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SOME SUMMATION THEOREMS AND TRANSFORMATIONS FOR Q-SERIES

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ABSTRACT. We introduce a double sum extension of a very well-poised series and extend to this the transformations of Bailey and Sears as well as the ${}_6\phi_5$ summation formula of F. H. Jackson and the q-Dixon sum. We also give q-integral representations of the double sum. Generalizations of the Nassrallah-Rahman integral are also found.

1**. Introduction.** One of the most important formulas in the theory of basic hypergeometric series is Bailey's transformation formula ([6], (2.10.10)):

$$(1.1)\quad {}_8\phi_7\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f \\ \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e, qa/f \end{matrix}; q, \frac{a^2q^2}{bcdef}\right)$$
$$= \frac{(qa, qa/de, qa/df, qa/ef; q)_\infty}{(qa/d, qa/e, qa/f, qa/def; q)_\infty} \times {}_4\phi_3\left(\begin{matrix} qa/bc, d, e, f \\ def/a, qa/b, qa/c \end{matrix}; q, q\right)$$
$$+ \frac{(qa, d, e, f, q^2a^2/bdef, q^2a^2/cdef, qa/bc; q)_\infty}{(qa/b, qa/c, qa/d, qa/e, qa/f, def/qa, q^2a^2/bcdef; q)_\infty} \times {}_4\phi_3\left(\begin{matrix} qa/de, qa/df, qa/ef, q^2a^2/bcdef \\ q^2a^2/bdef, q^2a^2/cdef, q^2a/def \end{matrix}; q, q\right).$$

The ${}_8\phi_7$ series on the left and the two ${}_4\phi_3$ series on the right hand side are special cases of the basic hypergeometric series ${}_r\phi_s$ defined by

$$(1.2)\quad {}_r\phi_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z\right) = {}_r\phi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, z)$$
$$:= \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(q, b_1, \dots, b_s; q)_n} [(-1)^n q^{n(n-1)/2}]^{1+s-r} z^n,$$

where

$$(1.3)\quad (a; q)_n := \prod_{k=1}^{n} (1 - aq^{k-1}),$$

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$$(a_1, a_2, \ldots, a_r; q)_n := \prod_{k=1}^{r} (a_k; q)_n,$$

and q is a complex parameter which is usually assumed to have modulus less than 1. In this paper we shall assume that q is real and $0 < q < 1$, although most of our results will remain valid for complex q with $|q| < 1$.

A basic hypergeometric series

$$_{r+1}\phi_r(a_1, \ldots, a_{r+1}; b_1, \ldots, b_r; q, z)$$

is said to be *balanced* if $qa_1a_2 \cdots a_{r+1} = b_1b_2 \cdots b_r$. It is called *very-well-poised* if $r \geq 2$ and $a_2 = q\sqrt{a_1}$, $a_3 = -q\sqrt{a_1}$, $b_1 = \sqrt{a_1}$, $b_2 = -\sqrt{a_1}$, $b_j = qa_1/a_j$, $j = 3, \ldots, r$. Note that the $_8\phi_7$ series on the left hand side of (1.1) is very-well-poised while the two $_4\phi_3$ series on the right hand side are balanced. By (2.10.19) of [6] this formula can also be rewritten as a q-integral.

The main result of this paper is the following extension of (1.1).

THEOREM 1.1.

(1.4)

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f; q)_n}{(q, \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e, qa/f; q)_n} \left(\frac{q^2a^2}{bcdef}\right)^n \\
&\quad \times {}_4\phi_3\left(\begin{matrix} q^{-n}, aq^n, g, h \\ b, c, qagh/bc \end{matrix}; q, q\right) \\
&\quad = \frac{(qa, qa/de, qa/df, qa/ef; q)_\infty}{(qa/d, qa/e, qa/f, qa/def; q)_\infty} \\
&\qquad \times {}_5\phi_4\left(\begin{matrix} qag/bc, qah/bc, d, e, f \\ qagh/bc, qa/b, qa/c, def/a \end{matrix}; q, q\right) \\
&\qquad + \frac{(qa, d, e, f, q^2a^2/bdef, q^2a^2/cdef; q)_\infty}{(qa/b, qa/c, qa/d, qa/e, qa/f, def/qa; q)_\infty} \\
&\qquad \times \frac{(qag/bc, qah/bc, q^2a^2gh/bcdef; q)_\infty}{(qagh/bc, q^2a^2g/bcdef, q^2a^2h/bcdef; q)_\infty} \\
&\qquad \times {}_5\phi_4\left(\begin{matrix} qa/de, qa/df, qa/ef, q^2a^2g/bcdef, q^2a^2h/bcdef \\ q^2a^2/bdef, q^2a^2/cdef, q^2a/def, q^2a^2gh/bcdef \end{matrix}; q, q\right).
\end{aligned}$$

The convergence of the two $_5\phi_4$ series on the right hand side is assured by our condition $|q| < 1$. When g or h equals 1 the $_4\phi_3$ series on the left hand side is 1 and (1.4) reduces to Bailey's formula (1.1) in which case the left hand side converges if $|q^2a^2/bcdef| < 1$, unless the series terminates. If $g \neq q^{-2}$, $h \neq q^{-1}$, and $|h| < |g|$, then one can show by using the terminating case of (1.1), that is, Watson's formula (2.5.1) of [6], that

$$_4\phi_3\left(\begin{matrix} q^{-n}, aq^n, g, h \\ b, c, qagh/bc \end{matrix}; q, q\right) \sim \frac{(h, b/g, c/g, qah/bc; q)_\infty}{(b, c, qagh/bc; q)_\infty} g^n, \tag{1.5}$$

as $n \to \infty$ [9], so the double series on the left hand side of (1.4) converges if $|q^2a^2g/bcdef| < 1$. Hence the condition of convergence of the infinite series on the left hand side of (1.4) is

$$\left|\frac{q^2a^2\max(|g|,|h|)}{bcdef}\right| < 1. \tag{1.6}$$

The special case $h = 0$ of (1.4) is of particular interest to us and we state it as a separate result.

COROLLARY 1.2.

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f; q)_n}{(q, \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e, qa/f; q)_n}\left(\frac{q^2a^2}{bcdef}\right)^n \\ &\times {}_3\phi_2\left(\begin{matrix} q^{-n}, aq^n, g \\ b, c\end{matrix}; q, q\right) \\ &= \frac{(qa, qa/de, qa/df, qa/ef; q)_\infty}{(qa/d, qa/e, qa/f, qa/def; q)_\infty} \\ &\quad\times {}_4\phi_3\left(\begin{matrix} qag/bc, d, e, f \\ def/a, qa/b, qa/c\end{matrix}; q, q\right) \\ &\quad+\frac{(qa, d, e, f, q^2a^2/bdef, q^2a^2/cdef, qag/bc; q)_\infty}{(qa/b, qa/c, qa/d, qa/e, qa/f, def/qa, q^2a^2g/bcdef; q)_\infty} \\ &\quad\times {}_4\phi_3\left(\begin{matrix} qa/de, qa/df, qa/ef, q^2a^2g/bcdef \\ q^2a^2/bdef, q^2a^2/cdef, q^2a/def\end{matrix}; q, q\right).\end{aligned} \tag{1.7}$$

Note that none of the two ${}_4\phi_3$ series on the right hand side is balanced unless $g = 1$. This is the formula that is directly connected with the nonsymmetric bilinear generating function:

$$K_t^{\mathbf{a},\boldsymbol{\alpha}}(x, y) := \sum_{n=0}^{\infty}\frac{(ab; q)_n}{(q; q)_n a^{2n}} t^n p_n(x; a, b)p_n(y; \alpha, \beta). \tag{1.8}$$

Here we have used the vector notation

$$\mathbf{a} = (a, b), \quad \boldsymbol{\alpha} = (\alpha, \beta). \tag{1.9}$$

Also the polynomials

$$p_n(x; a, b) = {}_3\phi_2\left(\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, 0\end{matrix}; q, q\right), \quad x = \cos\theta, \tag{1.10}$$

are the Al-Salam-Chihara polynomials that arose in a characterization problem considered in [1]. Their weight function was found later in [2] and [4]. They are q-analogues of Laguerre polynomials and are orthogonal with respect to an absolutely continuous measure when $-1 < a, b < 1$. The notation used here is as in [5] and [3] but differs from the one in [4] in a multiplicative factor. The present notation is more convenient for our purposes. The orthogonality relation of the Al-Salam-Chihara polynomials is

(1.11)
$$\int_0^\pi \frac{p_m(\cos\theta;a,b)p_n(\cos\theta;a,b)(e^{2i\theta},e^{-2i\theta};q)_\infty}{(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta};q)_\infty}\,d\theta = \frac{2\pi a^{2n}(q;q)_n}{(q,ab;q)_\infty(ab;q)_n}\delta_{m,n}.$$

The Poisson kernel of a sequence of orthonormal polynomials $\{p_n(x)\}$ is defined by

$$\sum_{n=0}^{\infty} p_n(x)p_n(y)t^n, \quad |t|<1, \tag{1.12}$$

so the Poisson kernel of the Al-Salam-Chihara polynomials is a constant multiple of the kernel K_t:

$$K_t(x,y)=\sum_{n=0}^{\infty}\frac{(ab;q)_n}{(q;q)_n a^{2n}}t^n p_n(x;a,b)p_n(y;a,b), \tag{1.13}$$

which is the symmetric case of (1.8), that is, the case $\alpha=a$, $\beta=b$. By (1.5) it is clear that the infinite series on the right hand side of (1.13) converges if $|t|<1$, but in (1.8) it converges if $|\alpha t/a|<1$ which is what we shall assume to hold.

Recently an explicit form of $K_t^{\mathbf{a},\boldsymbol{\alpha}}(x,y)$ when $ab=\alpha\beta$ was found independently in [3] and [7]. As we shall see later this really corresponds to setting $g=1$ in (1.7), so the kernel $K_t^{\mathbf{a},\boldsymbol{\alpha}}(x,y)$ can be expressed in terms of a very well-poised ${}_8\phi_7$ function. A year later Ismail and Stanton [8], in their study of linear and bilinear generating functions of the Al-Salam-Chihara polynomials, found the following representation for $K_t^{\mathbf{a},\boldsymbol{\alpha}}(x,y)$, without the restriction $ab=\alpha\beta$:

THEOREM 1.3.

$$\begin{aligned} K_t^{\mathbf{a},\boldsymbol{\alpha}}(\cos\theta,\cos\phi) &= \frac{(\alpha e^{-i\phi},\beta e^{-i\phi},\alpha te^{i\phi},\alpha bte^{i\phi}/a;q)_\infty}{(\alpha\beta,e^{-2i\phi},\alpha te^{i(\theta+\phi)}/a,\alpha te^{i(\phi-\theta)}/a;q)_\infty} \\ &\quad\times {}_4\phi_3\left(\begin{matrix}\alpha e^{i\phi},\beta e^{i\phi},\alpha te^{i(\theta+\phi)}/a,\alpha te^{i(\phi-\theta)}/a\\ qe^{2i\phi},\alpha te^{i\phi},\alpha bte^{i\phi}/a\end{matrix};q,q\right) \\ &\quad+\frac{(\alpha e^{i\phi},\beta e^{i\phi},\alpha te^{-i\phi},\alpha bte^{-i\phi}/a;q)_\infty}{(\alpha\beta,e^{2i\phi},\alpha te^{i(\theta-\phi)}/a,\alpha te^{-i(\theta+\phi)}/a;q)_\infty} \\ &\quad\times {}_4\phi_3\left(\begin{matrix}\alpha e^{-i\phi},\beta e^{-i\phi},\alpha te^{-i(\theta+\phi)}/a,\alpha te^{i(\theta-\phi)}/a\\ qe^{-2i\phi},\alpha te^{-i\phi},\alpha bte^{-i\phi}/a\end{matrix};q,q\right). \end{aligned} \tag{1.14}$$

Although the kernel on the left hand side of (1.14) is well-defined for all θ, ϕ in $[0,\pi]$ provided $\max(|a|,|b|,|c|,|d|)<1$ and $|\alpha t/a|<1$, the right hand side must be further restricted by $0<\phi<\pi$ because of the infinite products $(e^{\pm 2i\phi};q)_\infty$ in the denominators.

