section 1, for $k = 1, 2, \dots, n-2$, we have

$$(2) \qquad \min_{P \in \pi_{n,s}} \sigma(P^{(k)}) = 2s \cdot \left(\frac{n - k}{n} \cdot \frac{n - k - 1}{n - 1} \right)^{\frac{1}{2}}$$

and this is attained only for polynomials of the form

$$C(x - a)^{n-2} \{(x - a)^2 - s^2\} ,$$

with a and C real.

Proof. It will be enough to prove (2) for $k = 1$ and $n \ge 3$. For if (2) has been proved for $k = 1$ and all $n \ge 3$, then on successively applying it to $P^{(j-1)} (j = 1, 2, \dots, k)$ we have

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$$\sigma(P^{(j)}) \geq \sigma(P^{(j-1)}) \cdot \left(\frac{n-j-1}{n-j+1}\right)^{\frac{1}{2}}, \quad j = 1, 2, \ldots, k,$$

whence

(3) $$\sigma(P^{(k)}) \geq \sigma(P) \left(\prod_{j=1}^{k} \frac{n-j-1}{n-j+1}\right)^{\frac{1}{2}} = \sigma(P) \left(\frac{n-k}{n} \cdot \frac{n-k-1}{n-1}\right)^{\frac{1}{2}}$$

with equality only if $\sigma(P') = \sigma(P) \cdot \left(\frac{n-2}{n}\right)^{\frac{1}{2}}$. So we proceed to prove (2) for $k = 1$. We may assume, without loss of generality, that $s = 1$ and that the zeros of $P(x) \in \pi_{n,1}$ are located in $[-1, 1]$. Then

$$P(x) = C(x^2 - 1) \prod_{i=1}^{n-2} (x - x_i), \quad -1 \leq x_1 \leq x_2 \leq \ldots \leq x_{n-2} \leq 1.$$

It is clear that the minimum span of P' cannot be attained if $x_1 = -1$ and $x_{n-2} = +1$. So denoting the zeros of $P'(x)$ by $\xi_1 \leq \ldots \leq \xi_{n-1}$, it follows from Rolle's theorem that either (i) both ξ_1 and ξ_{n-1} are the extreme solutions of

(4) $$\frac{P'(\xi)}{P(\xi)} \equiv \frac{2\xi}{\xi^2 - 1} + \sum_{i=1}^{n-2} \frac{1}{\xi - x_i} = 0$$

or (ii) one of them is fixed (say $\xi_{n-1} = +1$) and the other ξ_1 is the smallest root of (4).

We will show that the minimum span of P' cannot be attained when $x_1 < x_{n-2}$. For if $x_1 < x_{n-2}$, then differentiating ξ_1 partially with respect to x_1 on the hyperplane $x_1 + x_{n-2} = d$ (d constant), we have from (4),

$$\frac{\partial \xi_1}{\partial x_1} \cdot \left[\sum_{i=1}^{n-2} \frac{1}{(\xi_1 - x_i)^2} + \frac{2(\xi_1^2 + 1)}{(\xi_1^2 - 1)^2} \right] = \frac{1}{(\xi_1 - x_1)^2} - \frac{1}{(\xi_1 - x_{n-2})^2} .$$

Since $\xi_1 < x_1 < x_{n-2}$, we see from the above that $\frac{\partial \xi_1}{\partial x_1} > 0$.

A similar argument shows that in case (i) as well as in (ii) $\frac{\partial \xi_{n-1}}{\partial x_1} \leq 0$, so that

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$$\frac{\partial}{\partial x_1}(\sigma(P')) = \frac{\partial \xi_{n-1}}{\partial x_1} - \frac{\partial \xi_1}{\partial x_1} < 0 .$$

This shows that the minimum span can occur only if $x_1 = x_2 \dots = x_{n-2} = \alpha$, $|\alpha| \leq 1$.

We now show that we must necessarily have $\alpha = 0$. Since if $P(x) = C(x^2 - 1)(x - \alpha)^{n-2}$, then $\sigma(P') = 2\sqrt{\frac{\alpha^2 + n(n-2)}{n^2}}$ which can be further decreased if $\alpha \neq 0$. This completes the proof of theorem.

The problem of computing the maximum span of derivatives of polynomials was first investigated by Robinson[4].

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