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ON TAILS OF PERPETUITIES

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Abstract

We establish an upper bound on the tails of a random variable that arises as a solution of a stochastic difference equation. In the nonnegative case our bound is similar to a lower bound obtained in Goldie and Grübel (1996).

Keywords: Perpetuity; stochastic difference equation; tail behavior

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1. Introduction

A random variable R satisfying the distributional identity

$$R \stackrel{\mathrm{D}}{=} MR + Q, \tag{1}$$

where (M, Q) are independent of R on the right-hand side and '$\stackrel{\mathrm{D}}{=}$' denotes equality in distribution, is referred to as perpetuity and plays an important role in applied probability. The main reason for this is that it appears as a limit in distribution of a sequence (R_n) given by

$$R_n \stackrel{\mathrm{D}}{=} M_n R_{n-1} + Q_n, \quad n \geq 1,$$

provided that the limit exists. (Here, (M_n, Q_n) is a sequence of independent and identically distributed (i.i.d.) random vectors distributed like (M, Q) and R_0 could be an arbitrary random variable; for convenience, we will set $R_0 = 0$.) A systematic study of the properties of such sequences was initiated by Kesten [5], and such studies continue to this day. Once the convergence in distribution of (R_n) is established, at the center of the investigation is the tail behavior of R. There are two distinctly different cases:

$$\mathrm{P}(|M| > 1) > 0 \quad \text{and} \quad \mathrm{P}(|M| \leq 1) = 1.$$

The first case results in R having a heavy tail distribution, that is,

$$\mathrm{P}(|R| > x) \sim Cx^{-\kappa}$$

for a suitably chosen constant κ and some constant C (see the original paper of Kesten [5] or [2]), while in the second case the tails of R are no heavier than exponential. This was observed by Goldie and Grübel [3]. Some subsequent work is given in [4], but the full picture in this case is not complete. The purpose of this note is to shed some additional light on this case by establishing a universal upper bound on the tails of $|R|$. In a special, but important, situation when Q and M (and, thus, also R) are nonnegative our bound is comparable to a lower bound obtained by Goldie and Grübel [3].

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2. Bounds on the tails

For a random variable M such that $|M| \le 1$ and $0 < \delta < 1$, define $p_\delta := \mathrm{P}(1-\delta \le |M| \le 1)$. Then, as has been shown in [3] (see also Equation (2.2) of [4]), if $0 \le M \le 1$ and $Q \equiv q$ (q being a positive constant), then, for $0 < c < 1$ and $x > q$, we have

$$\mathrm{P}(R > x) \ge \exp\biggl(\frac{\ln(1-c)}{\ln(1-cq/x)} \ln p_{cq/x}\biggr).$$

Since $\ln(1 - cq/x) \le -cq/x$ for any particular value of c, say $c = \frac{1}{2}$, this immediately gives

$$\mathrm{P}(R > x) \ge \exp\biggl(-\frac{\ln(1-c)}{cq} x \ln(p_{cq/x})\biggr) = \exp\biggl(\frac{2\ln 2}{q} x \ln p_{q/(2x)}\biggr).$$

Our aim here is to supply an upper bound of a similar form. While our result does not give the asymptotics of $\mathrm{P}(R > x)$ as $x \to \infty$, it shows that it essentially behaves like $\exp(c_1 x \ln(p_{c_2 q/x})/q)$ for some positive constants c_1 and c_2. Specifically, we prove the following result.

Proposition 1. *Assume that $|Q| \le q$ and $|M| \le 1$, and let R be given by (1). Then, for sufficiently large x,*

$$\mathrm{P}(|R| > x) \le \exp\biggl(\frac{1}{4q} x \ln p_{2q/x}\biggr).$$

Thus, if $Q \equiv q > 0$ and $0 \le M \le 1$, then

$$\exp\biggl(\frac{2\ln 2}{q} x \ln p_{q/(2x)}\biggr) \le \mathrm{P}(R > x) \le \exp\biggl(\frac{1}{4q} x \ln p_{2q/x}\biggr).$$