In Section 2 we shall give a proof of (1.4). In Section 3 we shall establish the connection between (1.14) and the special case (1.7) of (1.4). In Section 4 we shall consider the double series extension of the very-well-poised ${}_8\phi_7$ series, that is,

$$(1.15)\quad \phi(a;b,c;d,e,f;g;z) := \sum_{n=0}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f; q)_n}{(q, \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e, qa/f; q)_n} z^n \times {}_3\phi_2\left(\begin{matrix} q^{-n}, aq^n, g \\ b, c \end{matrix}; q, q\right),$$

so that the double series on the left hand side of (1.7) is $\phi(a;b,c;d,e,f;g;q^2a^2/bcdef)$. We shall discuss a q-integral representation and some useful properties and special cases of this function in Sections 4 and 5. In Section 6 we shall consider an extension of the Nassrallah-Rahman integral [10]:

$$(1.16)\quad \int_0^{\pi} \frac{(e^{2i\theta}, e^{-2i\theta}, be^{i\theta}, be^{-i\theta}; q)_\infty}{\prod_{k=1}^{5}(a_k e^{i\theta}, a_k e^{-i\theta}; q)_\infty}\, d\theta = \frac{2\pi(a_4a_5, a_1b, a_2b, a_3b, a_1a_2a_3a_4, a_1a_2a_3a_5; q)_\infty}{(q, a_1a_2a_3b; q)_\infty \prod_{1\le j<k\le 5}(a_ja_k; q)_\infty} \times {}_8W_7(a_1a_2a_3b/q; a_1a_2, a_1a_3, a_2a_3, b/a_4, b/a_5; q, a_4a_5),$$

where

$$(1.17)\quad \max(|a_j|) < 1, \quad j = 1, \ldots, 5,$$

$$(1.18)\quad a_1a_2a_3a_4a_5 = b,$$

and we have used the simpler notation for a very-well-poised series:

$$(1.19)\quad {}_{s+1}W_s(\alpha_1; \alpha_2, \ldots, \alpha_{s-1}; q, z) := {}_{s+1}\phi_s\left(\begin{matrix} \alpha_1, q\sqrt{\alpha_1}, -\sqrt{\alpha_1}, \alpha_2, \ldots, \alpha_{s-1} \\ \sqrt{\alpha_1}, -\sqrt{\alpha_1}, q\alpha_1/\alpha_2, \ldots, q\alpha_1/\alpha_{s-1} \end{matrix}; q, z\right).$$

When $a_5 = 0$, $b = 0$, the Nassrallah-Rahman integral (1.17) reduces to the well-known Askey-Wilson integral

$$(1.20)\quad \int_0^{\pi} w(\cos\theta; a_1, a_2, a_3, a_4)\, d\theta = \frac{2\pi(a_1a_2a_3a_4; q)_\infty}{(q;q)_\infty \prod_{1\le j<k\le 4}(a_ja_k; q)_\infty}$$

where the Askey-Wilson weight function w is

$$(1.21)\quad w(\cos\theta; a_1, a_2, a_3, a_4) := \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\prod_{j=1}^{4}(a_j e^{i\theta}, a_j e^{-i\theta}; q)_\infty}.$$

So Section 6 will deal with an extension of (1.16) to the functions (1.15).

2. **Proof of the main result.** In order to prove (1.4) we first use Sears' transformation formula (III.15) of [6] to get

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(2.1)

$$
\begin{aligned}
{}_4\phi_3&\left(\begin{matrix} q^{-n}, aq^n, g, h \\ b, c, qagh/bc \end{matrix}; q, q\right)\\
&= \frac{(qa/b, qa/c; q)_n}{(b, c; q)_n}\left(\frac{bc}{aq}\right)^n\\
&\quad \times {}_4\phi_3\left(\begin{matrix} q^{-n}, aq^n, qag/bc, qah/bc \\ qagh/bc, qa/b, qa/c \end{matrix}; q, q\right)\\
&= \frac{(qa/b, qa/c, h; q)_n}{(b, c, qa/h, qagh/bc; q)_n} g^n\\
&\quad \times {}_8\phi_7\left(\begin{matrix} a/h, q\sqrt{a/h}, -q\sqrt{a/h}, b/h, c/h, qag/bc, aq^n, q^{-n} \\ \sqrt{a/h}, -\sqrt{a/h}, qa/b, qa/c, bc/gh, q^{1-n}/h, q^{1+n}a/h \end{matrix}; q, q/g\right).
\end{aligned}
$$

Therefore, the left-hand side of (1.4) can be written as

$$
\begin{aligned}
(2.2)\quad &\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(1-aq^{2n})(d, e, f, bc/gh; q)_n}{(1-a)(qa/d, qa/e, qa/f, qagh/bc; q)_n}\left(\frac{q^2a^2g}{bcdef}\right)^n\\
&\quad \times \frac{(1-aq^{2m}/h)(a/h, b/h, c/h, qag/bc; q)_m}{(1-a/h)(q, qa/b, qa/c, bc/gh; q)_m}\\
&\quad \times \frac{(a; q)_{n+m}(h; q)_{n-m}}{(qa/h; q)_{m+n}(q; q)_{n-m}}\left(\frac{h}{g}\right)^m\\
&\qquad = \sum_{m=0}^{\infty} \frac{(1-aq^{2m}/h)(a/h, b/h, c/h, qag/bc; q)_m}{(1-a/h)(q, qa/b, qa/c, bc/gh; q)_m}\\
&\qquad\quad \times \frac{(d, e, f; q)_m (qa; q)_{2m}}{(qa/d, qa/e, qa/f, qagh/bc; q)_m (qa/h; q)_{2m}}\left(\frac{q^2a^2h}{bcdef}\right)^m\\
&\qquad\quad \times {}_8W_7\left(aq^{2m}; dq^m, eq^m, fq^m, bcq^m/gh, h; q, q^2a^2g/bcdef\right),
\end{aligned}
$$

where we interchanged the order of summation that can be justified by the inequality (1.6) which we shall assume to hold. The ${}_8\phi_7$ function above can be expressed as a sum of two ${}_4\phi_3$'s by virtue of (1.1) in the form

$$
\begin{aligned}
(2.3)\quad &\frac{(aq^{2m+1}, qa/de, qa/df, qa/ef; q)_\infty}{(q^{m+1}a/d, q^{m+1}a/e, q^{m+1}a/f, q^{1-m}a/def; q)_\infty}\\
&\quad \times {}_4\phi_3\left(\begin{matrix} q^{m+1}ag/bc, dq^m, eq^m, fq^m \\ q^{m+1}agh/bc, q^{1+2m}a/h, a/d, q^m def/a \end{matrix}; q, q\right)\\
&\quad + \frac{(aq^{2m+1}, q^{m+1}ag/bc, dq^m, eq^m, fq^m; q)_\infty}{(q^{1+m}agh/bc, q^{2m+1}a/h, q^{m+1}a/d, q^{m+1}a/e, q^{m+1}a/f; q)_\infty}\\
&\quad \times \frac{(q^2a^2gh/bcdef, q^{2+m}a^2/defh; q)_\infty}{(q^2a^2g/bcdef, q^{m-1}def/a; q)_\infty}\\
&\quad \times {}_4\phi_3\left(\begin{matrix} qa/de, qa/df, q^2a^2g/bcdef, qa/ef \\ q^2a^2gh/bcdef, q^{m+2}a^2/defh, q^{2-m}a/def \end{matrix}; q, q\right).
\end{aligned}
$$

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Hence the expression in (2.2) becomes

$$(2.4)\quad \frac{(qa, qa/de, qa/df, qa/ef; q)_\infty}{(qa/d, qa/e, qa/f, qa/def; q)_\infty}\phi_1 + \frac{(qa, qag/bc, d, e, f, q^2a^2gh/bcdef, q^2a^2/defh; q)_\infty}{(qagh/bc, qa/d, qa/e, qa/f, qa/h, q^2a^2g/bcdef, def/qa; q)_\infty}\phi_2,$$

where

$$(2.5)\quad \phi_1 := \sum_{m=0}^{\infty} \frac{(1-aq^{2m}/h)(a/h, b/h, c/h, d, e, f, qag/bc; q)_m}{(1-a/h)(q, qa/b, qa/c, qagh/bc, q^{1-m}a/def; q)_m} \times \frac{(q^2a^2h/bcdef)^m}{(qa/h; q)_{2m}}\, {}_4\phi_3\left(\begin{matrix} q^{m+1}ag/bc, dq^m, eq^m, fq^m \\ q^{m+1}gh/bc, q^{1+2m}a/h, q^m def/a \end{matrix}; q, q\right)$$

and

$$(2.6)\quad \phi_2 := \sum_{m=0}^{\infty} \frac{(1-aq^{2m}/h)(a/h, b/h, c/h, def/qa; q)_m}{(1-a/h)(q, qa/b, qa/c, q^2a^2/defh; q)_m}\left(\frac{q^2a^2h}{bcdef}\right)^m \times {}_4\phi_3\left(\begin{matrix} qa/de, aq/df, qa/ef, q^2a^2g/bcdef \\ q^2a^2gh/bcdef, q^{m+2}a^2/defh, q^{2-m}a/def \end{matrix}; q, q\right).$$

Writing out the ${}_4\phi_3$ in (2.5) as infinite series we get

$$(2.7)\quad \phi_1 = \sum_{m=0}^{\infty}\sum_{j=0}^{\infty} \frac{(1-aq^{2m}/h)(a/h, b/h, c/h; q)_m}{(1-a/h)(q, qa/b, qa/c; q)_m} \times \frac{(d, e, f, qag/bc; q)_{m+j}}{(qagh/bc, def/a; q)_{m+j}} \frac{(-q^2ah/bc)^m q^{\binom{m}{2}+j}}{(q; q)_j (qa/h; q)_{2m+j}}$$
$$= \sum_{k=0}^{\infty} q^k \frac{(d, e, f, qag/bc; q)_k}{(q, qagh/bc, qa/h, def/a; q)_k} \times {}_6\phi_5\left(\begin{matrix} a/h, q\sqrt{a/h}, -\sqrt{a/h}, b/h, c/h, q^{-k} \\ \sqrt{a/h}, -\sqrt{a/h}, qa/b, qa/c, q^{k+1}a/h \end{matrix}; q, \frac{ah}{bc}q^{k+1}\right)$$
$$= {}_5\phi_4\left(\begin{matrix} qag/bc, qah/bc, d, e, f \\ qagh/bc, qa/b, qa/c, def/a \end{matrix}; q, q\right),$$

where we used [6, (II.21)] to sum the very-well-poised ${}_6\phi_5$ series. In the same manner we find

$$(2.8)\quad \phi_2 = \sum_{k=0}^{\infty} q^k \frac{(qa/de, qa/df, qa/ef, q^2a^2g/bcdef; q)_k}{(q, q^2a^2gh/bcde, q^2a/def, q^2a^2/defh; q)_k} \times {}_6\phi_5\left(\begin{matrix} a/h, q\sqrt{a/h}, -\sqrt{a/h}, b/h, c/h, q^{-k-1}def/a \\ \sqrt{a/h}, -\sqrt{a/h}, qa/b, qa/c, q^{k+2}a^2/defh \end{matrix}; q, \frac{a^2h}{bcdef}q^{k+2}\right)$$
$$= \frac{(qa/h, qa/bc, q^2a^2/bdef, q^2a^2/cdef; q)_\infty}{(qa/b, qa/c, q^2a^2/defh, q^2a^2h/bcdef; q)_\infty} \times {}_5\phi_4\left(\begin{matrix} qa/de, qa/df, qa/ef, q^2a^2g/bcdef, q^2a^2h/bcdef \\ q^2a^2/bdef, q^2a^2/cdef, q^2a/def, q^2a^2gh/bcdef \end{matrix}; q, q\right).$$

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Substituting (2.7) and (2.8) in (2.4) we establish (1.4) and the proof of Theorem 1.1 is complete.

Apart from the Corollary 1.2 which is a special case of (1.4) and directly linked with the bilinear generating function for the Al-Salam-Chihara polynomials there is a limiting case of (1.4) that needs to be mentioned. This corresponds to the limit $h \longrightarrow \infty$, provided $|q/g| < 1$. The result is

COROLLARY 2.1.

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f; q)_n}{(q, \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e, qa/f; q)_n} \left(\frac{q^2a^2}{bcdef}\right)^n \\ &\quad \times {}_3\phi_2\left(\begin{matrix} q^{-n}, aq^n, g \\ b, c \end{matrix}; q, \frac{bc}{ag}\right) \\ &= \frac{(qa, qa/de, qa/df, qa/ef; q)_\infty}{(qa/d, qa/e, qa/f, qa/def; q)_\infty} \\ &\quad \times {}_4\phi_3\left(\begin{matrix} qag/bc, d, e, f \\ qa/b, qa/c, def/a \end{matrix}; q, q/g\right) \\ &\quad + \frac{(qa, d, e, f, qa/bc, bc/a; q)_\infty}{(qa/b, qa/c, qa/d, qa/e, qa/f, bc/ag; q)_\infty} \\ &\quad \times \frac{(q^2a^2/bdef, q^2a^2/cdef, bcdef/qa^2g; q)_\infty}{(q^2a^2/bcdef, bcdef/qa^2, def/qa; q)_\infty} \\ &\quad \times {}_4\phi_3\left(\begin{matrix} qa/de, qa/df, qa/ef, q^2a^2g/bcdef \\ q^2a^2/bdef, q^2a^2/cdef, q^2a/def \end{matrix}; q, q/g\right). \end{aligned} \tag{2.9}$$

PROOF. Replace h in (1.4) by q^{-m}, for a positive integer m, then let $m \longrightarrow \infty$. The result is (2.9) and the proof is complete.

3. **Proof of Theorem 1.3.** In this section we shall present a new proof of (1.14). This proof is not simpler that one in [8], but it gives a "Poisson kernel" motivation of our transformation (1.7) which is important for us in the present paper. The proof of (1.14) given here depends on the q-integral representation of the Al-Salam-Chihara polynomials [6, $c = d = 0$ in Example 7.34]

$$\begin{aligned} &p_n(\cos\theta; a, b) \\ &\quad = B(\theta) \frac{a^{2n}}{(ab;q)_n} \int_{qe^{i\theta}/a}^{qe^{-i\theta}/a} \frac{(aue^{i\theta}, aue^{-i\theta}; q)_\infty}{(abu/q; q)_\infty} (q/u; q)_n (u/q)^n \, d_q u, \end{aligned} \tag{3.1}$$

where

$$B(\theta) := \frac{2ia\sin\theta}{q(1-q)} \frac{(be^{i\theta}, be^{-i\theta}; q)_\infty}{(q, e^{2i\theta}, e^{-2i\theta}; q)_\infty}. \tag{3.2}$$

The q-integral in (3.1) is defined by

$$\begin{aligned} &\int_0^a f(u)\, d_q u := a(1-q) \sum_{m=0}^{\infty} q^m f(aq^m), \\ &\int_a^b f(u)\, d_q u := \int_0^b f(u)\, d_q u - \int_0^a f(u)\, d_q u. \end{aligned} \tag{3.3}$$

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In (3.3) f is assumed to be such that the series on the right hand side converge. Note the difference between (3.1) and q-integral representation in [8].

The proof of (1.14) consists of two parts. In the first part we work on the left-hand side and reduce it to a double sum. We then transform the double sum into the right-hand side of (1.14). Theorem 3.1 below provides the first step on this route.

THEOREM 3.1. *The kernel* $K_t^{\mathbf{a},\boldsymbol{\alpha}}$ *has the double sum representation*

$$
\begin{aligned}
(3.4)\qquad K_t^{\mathbf{a},\boldsymbol{\alpha}}(\cos\theta,\cos\phi)
&= \frac{(\alpha e^{-i\phi}, \beta e^{i\phi}, \alpha t e^{-i\phi}, \alpha b t e^{i\phi}/a, \alpha^2 t^2/a^2; q)_\infty}{(\alpha\beta, \alpha t e^{i(\theta+\phi)}/a, \alpha t e^{i(\theta-\phi)}/a, \alpha t e^{i(\phi-\theta)}/a, \alpha t e^{-i(\theta+\phi)}/a; q)_\infty} \\
&\times \sum_{k=0}^{\infty} (\beta e^{i\phi})^k \frac{(\alpha t e^{i(\theta-\phi)}/a, \alpha t e^{-i(\theta+\phi)}/a, \alpha t/\beta; q)_k}{(q, \alpha t e^{-i\phi}, \alpha^2 t^2/a^2; q)_k} \\
&\times {}_3\phi_2\left(\begin{matrix} \alpha t e^{i(\theta+\phi)}/a, \alpha t e^{i(\phi-\theta)}/a, bt/a \\ \alpha b t e^{i\phi}/a, \alpha^2 t^2 q^k/a^2 \end{matrix}; q, \alpha q^k e^{-i\phi}\right).
\end{aligned}
$$

PROOF. From (1.13) and (3.1) we get

$$
\begin{aligned}
(3.5)\qquad K_t^{\mathbf{a},\boldsymbol{\alpha}}(\cos\theta,\cos\phi) &= B(\theta)\int_{qe^{i\theta}/a}^{qe^{-i\theta}/a} \frac{(aue^{i\theta}, aue^{-i\theta}; q)_\infty}{(abu/q; q)_\infty} \\
&\times \left[\sum_{n=0}^{\infty} \frac{(q/u; q)_n}{(q; q)_n} (ut/q)^n p_n(\cos\phi; \alpha, \beta)\right] d_q u.
\end{aligned}
$$

The last series can be summed by the generating function [11], [8]

$$
\begin{aligned}
(3.6)\qquad \sum_{n=0}^{\infty} \frac{(v; q)_n}{(q; q)_n} t^n p_n(\cos\phi; \alpha, \beta) &= \frac{(\alpha^2 t, \beta e^{i\phi}, \alpha v t e^{-i\phi}; q)_\infty}{(\alpha\beta, \alpha t e^{i\phi}, \alpha t e^{-i\phi}; q)_\infty} \\
&\times {}_3\phi_2\left(\begin{matrix} \alpha e^{-i\phi}, \alpha t e^{-i\phi}, \alpha t v/\beta \\ \alpha t v e^{-i\phi}, \alpha^2 t \end{matrix}; q, \beta e^{i\phi}\right).
\end{aligned}
$$

This results in the following representation of the kernel

$$
\begin{aligned}
(3.7)\qquad K_t^{\mathbf{a},\boldsymbol{\alpha}}(\cos\theta,\cos\phi)
&= B(\theta)\frac{(\alpha t e^{-i\phi}, \beta e^{i\phi}; q)_\infty}{(\alpha\beta; q)_\infty} \sum_{k=0}^{\infty} (\beta e^{i\phi})^k \frac{(\alpha e^{-i\phi}, \alpha t/\beta; q)_k}{(q, \alpha t e^{-i\phi}; q)_k} \\
&\times \int_{qe^{i\theta}/a}^{qe^{-i\theta}/a} \frac{(aue^{i\theta}, aue^{-i\theta}, \alpha^2 t u q^{k-1}; q)_\infty}{(abu/q, \alpha t u q^{-1} e^{i\phi}, \alpha t u q^{k-1} e^{-i\phi}; q)_\infty}\, d_q u.
\end{aligned}
$$

Now we can use the limiting case of [6, (2.10.19)] in the form

$$
\begin{aligned}
(3.8)\qquad \int_a^b \frac{(qt/a, qt/b, ct; q)_\infty}{(dt, et, ft; q)_\infty}\, d_q t &= b(1-q)\frac{(q, a/b, qb/a, c/e, abde, abef; q)_\infty}{(ad, ae, af, bd, be, bf; q)_\infty} \\
&\times {}_3\phi_2\left(\begin{matrix} ae, be, abdef/c \\ abde, abef \end{matrix}; q, c/e\right),
\end{aligned}
$$

and the representation (3.7) to obtain the right-hand side of (3.4). This completes the proof of Theorem 3.1.

We next transform the right-hand side of (3.4) to the right-hand side of (1.14).

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PROOF OF (1.14). Change the order of summations in (3.4) to get

$$(3.9)\qquad \sum_{k=0}^{\infty}(\beta e^{i\phi})^k \frac{(\alpha t e^{i(\theta-\phi)}/a, \alpha t e^{-i(\theta+\phi)}/a, \alpha t/\beta; q)_k}{(q, \alpha t e^{-i\phi}, \alpha^2 t^2/a^2; q)_k} \times {}_3\phi_2\left(\begin{matrix}\alpha t e^{i(\theta+\phi)}/a, \alpha t e^{i(\phi-\theta)}/a, bt/a \\ \alpha b t e^{i\phi}/a, \alpha^2 t^2 q^k/a^2\end{matrix}; q, \alpha q^k e^{-i\phi}\right)$$
$$= \sum_{j=0}^{\infty}(\alpha e^{-i\phi})^j \frac{(\alpha t e^{i(\phi-\theta)}/a, \alpha t e^{i(\theta+\phi)}/a, tb/a; q)_j}{(q, \alpha b t e^{i\phi}/a, \alpha^2 t^2/a^2; q)_j} \times {}_3\phi_2\left(\begin{matrix}\alpha t e^{i(\theta-\phi)}/a, \alpha t e^{-i(\theta+\phi)}/a, \alpha t/\beta \\ \alpha t e^{-i\phi}, \alpha^2 t^2 q^j/a^2\end{matrix}; q, \beta q^j e^{i\phi}\right).$$