Proof. If $\mathrm{P}(|M| = 1) > 0$ then, as was proved in [3], R has tails bounded by those of an exponential variable, so we assume that $|M|$ has no atom at 1. Fix $0 < \delta < 1$, and define a sequence (T_k) as follows:

$$T_0 = 0, \qquad T_m = \inf\{k \ge 1 \colon |M_{T_{m-1}+k}| \le 1-\delta\}, \quad m \ge 1.$$

Then the T_ks are i.i.d. random variables, each having a geometric distribution with parameter $1 - p_\delta$. Furthermore, $|M_k| \le 1-\delta$ if $k = T_1 + \cdots + T_i$ for some $i \ge 1$ and $|M_k| \le 1$ otherwise. Therefore,

$$\prod_{k=1}^{m} |M_k| \le (1-\delta)^j \quad \text{for } T_1 + \cdots + T_j \le m < T_1 + \cdots + T_j + T_{j+1}.$$

This in turn implies that

$$\biggl|\sum_{k\ge 1}\prod_{j=1}^{k-1} M_j\biggr| \le \sum_{k\ge 1}\prod_{j=1}^{k-1} |M_j| \le T_1 + (1-\delta)T_2 + (1-\delta)^2 T_3 + \cdots = \sum_{k\ge 1}(1-\delta)^{k-1} T_k.$$

Therefore, if $|Q| \le q$, we obtain

$$\mathrm{P}(|R| > x) \le \mathrm{P}\biggl(\sum_{k\ge 1}\prod_{j=1}^{k-1} |M_j| \ge \frac{x}{q}\biggr) \le \mathrm{P}\biggl(\sum_{k\ge 1} T_k (1-\delta)^{k-1} \ge \frac{x}{q}\biggr). \tag{2}$$

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To bound the latter probability, we use a widely known argument (our calculations follow [1, Proof of Proposition 2]). First, if T is a geometric variable with parameter $1-p$ then

$$\mathrm{E}\,\mathrm{e}^{\lambda T} = \sum_{j=1}^{\infty} \mathrm{e}^{\lambda j}\,\mathrm{P}(T=j) = \sum_{j=1}^{\infty} \mathrm{e}^{\lambda j} p^{j-1}(1-p) = \frac{\mathrm{e}^{\lambda}(1-p)}{1-\mathrm{e}^{\lambda}p} = \frac{\mathrm{e}^{\lambda}}{1-p(\mathrm{e}^{\lambda}-1)/(1-p)},$$

provided that $\mathrm{e}^{\lambda}p < 1$. Thus, writing t in place of x/q on the right-hand side of (2), for $\lambda > 0$, we have

$$\begin{aligned}\mathrm{P}\Big(\sum_{k\geq 1}(1-\delta)^{k-1}T_k \geq t\Big) &= \mathrm{P}\Big(\exp\Big(\lambda\sum_{k\geq 1}(1-\delta)^{k-1}T_k\Big) \geq \mathrm{e}^{\lambda t}\Big)\\ &\leq \mathrm{e}^{-\lambda t}\,\mathrm{E}\exp\Big(\lambda\sum_{k\geq 1}T_k(1-\delta)^{k-1}\Big).\end{aligned}$$

If λ satisfies $\mathrm{e}^{\lambda}p < 1$ then $p\mathrm{e}^{\lambda(1-\delta)^{k-1}} < 1$ for every $k \geq 1$ as well, and, by the independence of (T_k), the expectation on the right-hand side is

$$\prod_{k=1}^{\infty} \frac{\mathrm{e}^{\lambda(1-\delta)^{k-1}}}{1-p(\mathrm{e}^{\lambda(1-\delta)^{k-1}}-1)/(1-p)} = \mathrm{e}^{\lambda/\delta}\prod_{k=1}^{\infty}\frac{1}{1-p(\mathrm{e}^{\lambda(1-\delta)^{k-1}}-1)/(1-p)}. \tag{3}$$