We now invoke [6, (III.34)] to see that the ${}_3\phi_2$ on the right-hand side of (3.9) is the following linear combination of ${}_3\phi_2$'s

$$\frac{(\alpha t q^j e^{i(\phi-\theta)}/a, \alpha t q^j e^{i(\theta+\phi)}/a; q)_\infty}{(\alpha^2 t^2 q^j/a^2, q^j e^{2i\phi}; q)_\infty} \times {}_3\phi_2\left(\begin{matrix}\beta e^{-i\phi}, \alpha t e^{i(\theta-\phi)}/a, \alpha t e^{-i(\theta+\phi)}/a \\ \alpha t e^{-i\phi}, q^{1-j} e^{-2i\phi}\end{matrix}; q, q\right)$$
$$+ \frac{(\beta e^{-i\phi}, \alpha t e^{i(\theta-\phi)}/a, \alpha t e^{-i(\theta+\phi)}/a, \alpha t q^j e^{i\phi}; q)_\infty}{(\alpha t e^{-i\phi}, \alpha^2 t^2 q^j/a^2, q^{-j} e^{-2i\phi}, \beta q^j e^{i\phi}; q)_\infty} \times {}_3\phi_2\left(\begin{matrix}\alpha t q^j e^{i(\phi-\theta)}/a, \alpha t q^j e^{i(\theta+\phi)}/a, \beta q^j e^{i\phi} \\ \alpha t q^j e^{i\phi}, q^{j+1} e^{2i\phi}\end{matrix}; q, q\right),$$

and the kernel (3.4) now takes the form

$$(3.10)\qquad K_t^{\mathbf{a},\boldsymbol{\alpha}}(\cos\theta, \cos\phi) = \frac{(\alpha e^{-i\phi}, \beta e^{i\phi}, \alpha t e^{-i\phi}, \alpha b t e^{i\phi}/a; q)_\infty}{(\alpha\beta, \alpha t e^{i(\theta-\phi)}/a, \alpha t e^{-i(\theta+\phi)}/a, e^{2i\phi}; q)_\infty} \times \sum_{k=0}^{\infty}(\alpha e^{-i\phi})^k \frac{(e^{2i\phi}, bt/a; q)_k}{(q, \alpha b t e^{i\phi}/a; q)_k} \times {}_3\phi_2\left(\begin{matrix}\beta e^{-i\phi}, \alpha t e^{i(\theta-\phi)}/a, \alpha t e^{-i(\theta+\phi)}/a \\ \alpha t e^{-i\phi}, q^{1-j} e^{-2i\phi}\end{matrix}; q, q\right)$$
$$+ \frac{(\alpha e^{-i\phi}, \beta e^{-i\phi}, \alpha t e^{i\phi}, \alpha b t e^{i\phi}/a; q)_\infty}{(\alpha\beta, \alpha t e^{i(\theta+\phi)}/a, \alpha t e^{i(\phi-\theta)}/a, e^{-2i\phi}; q)_\infty} \times \sum_{k=0}^{\infty}(-\alpha e^{i\phi})^k q^{k(k+1)/2} \frac{(\alpha t e^{i(\phi-\theta)}/a, \alpha t e^{i(\theta+\phi)}/a, bt/a, \beta e^{i\phi}; q)_k}{(q, \alpha t e^{i\phi}, \alpha b t e^{i\phi}/a, q e^{2i\phi}; q)_k} \times {}_3\phi_2\left(\begin{matrix}\alpha t q^k e^{i(\phi-\theta)}/a, \alpha t q^k e^{i(\theta+\phi)}/a, \beta q^k e^{i\phi} \\ \alpha t q^k e^{i\phi}, q^{1+k} e^{2i\phi}\end{matrix}; q, q\right).$$

The first series on the right-hand side of (3.10) is

(3.11)
$$\sum_{j=0}^{\infty}\frac{(\beta e^{-i\phi}, \alpha te^{i(\theta-\phi)}/a, \alpha te^{-i(\theta+\phi)}/a;q)_j}{(q,\alpha te^{-i\phi}, qe^{-2i\phi};q)_j}q^j \times {}_2\phi_1\left(\begin{matrix} bt/a, q^{-j}e^{2i\phi} \\ \alpha bte^{i\phi}/a\end{matrix}; q, \alpha q^j e^{-i\phi}\right) = \frac{(\alpha e^{i\phi}, \alpha bte^{-i\phi}/a;q)_\infty}{(\alpha e^{-i\phi}, \alpha bte^{i\phi}/a;q)_\infty} \times {}_4\phi_3\left(\begin{matrix}\alpha e^{-i\phi}, \beta e^{-i\phi}, \alpha te^{-i(\theta+\phi)}/a, \alpha te^{i(\theta-\phi)}/a \\ qe^{-2i\phi}, \alpha te^{-i\phi}, \alpha bte^{-i\phi}/a\end{matrix}; q,q\right),$$

by [6, (II.8)]. The second series in (3.10) is

(3.12)
$$\sum_{k,j=0}^{\infty}(-\alpha e^{i\phi})^k q^{j+k(k+1)/2}\frac{(\alpha te^{i(\phi-\theta)}/a, \alpha te^{i(\theta+\phi)}/a, \beta e^{i\phi};q)_{j+k}}{(qe^{2i\phi}, \alpha te^{i\phi};q)_{j+k}(q;q)_j} \times \frac{(bt/a;q)_k}{(q,\alpha bte^{i\phi}/a;q)_k} = \sum_{m=0}^{\infty}\frac{(\alpha te^{i(\phi-\theta)}/a, \alpha te^{i(\theta+\phi)}/a, \beta e^{i\phi};q)_m}{(q,qe^{2i\phi},\alpha te^{i\phi};q)_m}q^m \times {}_2\phi_1\left(\begin{matrix}q^{-m}, bt/a \\ \alpha bte^{i\phi}/a\end{matrix}; q, \alpha q^m e^{i\phi}\right) = {}_4\phi_3\left(\begin{matrix}\alpha e^{i\phi}, \beta e^{i\phi}, \alpha te^{i(\phi-\theta)}/a, \alpha te^{i(\theta+\phi)}/a \\ qe^{2i\phi}, \alpha te^{i\phi}, \alpha bte^{i\phi}/a\end{matrix}; q,q\right).$$

Substituting from (3.11) and (3.12) in (3.10) we establish (1.14) and the proof is complete.

We have already mentioned that the ${}_4\phi_3$'s in (1.14) become balanced when $ab = \alpha\beta$ and the right-hand side of (1.14) can be transformed to a single function, a very well-poised ${}_8\phi_7$ function by the Bailey transform [6, (III.36)]. In the general case equating the right-hand sides of (3.4) and (1.14) gives us a transformation formula (1.7) expressing a double sum as a linear combination of two ${}_4\phi_3$'s. To give a motivation of this transformation as the nonsymmetric Poisson kernel for the Al-Salam-Chihara polynomials we first need to cast the right-hand side of (3.4) in a more convenient form, a task we now proceed to do.

The use of (III.36) in [6] allows us to rewrite (3.4) in the equivalent form

(3.13)
$$K_t^{\mathbf{a},\boldsymbol{\alpha}}(\cos\theta,\cos\phi) = \frac{(be^{i\theta}, \beta e^{i\phi}, \alpha^2 te^{-i\theta}/a, \alpha te^{-i\phi}, \alpha^2t^2/a^2;q)_\infty}{(\alpha\beta, \alpha te^{i(\theta+\phi)}/a, \alpha te^{i(\theta-\phi)}/a, \alpha te^{i(\phi-\theta)}/a, \alpha te^{-i(\theta+\phi)}/a;q)_\infty} \times \sum_{k=0}^{\infty}(\beta e^{i\phi})^k\frac{(\alpha te^{i(\theta-\phi)}/a, ate^{-i(\theta+\phi)}/a, \alpha e^{-i\phi}, \alpha t/\beta;q)_k}{(q,\alpha^2t^2/a^2, \alpha^2te^{-i\theta}/a, \alpha te^{-i\phi};q)_k} \times {}_3\phi_2\left(\begin{matrix}\alpha te^{i(\phi-\theta)}/a, \alpha tq^ke^{-i(\theta+\phi)}/a, \alpha^2tq^k/ab \\ \alpha^2tq^ke^{-i\theta}/a, \alpha^2t^2q^k/a^2\end{matrix}; q, be^{i\theta}\right).$$

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We then use [6, (3.2.11)] to transform the ${}_3\phi_2$ in (3.13), then express the resulting double sum in the form

$$\begin{aligned}
&\frac{(\alpha bte^{-i\phi}/a, \alpha^2te^{i\theta}/a;q)_\infty}{(be^{i\theta}, \alpha^3t^2q^{2k}e^{-i\phi}/a^2;q)_\infty} \\
&\quad\times\sum_{k,j=0}^{\infty}\frac{(\alpha te^{i(\theta-\phi)}/a, \alpha te^{-i(\theta+\phi)}/a, \alpha e^{-i\phi}, \alpha t/\beta;q)_{k+j}}{(\alpha^2te^{-i\theta}/a, \alpha^2t^2/a^2, \alpha tbe^{-i\phi}/a, \alpha^2te^{i\theta}/a;q)_{k+j}} \\
&\quad\times\frac{(\beta e^{i\phi})^k}{(q, \alpha te^{-i\phi};q)_k}\frac{(1-\alpha^3t^2q^{2k+2j-1}e^{-i\phi}/a^2)}{(1-\alpha^3t^2q^{2k-1}e^{-i\phi}/a^2)} \\
&\quad\times\frac{(\alpha^3t^2q^{2k-1}e^{-i\phi}/a^2, \alpha^2tq^k/ab;q)_j}{(q, \alpha tq^k/\beta;q)_j}q^{j(j-1)/2}(-\alpha bte^{i\phi}/a)^j \\
&=\frac{(\alpha bte^{-i\phi}/a, \alpha^2te^{i\theta}/a;q)_\infty}{(be^{i\theta}, \alpha^3t^2e^{-i\phi}/a^2;q)_\infty} \\
&\quad\times\sum_{n=0}^{\infty}q^{n(n-1)/2}(-\alpha bte^{i\phi}/a)^n\frac{(1-\alpha^3t^2q^{2n-1}e^{-i\phi}/a^2)}{(1-\alpha^3t^2q^{-1}e^{-i\phi}/a^2)} \\
&\quad\times\frac{(\alpha^3t^2q^{-1}e^{-i\phi}/a^2, \alpha te^{i(\theta-\phi)}/a, \alpha te^{-i(\theta+\phi)}/a, \alpha e^{-i\phi}, \alpha^2t/ab;q)_n}{(q, \alpha^2te^{-i\theta}/a, \alpha^2t^2/a^2, \alpha bte^{-i\phi}/a, \alpha^2te^{i\theta}/a;q)_n} \\
&\quad\times{}_3\phi_2\left(\begin{matrix}q^{-n}, \alpha^3t^2q^{n-1}e^{-i\phi}/a^2, \alpha t/\beta \\ \alpha te^{-i\phi}, \alpha^2t/ab\end{matrix};q, \beta aq/\alpha bt\right).
\end{aligned}$$

Finally we use [6, (III.13)] to transform the last ${}_3\phi_2$ and arrive at the following result.

THEOREM 3.2. *The kernel $K_t^{\mathbf{a},\boldsymbol{\alpha}}$ of (1.8) has the alternate double sum representation*

(3.14)

$$\begin{aligned}
&K_t^{\mathbf{a},\boldsymbol{\alpha}}(\cos\theta, \cos\phi) \\
&=\frac{(\beta e^{i\phi}, \alpha^2te^{i\theta}/a, \alpha^2te^{-i\theta}/a, \alpha te^{-i\phi}, \alpha bte^{-i\phi}/a, \alpha^2t^2/a^2;q)_\infty}{(\alpha\beta, \alpha^3t^2e^{-i\phi}/a^2, \alpha te^{i(\theta+\phi)}/a, \alpha te^{i(\theta-\phi)}/a, \alpha te^{i(\phi-\theta)}/a, \alpha te^{-i(\theta+\phi)}/a;q)_\infty} \\
&\quad\times\sum_{n=0}^{\infty}(abe^{i\phi}/\alpha)^n\frac{(1-\alpha^3t^2q^{2n-1}e^{-i\phi}/a^2)}{(1-\alpha^3t^2q^{-1}e^{-i\phi}/a^2)} \\
&\quad\times\frac{(\alpha^3t^2q^{-1}e^{-i\phi}/a^2, \alpha te^{i(\theta-\phi)}/a, \alpha te^{-i(\theta+\phi)}/a, \alpha e^{-i\phi}, \alpha^2t/ab, \alpha^2t/a^2;q)_n}{(q, \alpha^2te^{-i\theta}/a, \alpha^2t^2/a^2, \alpha bte^{-i\phi}/a, \alpha^2te^{i\theta}/a, \alpha te^{-i\phi};q)_n} \\
&\quad\times{}_3\phi_2\left(\begin{matrix}q^{-n}, \alpha^3t^2q^{n-1}e^{-i\phi}/a^2, \alpha\beta/ab \\ \alpha^2t/ab, \alpha^2t/a^2\end{matrix};q, q\right).
\end{aligned}$$

We note that interchanging sums in the argument leading to Theorem 3.2 can be easily justified until we reach the last ${}_3\phi_2$. The justification of the last step follows from the asymptotic formula, [9]

(3.15)
$$ {}_3\phi_2\left(\begin{matrix}q^{-n}, abq^{n+1}, x \\ qa, qc\end{matrix};q, q\right)\sim\frac{x^n(qa/x, qc/x;q)_\infty}{(qa, qc;q)_\infty},$$

as $n \longrightarrow \infty$, for fixed x, $x \neq 0, aq^{m+1}, cq^{m+1}$, $m = 0, 1, \ldots$. The asymptotic formula (3.15), which can be seen as a special case of (1.5), also shows that the right-hand side of (3.14) converges for $|\beta e^{i\phi}| < 1$ except when one of the factors in the denominators on the right-hand side vanishes. This provides analytic continuation for the kernel on the left-hand side of (3.14).

We conclude this section by stating symmetry relations for the kernel (1.8).

THEOREM 3.3. *The kernel $K_t^{\mathbf{a},\boldsymbol{\alpha}}(x,y)$ has the properties*

$$K_{bt/a}^{\mathbf{b},\boldsymbol{\beta}}(x,y) = K_t^{\mathbf{a},\boldsymbol{\alpha}}(x,y), \tag{3.16}$$

$$K_{\alpha bt/\beta a}^{\mathbf{b},\boldsymbol{\alpha}}(x,y) = K_t^{\mathbf{a},\boldsymbol{\alpha}}(x,y), \tag{3.17}$$

and

$$K_t^{\mathbf{a},\boldsymbol{\alpha}}(x,y) = \frac{(ab;q)_\infty}{(\alpha\beta;q)_\infty} \sum_{k=0}^{\infty} \frac{(\alpha\beta/ab;q)_k}{(q;q)_k} (ab)^k K_{\alpha^2 tq^k/a^2}^{\boldsymbol{\alpha},\mathbf{a}}(y,x). \tag{3.18}$$

Here $\mathbf{a} = (a,b)$, $\mathbf{b} = (b,a)$, $\boldsymbol{\alpha} = (\alpha,\beta)$, *and* $\boldsymbol{\beta} = (\beta,\alpha)$.

PROOF. Use the symmetry relation

$$p_n(x;a,b) = (a/b)^n p_n(x;b,a), \tag{3.19}$$

to get (3.16) and (3.17). Apply the q-binomial theorem to $(ab;q)_n/(q;q)_n$ then rearrange the sums in (3.18).

4. **Some properties of** $\phi(a;b,c,d,e,f;g;z)$**.** When $g = 1$ the function $\phi(a;b,c;d,e,f;g;q^2a^2/bcdef)$ defined by (1.15) becomes a very well-poised ${}_8\phi_7$.

THEOREM 4.1. *A q-integral form of (1.7) is*

$$\begin{aligned} \int_a^b \frac{(qu/a, qu/b, cu, du;q)_\infty}{(eu, fu, ru, ghu;q)_\infty}\, d_q u &= b(1-q)\frac{(q, a/b, qb/a, cd/eh, cd/fh, cd/rh, bc, bd;q)_\infty}{(bcd/h, ae, af, ar, be, bf, br, bgh;q)_\infty} \\ &\quad \times \phi(bcd/qh; c/h, d/h; be, bf, br; g; ah), \end{aligned} \tag{4.1}$$

provided that cd = abefrh.

Theorem 4.1 is an extension of the q-integral representation (2.10.19) in [6].

The next theorem states two transformation formulas for our ϕ functions.

THEOREM 4.2. *The function $\phi(a;b,c;d,e,f;g;q^2a^2/bcdef)$ of (4.1) obeys the following transformation rules:*

$$\begin{aligned} \phi(a;b,c;d,e,f;g;q^2a^2/bcdef) &= \frac{(qa, qa/de, q^2a^2g/bcdf, q^2a^2g/bcef;q)_\infty}{(qa/d, qa/e, q^2a^2g/bcf, q^2a^2g/bcdef;q)_\infty} \\ &\quad \times \phi(qa^2g/bcf; qag/bf, qag/cf; d, e, qag/bc; g; qa/deg), \end{aligned} \tag{4.2}$$

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and

$$\begin{aligned}(4.3)\qquad &\phi(a;b,c;d,e,f;g;q^2a^2/bcdef)\\ &= \frac{(qa,q^2a^2/bdef,q^2a^2/cdef,qag/bc;q)_\infty}{(qa/b,qa/c,q^2a^2/def,q^2a^2g/bcdef;q)_\infty}\\ &\quad\times\phi(qa^2/def;b,c;qa/de,qa/df,qa/ef;g;qa/bc),\end{aligned}$$

which are independent if $g\neq 1$, *but reduce to the transformation (III.23) of [6] when* $g=1$.

PROOF. Using (III.23) of [6] on the ${}_8\phi_7$ on the right-hand side of (2.2), we have

$$\begin{aligned}(4.4)\qquad &{}_8W_7\Big(aq^{2m};dq^m,eq^m,fq^m,bcq^m/gh,h;q,\frac{q^2a^2g}{bcdef}\Big)\\ &= \frac{(qa,qa/de,q^2a^2g/bcdf,q^2a^2g/bcef;q)_\infty}{(qa/d,qa/e,q^2a^2g/bcdef,q^2a^2g/bcf;q)_\infty}\\ &\quad\times\frac{(qa/d,qa/e;q)_m(q^2a^2g/bcf;q)_{2m}}{(q^2a^2g/bcef,q^2a^2g/bcdf;q)_m(qa;q)_{2m}}\\ &\quad\times{}_8W_7\Big(q^{2m+1}\frac{a^2g}{bcf};q\frac{agh}{bcf},q^{m+1}\frac{a}{fh},q^{m+1}\frac{ag}{bc},dq^m,eq^m;q\frac{qa}{de}\Big).\end{aligned}$$

Assuming that $|qa/de|<1$, we now substitute this into (2.2), interchange the order of summation again, convert the infinite ${}_8\phi_7$ series back into a balanced ${}_4\phi_3$ series. Finally we take the limit $h\longrightarrow 0$ and obtain (4.2). A similar line of argument also yields (4.3). This proves the theorem.

THEOREM 4.3. *We have the following three term transformation formula for a* ϕ *series*

$$\begin{aligned}(4.5)\qquad &\phi(a;b,c;d,e,f;g;q^2a^2/bcdef)\\ &= \frac{(qa,qa/de,qa/df,qa/ef;q)_\infty}{(qa/b,qa/d,qa/e,qa/f;q)_\infty}\\ &\quad\times\frac{(bde/a,bdf/a,bef/a,b/a,qa/b;q)_\infty}{(bd/a,be/a,bf/a,qa/def,bdef/a;q)_\infty}\\ &\quad\times\phi(bdef/qa;b,bcdef/qa^2;d,e,f;g;q/c)\\ &\quad+\frac{b}{a}\frac{(qa,d,e,f,qb/a,qb/c,qb/d,qb/e,qb/f;q)_\infty}{(qa/b,qa/c,qa/d,qa/e,qa/f,bd/a,be/a,bf/a;q)_\infty}\\ &\quad\times\frac{(qa^2/bdef,bdef/a^2,qag/bc;q)_\infty}{(qb^2/a,qa/def,def/a,qg/c;q)_\infty}\\ &\quad\times\phi(b^2/a;b,bc/a;bd/a,be/a,bf/a;g;q^2a^2/bcdef).\end{aligned}$$

PROOF. First note that (3.3) and (1.7) imply

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(4.6) $$\phi(a;b,c;d,e,f;g;q^2a^2/bcdef) = \frac{aq-def}{q(1-q)adef}\frac{(qa,d,e,f,qa/de,qa/df,qa/ef,qag/bc;q)_\infty}{(q,qa/b,qa/c,qa/d,qa/e,qa/f,qa/def,def/qa;q)_\infty} \times \int_{qa}^{def}\frac{(u/a,qu/def,qau/bdef,qau/cdef;q)_\infty}{(u/de,u/df,u/ef,qagu/bcdef;q)_\infty}\,d_qu.$$

Let $f(u)$ denote the integrand in the above q-integral. Use

(4.7) $$\int_{qa}^{def} f(u)\,d_qu = \int_{qa}^{bdef/a} f(u)\,d_qu + \int_{bdef/a}^{def} f(u)\,d_qu$$

and the q-integral relationships

(4.8) $$\int_{qa}^{bdef/a} f(u)\,d_qu = \frac{(1-q)bdef(q,qb/a,qb/c,qb/d,qb/e,qb/f,qa^2/bdef,bdef/a^2;q)_\infty}{a(qb^2/a,bd/a,be/a,bf/a,qa/de,qa/df,qa/ef,qg/c;q)_\infty} \times \phi(b^2/a;b,bc/a;bd/a,be/a,bf/a;g;q^2a^2/bcdef),$$

and

(4.9) $$\int_{bdef/a}^{def} f(u)\,d_qu = \frac{(1-q)def(q,qa/c,def/a,bef/a,bdf/a,bde/a,b/a,qa/b;q)_\infty}{(bdef/a,d,e,f,bd/a,be/a,bf/a,qag/bc;q)_\infty} \times \phi(bdef/qa;b,bcdef/qa^2;d,e,f;g;q/c),$$

to get (4.5) and the proof is complete.

When $g=1$ the special case $qa^2=bcdef$ of the transformation (4.5) gives Bailey's summation formula á la the proof of (2.11.7) in [6]. An extension of Bailey's formula to the case $g\neq 1$ will now be stated as our next theorem.