Now, choose $\lambda > 0$ so that $p(\mathrm{e}^{\lambda}-1)/(1-p) \leq \frac{1}{2}$. Then, as $1/(1-u) \leq \mathrm{e}^{2u}$ for $0 \leq u \leq \frac{1}{2}$, for every $k \geq 1$, we obtain

$$\frac{1}{1-p(\mathrm{e}^{\lambda(1-\delta)^{k-1}}-1)/(1-p)} \leq \exp\Big(2\frac{p}{1-p}(\mathrm{e}^{\lambda(1-\delta)^{k-1}}-1)\Big).$$

Therefore, the rightmost product in (3) is bounded by

$$\exp\Big(2\frac{p}{1-p}\sum_{k\geq 1}(\mathrm{e}^{\lambda(1-\delta)^{k-1}}-1)\Big).$$

We bound the sum in the exponent as follows:

$$\begin{aligned}\sum_{k\geq 1}\sum_{j\geq 1}\frac{\lambda^j(1-\delta)^{(k-1)j}}{j!} &= \sum_{j\geq 1}\frac{\lambda^j}{j!}\sum_{k\geq 1}(1-\delta)^{j(k-1)}\\ &= \sum_{j\geq 1}\frac{\lambda^j}{j!}\frac{1}{1-(1-\delta)^j}\\ &\leq \frac{1}{\delta}\sum_{j\geq 1}\frac{\lambda^j}{j!}\\ &= \frac{\mathrm{e}^{\lambda}-1}{\delta}.\end{aligned}$$

Combining the above estimates we obtain

$$\mathrm{P}(|R| > qt) \leq \exp\Big(-t\lambda + \frac{\lambda}{\delta} + \frac{2p}{1-p}\frac{\mathrm{e}^{\lambda}-1}{\delta}\Big), \tag{4}$$

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provided that λ satisfies the required conditions, that is,

$$e^{\lambda} p < 1 \quad \text{and} \quad \frac{p}{1-p}(e^{\lambda} - 1) \leq \frac{1}{2}.$$

Clearly, both are satisfied when $e^{\lambda} p \leq \frac{1}{2}$.

We complete the proof by making a suitable choice of λ. Since we are assuming that $|M|$ has no atom at 1 and we are interested in large x, we may assume that δ is small enough so that $p_\delta < \frac{1}{3}$. This condition implies that $2p_\delta/(1-p_\delta) < 3p_\delta$, so that the last term in the exponent of (4) is bounded by $3p_\delta(e^{\lambda}-1)/\delta$. Now let $t = 2/\delta$. Then (4) becomes

$$P(|R| > qt) \leq \exp\biggl(-\lambda\frac{2}{\delta} + \frac{\lambda}{\delta} + \frac{2p_\delta}{1-p_\delta}\frac{e^{\lambda}-1}{\delta}\biggr) \leq \exp\biggl(-\frac{1}{\delta}(\lambda - 3p_\delta(e^{\lambda}-1))\biggr).$$

Set $\lambda = \ln(1/3p_\delta)$ so that $e^{\lambda} p_\delta = \frac{1}{3}$. This choice of λ is within the constraints and maximizes the value of $\lambda - 3p_\delta(e^{\lambda}-1)$, this maximal value being

$$\ln\biggl(\frac{1}{3p_\delta}\biggr) - 3p_\delta\biggl(\frac{1}{3p_\delta} - 1\biggr) = \ln\biggl(\frac{1}{p_\delta}\biggr) - (1 + \ln 3) + 3p_\delta \geq \frac{1}{2}\ln\biggl(\frac{1}{p_\delta}\biggr),$$

with the inequality valid for sufficiently small p_δ (less than $e^{-2}/9$ for example). Thus, using $t = 2/\delta$, we finally obtain

$$P(|R| > qt) \leq \exp\biggl(-\frac{1}{2\delta}\ln\biggl(\frac{1}{p_\delta}\biggr)\biggr) = \exp\biggl(\frac{t}{4}\ln p_{2/t}\biggr),$$

or, in terms of x,

$$P(|R| > x) \leq \exp\biggl(\frac{x}{4q}\ln p_{2q/x}\biggr).$$

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