THEOREM 4.4. *The special case $qa^2=bcdef$ of (4.5) is*

(4.10)
$$\phi(a;b,c;d,e,f;g;q) = \frac{(qa,qa/cd,qa/ce,qa/cf,qa/de,qa/df,qa/ef,b/a;q)_\infty}{(qa/c,qa/d,qa/e,qa/f,bc/a,bd/a,be/a,bf/a;q)_\infty} \times \phi(a/c;b,1;d,e,f;g;q/c) + \frac{b(qa,c,d,e,f,qb/a,qb/c,qb/d,qb/e,qb/f,q/c,qag/bc;q)_\infty}{a(qa/b,qa/c,qa/d,qa/e,qa/f,bc/a,bd/a,be/a,bf/a,qb^2/a,qg/c,qa/bc;q)_\infty} \times \phi(b^2/a;b,bc/a;bd/a,be/a,bf/a;g;q),$$

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where

$$\begin{aligned}(4.11)\quad &\phi(a/c;b,1;d,e,f;g;q/c)\\ &= 1+\frac{q^2(1-q^2a/c)(1-d)(1-e)1-f)(1-g)}{c(1-q)^2(1-qa/c)(1-qa/bc)(1-qa/cd)(1-qa/ce)(1-qa/cf)}\\ &\times\sum_{n=0}^{\infty}\frac{(qa/c,q^2\sqrt{a/c},-q^2\sqrt{a/c},q,qb,qd,qe,qf;q)_n}{(q^2,q\sqrt{a/c},-q\sqrt{a/c},q^2a/c,q^2a/cd,q^2a/ce,q^2a/cf;q)_n}\Big(\frac{q}{c}\Big)^n\\ &\times(1-q^{-n-1})(1-aq^{n+1}/c)\,{}_3\phi_2\left(\begin{matrix}q^{-n},aq^{n+2}/c,qg\\ q^2,qb\end{matrix};q,q\right).\end{aligned}$$

PROOF. The limiting case $\phi(v;b,1;d,e,f;g;q^2v^2/bdef)$ of the function (1.15) is

$$\begin{aligned}(4.12)\quad &\phi(v;b,1;d,e,f;g;q^2v^2/bdef)\\ &= 1+\frac{(1-q^2v)(1-d)(1-e)1-f)(1-g)}{(1-q)^2(1-qv)(1-qv/b)(1-qv/d)(1-qv/e)(1-qv/f)}\Big(\frac{q^3v^2}{bdef}\Big)\\ &\times\sum_{n=0}^{\infty}\frac{(qv,q^2\sqrt{v},-q^2\sqrt{v},q,qb,qd,qe,qf;q)_n}{(q^2,q\sqrt{v},-q\sqrt{v},q^2v,q^2v/b,q^2v/d,q^2v/e,q^2v/f;q)_n}\Big(\frac{q^2v^2}{bdef}\Big)^n\\ &\times(1-q^{-n-1})(1-vq^{n+1})\,{}_3\phi_2\left(\begin{matrix}q^{-n},vq^{n+2},qg\\ q^2,qb\end{matrix};q,q\right).\end{aligned}$$

Now put $qa^2 = bcdef$ in (4.5) and apply (4.12) with $v = a/c$ to establish (4.10) and (4.11) and we have completed the proof.

When $f = q^{-m}$, $m = 0,1,\ldots$, the second term on the right-hand side of (4.10) vanishes and we obtain an extension of Jackson's summation formula for a terminating balanced ${}_8\phi_7$, [6, (II.22)]. This is our next theorem.

THEOREM 4.5. *The following terminating summation formula holds*

(4.13)

$$\begin{aligned}&\sum_{n=0}^{m}\frac{(a,q\sqrt{a},-q\sqrt{a},b,c,d,e,q^{-m};q)_n}{(q,\sqrt{a},-\sqrt{a},qa/b,qa/c,qa/d,qa/e,q^{m+1}a;q)_n}q^n\\ &\quad\times{}_3\phi_2\left(\begin{matrix}q^{-n},aq^n,g\\ b,c\end{matrix};q,q\right)\\ &=\frac{(qa,qa/bc,qa/bd,qa/cd;q)_m}{(qa/b,qa/c,qa/d,qa/bcd;q)_m}\\ &\quad\times\Bigg[1+\frac{q^2(1-q^2a/c)(1-d)(1-e)(1-q^{-m})(1-g)}{c(1-q)^2(1-qa/c)(1-qa/bc)(1-qa/cd)(1-qa/ce)(1-aq^{m+1}/c)}\\ &\quad\times\sum_{n=0}^{m-1}\frac{(qa/c,q^2\sqrt{a/c},-q^2\sqrt{a/c},q,qb,qd,qe,q^{1-m};q)_n}{(q^2,q\sqrt{a/c},-q\sqrt{a/c},q^2a/c,q^2a/cd,q^2a/ce,aq^{m+2}/c;q)_n}\Big(\frac{q}{c}\Big)^n\\ &\quad\times(1-q^{-n-1})(1-aq^{n+1}/c)\,{}_3\phi_2\left(\begin{matrix}q^{-n},aq^{n+2}/c,qg\\ q^2,qb\end{matrix};q,q\right)\Bigg].\end{aligned}$$

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5. **Some special cases.** We now consider special and limiting cases of the transformation formula (1.7). When $g = 1$ we have Bailey's result (1.1). The next interesting special case is when $qa = de$. The result is Theorem 5.1.

THEOREM 5.1. *The following summation theorem holds*

$$\sum_{n=0}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}, b, c, d; q)_n}{(q, \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d; q)_n} \left(\frac{qa}{bcd}\right)^n {}_3\phi_2\left(\begin{matrix} q^{-n}, aq^n, g \\ b, c \end{matrix}; q, q\right) = \frac{(qa, qa/bd, qa/cd, qag/bc; q)_\infty}{(qa/b, qa/c, qa/d, qag/bcd; q)_\infty}. \tag{5.1}$$

It follows directly from (4.3) because

$$\phi(a; b, c; d, e, f; q, z) = 1$$

if one of d, e, f equals 1. The summation theorem (5.1) is an extension of the q-Dougall sum which evaluates the sum of a very well-poised ${}_6\phi_5$ function [6, (II.20)]. The q-Dougall sum corresponds to the case $g = 1$.

By letting $d = \sqrt{a}$ in (5.1) we establish:

THEOREM 5.2. *We have*

$$\sum_{n=0}^{\infty} \frac{(a, -q\sqrt{a}, b, c; q)_n}{(q, -\sqrt{a}, qa/b, qa/c; q)_n} \left(\frac{q\sqrt{a}}{bc}\right)^n {}_3\phi_2\left(\begin{matrix} q^{-n}, aq^n, g \\ b, c \end{matrix}; q, q\right) = \frac{(qa, q\sqrt{a}/b, q\sqrt{a}/c, qag/bc; q)_\infty}{(q\sqrt{a}, qa/b, qa/c, q\sqrt{a}g/bc; q)_\infty}. \tag{5.2}$$

The q-Dixon sum [6, (II.13)] corresponds to the special value $g = 1$ of (5.2).

If we let $d \longrightarrow \infty$ in (5.1) we get

$$\sum_{n=0}^{\infty} \frac{(a, q\sqrt{a}, -q\sqrt{a}, b, c; q)_n}{(q, \sqrt{a}, -\sqrt{a}, qa/b, qa/c; q)_n} \left(-\frac{qa}{bc}\right)^n q^{n(n-1)/2} \times {}_3\phi_2\left(\begin{matrix} q^{-n}, aq^n, g \\ b, c \end{matrix}; q, q\right) = \frac{(qa, qag/bc; q)_\infty}{(qa/b, qa/c; q)_\infty}. \tag{5.3}$$

It is of interest to note that the q-Dougall sum [6, (II.20)] follows from (5.1) when $qag = bc$ because in this case the ${}_3\phi_2$ on the left-hand side of (1.7) becomes balanced and can be summed by [6, (II.2)].

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The special case $qa = cd$ of (1.7) is of interest and reads

$$
\begin{aligned}
(5.4)\qquad &\sum_{n=0}^{\infty}\frac{(a, q\sqrt{a}, -q\sqrt{a}, b, e, f; q)_n}{(q, \sqrt{a}, -\sqrt{a}, qa/b, qa/e, qa/f; q)_n}\left(\frac{qa}{bef}\right)^n \\
&\quad\times {}_3\phi_2\left(\begin{matrix} q^{-n}, aq^n, g \\ b, c \end{matrix}; q, q\right) \\
&= \frac{(qa, c/e, c/f, qa/ef; q)_\infty}{(c, qa/e, qa/f, c/ef; q)_\infty} \\
&\quad\times {}_3\phi_2\left(\begin{matrix} qag/bc, e, f \\ qef/c, qa/b \end{matrix}; q, q\right) \\
&\quad+\frac{(qa, e, f, qac/bef, qa/ef, qag/bc; q)_\infty}{(c, qa/b, qa/e, qa/f, ef/c, qag/bef; q)_\infty} \\
&\quad\times {}_3\phi_2\left(\begin{matrix} c/e, c/f, qag/bef \\ qac/bef, qc/ef \end{matrix}; q, q\right).
\end{aligned}
$$

When $g = 1$ the ${}_3\phi_2$ on the left-hand side of (5.4) becomes 1 and the remaining series can be summed as a very well-poised ${}_6\phi_5$ series, [6, (II.20)]. The result is the nonterminating q-Saalschütz sum [6, (2.10.11)]. For general g the right-hand side of (5.2) can be transformed by [6, (III.34)] and results in a transformation which we shall state as a separate theorem.

THEOREM 5.3. *We have*

$$
\begin{aligned}
(5.5)\qquad &\sum_{n=0}^{\infty}\frac{(a, q\sqrt{a}, -q\sqrt{a}, b, e, f; q)_n}{(q, \sqrt{a}, -\sqrt{a}, qa/b, qa/e, qa/f; q)_n}\left(\frac{qa}{bef}\right)^n \\
&\quad\times {}_3\phi_2\left(\begin{matrix} q^{-n}, aq^n, g \\ b, c \end{matrix}; q, q\right) \\
&= \frac{(qa, qa/ef; q)_\infty}{(qa/e, qa/f; q)_\infty}\,{}_3\phi_2\left(\begin{matrix} c/g, e, f \\ qa/b, c \end{matrix}; q, \frac{qag}{bef}\right) \\
&= \frac{(qa, qa/be, qa/ef, qag/bf; q)_\infty}{(qa/b, qa/e, qa/f, qag/bef; q)_\infty} \\
&\quad\times {}_3\phi_2\left(\begin{matrix} e, c/f, g \\ c, qag/bf \end{matrix}; q, \frac{qa}{be}\right).
\end{aligned}
$$

To go from the middle term to the last term in (5.5) we used [6, (III.9)]. When $c = e$ we can sum the right-hand side of (3.6) by the q-Gauss summation formula [6, (II.8)]. This gives an alternate proof of (5.1).

Another interesting limiting case of (1.7) is the case $f \longrightarrow \infty$. Replace f by fq^{-m} and let $m \longrightarrow \infty$. The result is

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$$\sum_{n=0}^{\infty}\frac{(a,q\sqrt{a},-q\sqrt{a},b,c,d,e;q)_n}{(q,\sqrt{a},-\sqrt{a},qa/b,qa/c,qa/d,qa/e;q)_n}q^{n(n-1)/2}\left(-\frac{q^2a^2}{bcde}\right)^n$$
$$\times\,{}_3\phi_2\left(\begin{matrix}q^{-n},aq^n,g\\ b,c\end{matrix};q,q\right)$$
$$=\frac{(qa,qa/de;q)_\infty}{(qa/d,qa/e;q)_\infty}$$
$$\times\,{}_3\phi_2\left(\begin{matrix}qag/bc,d,e\\ qa/b,qa/c\end{matrix};q,\frac{qa}{de}\right)$$
$$+\frac{(qa,d,e,qag/bc,q^2a/de;q)_\infty}{(q,qa/b,qa/c,qa/d,qa/e;q)_\infty}$$
$$\times\lim_{m\to\infty}\frac{(fq^{-m};q)_\infty}{(defq^{-1-m}/a;q)_\infty}.$$

We have used the q-binomial theorem to evaluate the second term on the right-hand side of (1.7). On the other hand

$$\frac{(fq^{-m};q)_\infty}{(defq^{-1-m}/a;q)_\infty}=\frac{(f;q)_\infty}{(def/qa;q)_\infty}\frac{(q/f;q)_m}{(q^2a/def;q)_m}\left(\frac{qa}{de}\right)^m,$$

hence

$$\lim_{m\to\infty}\frac{(fq^{-m};q)_\infty}{(defq^{-1-m}/a;q)_\infty}=0,\quad\text{if } |qa/de|<1.$$

Thus we proved:

THEOREM 5.4. *If* $|qa/de|<1$ *then*

(5.6)
$$\sum_{n=0}^{\infty}\frac{(a,q\sqrt{a},-q\sqrt{a},b,c,d,e;q)_n}{(q,\sqrt{a},-\sqrt{a},qa/b,qa/c,qa/d,qa/e;q)_n}q^{n(n-1)/2}\left(-\frac{q^2a^2}{bcde}\right)^n$$
$$\times\,{}_3\phi_2\left(\begin{matrix}q^{-n},aq^n,g\\ b,c\end{matrix};q,q\right)$$
$$=\frac{(qa,qa/de;q)_\infty}{(qa/d,qa/e;q)_\infty}$$
$$\times\,{}_3\phi_2\left(\begin{matrix}qag/bc,d,e\\ qa/b,qa/c\end{matrix};q,\frac{qa}{de}\right).$$

The above theorem is a double series extension of [6, (3.2.11)], which is already an extension of a transformation connecting a ${}_3F_2$ at $x=1$ and a ${}_6F_5$ at $x=-1$. Observe that the ${}_3\phi_2$ on the right-hand side of (5.6) is a general ${}_3\phi_2$ unlike the type *II* ${}_3\phi_2$ appearing in [6, (3.2.11)]. Therefore Theorem 5.6 provides an analytic continuation of the general ${}_3\phi_2$ function to a function meromorphic in the complex plane. In fact we have

(5.7)

$$
{}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, \frac{de}{abc}z\right) = \frac{(dez/ab, dez/ac; q)_\infty}{(dez/a, dez/abc; q)_\infty}
$$
$$
\times \sum_{n=0}^{\infty} \frac{(dez/qa, \sqrt{qdez/a}, -\sqrt{qdez/a}, ez/a, dz/a, b, c; q)_n}{(q, \sqrt{dez/qa}, -\sqrt{dez/qa}, d, e, dez/ab, dez/ac; q)_n} q^{n(n-1)/2}\left(-\frac{de}{bc}\right)^n
$$
$$
\times {}_3\phi_2\left(\begin{matrix} q^{-n}, dezq^{n-1}/a, z \\ ez/a, dz/a \end{matrix}; q, q\right).
$$

The limiting case $c \longrightarrow \infty$ of (5.7) is worth recording. It is

$$
(5.8) \quad {}_2\phi_2\left(\begin{matrix} a, b \\ d, e \end{matrix}; q, \frac{de}{ab}z\right) = \frac{(dez/ab; q)_\infty}{(dez/a; q)_\infty}
$$
$$
\times \sum_{n=0}^{\infty} \frac{(dez/qa, \sqrt{qdez/a}, -\sqrt{qdez/a}, ez/a, dz/a, b; q)_n}{(q, \sqrt{dez/qa}, -\sqrt{dez/qa}, d, e, dez/ab; q)_n} q^{n(n-1)}\left(\frac{de}{b}\right)^n
$$
$$
\times {}_3\phi_2\left(\begin{matrix} q^{-n}, dezq^{n-1}/a, z \\ ez/a, dz/a \end{matrix}; q, q\right).
$$

Another interesting special case of (5.7) is the case

$$
d = qa/b, \quad e = qa/c, \quad z = bcx/q.
$$

In this case the well-poised ${}_3\phi_2$ on the left-hand side of (5.7) can be transformed by [6, (III.35)], originally due to Gasper and Rahman. This gives

$$
(5.9) \quad {}_3\phi_2\left(\begin{matrix} a, b, c \\ qa/b, qa/c \end{matrix}; q, \frac{qax}{bc}\right) = \frac{(qax/b, qax/c; q)_\infty}{(qax, qax/bc; q)_\infty}
$$
$$
\times \sum_{n=0}^{\infty} \frac{(ax, q\sqrt{ax}, -q\sqrt{ax}, b, c, bx, cx; q)_n}{(q, \sqrt{ax}, -\sqrt{ax}, qax/b, qax/c, qa/b, qa/c; q)_n} q^{n(n-1)/2}\left(-\frac{q^2a^2}{b^2c^2}\right)^n
$$
$$
\times {}_3\phi_2\left(\begin{matrix} q^{-n}, axq^n, bcx/q \\ bx, cx \end{matrix}; q, q\right)
$$
$$
= \frac{(ax; q)_\infty}{(x; q)_\infty} {}_5\phi_4\left(\begin{matrix} \sqrt{a}, -\sqrt{a}, \sqrt{qa}, -\sqrt{qa}, qa/bc \\ qa/b, qa/c, ax, q/x \end{matrix}; q, q\right)
$$
$$
+ \frac{(a, qa/bc, qax/b, qax/c; q)_\infty}{(qa/b, qa/c, qax/bc, 1/x; q)_\infty}
$$
$$
\times {}_5\phi_4\left(\begin{matrix} x\sqrt{a}, -x\sqrt{a}, x\sqrt{qa}, -x\sqrt{qa}, qax/bc \\ qax/b, qax/c, qx, ax^2 \end{matrix}; q, q\right).
$$

For completeness we mention the terminating version of (1.7) as a separate theorem.

THEOREM 5.5. *We have the following extension of Watson's transformation formula*

$$
(5.10) \quad \sum_{n=0}^{m} \frac{(a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-m}; q)_n}{(q, \sqrt{a}, -\sqrt{a}, qa/b, qa/c, qa/d, qa/e, aq^{m+1}; q)_n} \left(\frac{q^{2+m}a^2}{bcde}\right)^n
$$
$$
\times {}_3\phi_2\left(\begin{matrix} q^{-n}, aq^n, g \\ b, c \end{matrix}; q, q\right)
$$
$$
= \frac{(qa, qa/de; q)_m}{(qa/d, qa/e; q)_m} {}_4\phi_3\left(\begin{matrix} qag/bc, d, e, q^{-m} \\ q^{-m}de/a, qa/b, qa/c \end{matrix}; q, q\right).
$$

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PROOF. Let $f = q^{-m}$ in (1.7) and simplify the result.

6**. Integral representations.** Our next aim is a generalization of Nassrallah and Rahman's extension (1.16) of the Askey-Wilson integral.

It is straightforward to use the Askey-Wilson integral (1.20) and obtain the following integral representation

$$\begin{aligned}\int_0^\pi {}_4\phi_3\left(\begin{matrix} de^{i\theta}, de^{-i\theta}, f, u \\ g, h, qfd^2u/gh \end{matrix}; q, q\right) w(\cos\theta; a, b, c, d)\, d\theta \\ = \frac{2\pi(abcd;q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty} \\ \times {}_5\phi_4\left(\begin{matrix} f, u, ad, bd, cd \\ g, h, qfd^2u/gh, abcd \end{matrix}; q, q\right).\end{aligned} \tag{6.1}$$

We similarly write a companion identity where the vector (d, f, u, g, h) is transformed to the vector $(qa/g, qf/g, qu/g, q^2/g, qh/g)$. We then transform the combination

$$\begin{aligned}&\frac{2\pi(abcd;q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty} \\ &\times {}_5\phi_4\left(\begin{matrix} f, u, ad, bd, cd \\ g, h, qfd^2u/gh, abcd \end{matrix}; q, q\right) \\ &+ \frac{(q/g, f, u, qh/g, q^2d^2fu/g^2h; q)_\infty}{(g/q, h, qf/g, qu/g, qfd^2u/gh; q)_\infty} \\ &\times \frac{2\pi(qabcd/g;q)_\infty}{(q, ab, ac, qad/g, bc, qbd/g, qcd/g; q)_\infty} \\ &\times {}_5\phi_4\left(\begin{matrix} qf/g, qu/g, qbd/g, qcd/g, qad/g \\ q^2/g, qh/g, qabcd/g, q^2d^2fu/g^2h \end{matrix}; q, q\right),\end{aligned} \tag{6.2}$$

and substitute for the ${}_4\phi_3$'s from (6.1) and its companion formula. The result is

$$\begin{aligned}&\frac{(q/g, qd^2/g; q)_\infty}{(qu/g, qd^2u/g; q)_\infty} \\ &\times \int_0^\pi {}_8W_7\left(\frac{d^2u}{g}; \frac{h}{f}, q\frac{d^2u}{gh}, u, de^{i\theta}, de^{-i\theta}; q, q\frac{f}{g}\right) \\ &\times w(\cos\theta; a, b, c, d)\frac{(qdue^{i\theta}/g, qdue^{-i\theta}/g; q)_\infty}{(qde^{i\theta}/g, qde^{-i\theta}/g; q)_\infty}\, d\theta.\end{aligned} \tag{6.3}$$

This leads to our next result.

THEOREM 6.1. *We have*

$$\begin{aligned}&\int_0^\pi {}_8W_7(dg/q; h, r, g/f, de^{i\theta}, de^{-i\theta}; q, gf/hr) \\ &\times w(\cos\theta; a, b, c, d)\frac{(ge^{i\theta}, ge^{-i\theta}; q)_\infty}{(fe^{i\theta}, fe^{-i\theta}; q)_\infty}\, d\theta\end{aligned} \tag{6.4}$$

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$$= \frac{2\pi(abcd, dg, g/d;q)_\infty}{(q, ab, ac, ad, bc, bd, cd, df, f/d;q)_\infty}$$
$$\times {}_5\phi_4\left(\begin{matrix} ad, bd, cd, g/f, dg/hr \\ abcd, qd/f, dg/h, dg/r \end{matrix}; q, q\right)$$
$$+ \frac{2\pi(abcf, dg, g/f;q)_\infty}{(q, ab, ac, af, bc, bf, cf, df, d/f;q)_\infty}$$
$$\times \frac{(fg/h, fg/r, dg/hr;q)_\infty}{(dg/h, dg/r, fg/hr;q)_\infty}$$
$$\times {}_5\phi_4\left(\begin{matrix} af, bf, cf, g/d, gf/hr \\ abcf, qf/d, gf/h, gf/r \end{matrix}; q, q\right).$$

Now we can use the Nassrallah-Rahman integral to replace the ${}_8\phi_7$ in (6.4) by its integral representation (1.16). This reduces the left-hand side of (6.4) to a double integral and we have proved the following theorem.

THEOREM 6.2. *We have*

$$(6.5)\quad \frac{(q;q)_\infty^2}{(2\pi)^2}\int_0^\pi\int_0^\pi \frac{(dre^{i\phi}, dre^{-i\phi}, fre^{i\theta}, fre^{-i\theta}, gre^{i\theta}, gre^{-i\theta};q)_\infty}{(re^{i(\theta+\phi)}, re^{i(\theta-\phi)}, re^{i(\phi-\theta)}, re^{-i(\theta+\phi)}, de^{i\theta}, de^{-i\theta};q)_\infty}$$
$$\times \frac{(e^{2i\phi}, e^{-2i\phi};q)_\infty}{(fe^{i\phi}, fe^{-i\phi}, ge^{i\phi}, ge^{-i\phi}, he^{i\phi}, he^{-i\phi};q)_\infty}$$
$$\times \frac{(e^{2i\theta}, e^{-2i\theta};q)_\infty\, d\theta\, d\phi}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta};q)_\infty}$$
$$= \frac{(abchr, dr/h, fhr^2, ghr^2;q)_\infty}{(r^2, ab, ac, bc, fh, gh, ahr, bhr, chr;q)_\infty}$$
$$\times {}_5\phi_4\left(\begin{matrix} r^2, ahr, bhr, chr, fghr/d \\ qhr/d, ghr^2, fhr^2, abchr \end{matrix}; q, q\right)$$
$$+ \frac{(abcd, dfr, dgr, fghr/d;q)_\infty}{(ab, ac, ad, bc, bd, cd, fg, fh, gh, hr/d;q)_\infty}$$
$$\times {}_5\phi_4\left(\begin{matrix} ad, bd, cd, fg, dr/h \\ abcd, qd/hr, dfr, dgr \end{matrix}; q, q\right).$$

We next reformulate (6.5) in such a way that the right-hand side is a double series.

THEOREM 6.3. *A double series form of (6.5) is*

$$(6.6)\quad \frac{(q;q)_\infty^2}{(2\pi)^2}\int_0^\pi\int_0^\pi \frac{(dre^{i\phi}, dre^{-i\phi}, fre^{i\theta}, fre^{-i\theta}, gre^{i\theta}, gre^{-i\theta};q)_\infty}{(re^{i(\theta+\phi)}, re^{i(\theta-\phi)}, re^{i(\phi-\theta)}, re^{-i(\theta+\phi)}, de^{i\theta}, de^{-i\theta};q)_\infty}$$
$$\times \frac{(e^{2i\phi}, e^{-2i\phi};q)_\infty}{(fe^{i\phi}, fe^{-i\phi}, ge^{i\phi}, ge^{-i\phi}, he^{i\phi}, he^{-i\phi};q)_\infty}$$
$$\times \frac{(e^{2i\theta}, e^{-2i\theta};q)_\infty\, d\theta\, d\phi}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta};q)_\infty}$$
$$= \frac{(abcd, dfr, dgr;q)_\infty}{(ab, ac, ad, bc, bd, cd, fg, fh, gh;q)_\infty}$$

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$$\times \frac{(fghr/d, abdhr, acdhr, bcdhr; q)_\infty}{(ahr, bhr, chr, abcd^2hr; q)_\infty}$$
$$\times \sum_{n=0}^{\infty} \frac{(abcd^2hr/q, \sqrt{abcd^2hrq}, -\sqrt{abcd^2hrq}; q)_n}{(q, \sqrt{abcd^2hr/q}, -\sqrt{abcd^2hr/q}; q)_n}$$
$$\times \frac{(ad, bd, cd, dhr, abcdh/g; q)_n}{(bcdhr, acdhr, abdhr, abcd, dgr; q)_n} \Big(\frac{gr}{d}\Big)^n$$
$$\times {}_4\phi_3\left(\begin{matrix} q^{-n}, abcd^2hrq^{n-1}, fh, dr/g \\ dfr, dhr, abcdh/g \end{matrix}; q, q\right).$$

Two special cases of Theorem 6.3 are worth noting. First the case $dr = h$ of (6.6) is

$$(6.7)\qquad \frac{(q;q)_\infty^2}{(2\pi)^2} \int_0^\pi \int_0^\pi \frac{(fre^{i\theta}, fre^{-i\theta}, gre^{i\theta}, gre^{-i\theta}; q)_\infty}{(re^{i(\theta+\phi)}, re^{i(\theta-\phi)}, re^{i(\phi-\theta)}, re^{-i(\theta+\phi)}, de^{i\theta}, de^{-i\theta}; q)_\infty}$$
$$\times \frac{(e^{2i\phi}, e^{-2i\phi}; q)_\infty}{(fe^{i\phi}, fe^{-i\phi}, ge^{i\phi}, ge^{-i\phi}; q)_\infty}$$
$$\times \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty \, d\theta \, d\phi}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}; q)_\infty}$$
$$= \frac{(abcd, fgr^2; q)_\infty}{(ab, ac, ad, bc, bd, fg, r^2; q)_\infty}.$$

The second case $dr = g$ is

$$(6.8)\qquad \frac{(q;q)_\infty^2}{(2\pi)^2} \int_0^\pi \int_0^\pi \frac{(fre^{i\theta}, fre^{-i\theta}, dr^2e^{i\theta}, dr^2e^{-i\theta}; q)_\infty}{(re^{i(\theta+\phi)}, re^{i(\theta-\phi)}, re^{i(\phi-\theta)}, re^{-i(\theta+\phi)}, de^{i\theta}, de^{-i\theta}; q)_\infty}$$
$$\times \frac{(e^{2i\phi}, e^{-2i\phi}; q)_\infty}{(fe^{i\phi}, fe^{-i\phi}, he^{i\phi}, he^{-i\phi}; q)_\infty}$$
$$\times \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty \, d\theta \, d\phi}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}; q)_\infty}$$
$$= \frac{(abcd, d^2r^2, fhr^2; q)_\infty}{(ab, ac, ad, bc, bd, cd, fh; q)_\infty}$$
$$\times \frac{(abdhr, acdhr, bcdhr; q)_\infty}{(ahr, bhr, chr, dhr, abcd^2hr; q)_\infty}$$
$$\times {}_8W_7(abcd^2hr/q; ad, bd, cd, dhr, abch/r; q, r^2)$$
$$= \frac{(adr^2, bdr^2, cdr^2, fhr^2; q)_\infty}{(ab, ac, ad, bc, bd, cd, fh; q)_\infty}$$
$$\times \frac{(abchr, abcd; q)_\infty}{(ahr, bhr, chr, abcdr^2, r^2; q)_\infty}$$
$$\times {}_8W_7(abcdr^2/q; ab, ac, bc, dr/h, r^2; q, dhr).$$

In the limit $r \longrightarrow 1^-$ formulas (6.7) and (6.8) reduce to the Askey-Wilson integral (1.20).

Denote the left-hand side of (1.4) by

$$\Phi\Big(a; b, c; d, e, f; g, h; q, \frac{q^2a^2}{bcde}\Big), \tag{6.9}$$

(*cf.* (1.15)). Following the outlines of the proof of Theorem 6.1 we obtain the following result.

THEOREM 6.4. *We have*

$$\begin{aligned}
&\int_0^\pi \Phi(dg/q; h, r; g/f, de^{i\theta}, de^{-i\theta}; \lambda, \mu; q, gf/hr) \\
&\qquad \times w(\cos\theta; a, b, c, d)\frac{(ge^{i\theta}, ge^{-i\theta}; q)_\infty}{(fe^{i\theta}, fe^{-i\theta}; q)_\infty}\, d\theta \\
&\quad = \frac{2\pi(abcd, dg, g/d; q)_\infty}{(q, ab, ac, ad, bc, bd, cd, df, f/d; q)_\infty} \\
&\qquad \times {}_6\phi_5\left(\begin{matrix} ad, bd, cd, g/f, dg\lambda/hr, dg\mu/hr \\ abcd, qd/f, dg/h, dg/r, dg\lambda\mu/hr \end{matrix}; q, q\right) \\
&\qquad + \frac{2\pi(abcf, dg, g/f; q)_\infty}{(q, ab, ac, af, bc, bf, cf, df, d/f; q)_\infty} \\
&\qquad \times \frac{(fg/h, fg/r, dg\lambda/hr, dg\mu/hr, fg\lambda\mu/hr; q)_\infty}{(dg/h, dg/r, fg\lambda/hr, fg\mu/hr, dg\lambda\mu/hr; q)_\infty} \\
&\qquad \times {}_6\phi_5\left(\begin{matrix} af, bf, cf, g/d, fg\lambda/hr, fg\mu/hr \\ abcf, qf/d, fg/h, fg/r, fg\lambda\mu/hr \end{matrix}; q, q\right).
\end{aligned} \tag{6.10}$$

Equation (6.4) is the special case $\lambda = 1$ or $\mu = 1$ of (6.10).

By combining (6.6) and (6.10) we establish our next theorem.

THEOREM 6.5. *We have*

$$\begin{aligned}
&\frac{(q;q)_\infty^3}{(2\pi)^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{(cse^{i\psi}, cse^{-i\psi}, dre^{i\phi}, dre^{-i\phi}; q)_\infty}{(se^{i(\theta+\psi)}, se^{i(\theta-\psi)}, se^{i(\psi-\theta)}, se^{-i(\theta+\psi)}; q)_\infty} \\
&\qquad \times \frac{(fre^{i\theta}, fre^{-i\theta}, gre^{i\theta}, gre^{-i\theta}; q)_\infty}{(re^{i(\theta+\phi)}, re^{i(\theta-\phi)}, re^{i(\phi-\theta)}, re^{-i(\theta+\phi)}; q)_\infty} \\
&\qquad \times \frac{(e^{2i\psi}, e^{-2i\psi}; q)_\infty}{(\alpha e^{i\psi}, \alpha e^{-i\psi}, \beta e^{i\psi}, \beta e^{-i\psi}, \gamma e^{i\psi}, \gamma e^{-i\psi}; q)_\infty} \\
&\qquad \times \frac{(e^{2i\phi}, e^{-2i\phi}; q)_\infty}{(fe^{i\phi}, fe^{-i\phi}, ge^{i\phi}, ge^{-i\phi}, he^{i\phi}, he^{-i\phi}; q)_\infty} \\
&\qquad \times \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty\, d\theta\, d\phi\, d\psi}{(ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}; q)_\infty} \\
&\quad = \frac{(\alpha\beta\gamma ds, cds^2, dfr, dgr, fghr/d; q)_\infty}{(\alpha\beta, \alpha\gamma, \beta\gamma, cd, fg, fh, gh, hr/d, \alpha ds, \beta ds, \gamma ds, s^2; q)_\infty} \\
&\qquad \times {}_6\phi_5\left(\begin{matrix} \alpha ds, \beta ds, \gamma ds, cd, fg, dr/h \\ \alpha\beta\gamma ds, cds^2, qd/hr, dfr, dgr \end{matrix}; q, q\right)
\end{aligned} \tag{6.11}$$

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$$+\frac{(\alpha\beta\gamma hrs, chrs^2, dr/h, fhr^2, ghr^2; q)_\infty}{(\alpha\beta, \alpha\gamma, \beta\gamma, fh, gh, chr, \alpha hrs, \beta hrs, \gamma hrs, r^2, s^2; q)_\infty}$$
$$\times {}_6\phi_5\left(\begin{matrix}\alpha hrs, \beta hrs, \gamma hrs, chr, fghr/d, r^2\\ \alpha\beta\gamma hrs, qhr/d, fhr^2, ghr^2, chrs^2\end{matrix}; q, q\right).$$

In the same manner, by induction, one can evaluate a similar n-fold integral. We would like to leave details to the reader.

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