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AN INFINITE CLASS OF HADAMARD MATRICES

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Abstract

An infinite class of T-matrices is constructed using Golay sequences. A list is given with new Hadamard matrices of order $2^t \cdot q$, q odd, $q < 10000$, improving the known values of t.

Finally T-matrices are given of order $2m + 1$, for small values of $m \leq 12$ which do not coincide with those generated by Turyn sequences.

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1. Introduction

Geramita and Seberry [4, pages 415–425] gave a list of the smallest known values of t for which a Hadamard matrix of order $2^t \cdot q$ exists. This list was originally published by Seberry [7] who had proved earlier (Seberry Wallis [11]) that there exists an integer $t \leq 2[\log_2(q-3)]$ such that an Hadamard matrix exists for every order $2^s \cdot q$ where $s \geq t$.

This list was improved for some values of t and q; see Seberry [9], Agayan and Sarukhanyan [1], Sawade [6], Yamada [13, 14], and Yang [15, 16, 17].

In this paper we improve this list by giving smaller values of t for q odd, $q < 10000$. These values of q, t are given in Table 1, where the previously smallest values of t, denoted by t', are also given for comparison.

We also extend the results of Turyn [12] who constructed T-matrices of order $2m+1$ and $4m+3$ (see also Geramita and Seberry [4, pages 141–145]).

In our construction we drop Turyn's assumptions on skewness and symmetry, as will be explained in Section 2.

Some definitions are now in order.

(i) A $(+1,-1)$ matrix of order n is called a Hadamard matrix if it satisfies $HH^T = H^T H = nI_n$.

(ii) Four $(+1,-1)$ matrices W_1, W_2, W_3, W_4 of order w which satisfy $MN^T = NM^T$, for $M \neq N$, and $M, N \in \{W_1, W_2, W_3, W_4\}$, and $\sum_{i=1}^{4} W_i W_i^T = 4wI_w$, are called Williamson type matrices of order w.

(iii) A Baumert-Hall array of order t, denoted by $BH[4t]$, is a $4t \times 4t$ array with entries from the set of commuting variables $\{\pm x_1, \pm x_2, \pm x_3, \pm x_4\}$, if $\pm x_i$ occurs t times in each row (column) of $BH[4t]$ and if the rows (columns) of $BH[4t]$ are formally orthogonal.

(iv) Four circulant $(0,1,-1)$ matrices X_1, X_2, X_3, X_4 of order t which satisfy $X_i * X_j = 0$ for $i \neq j$ ($*$ denotes Hadamard product) and $\sum_{i=1}^{4} X_i X_i^T = tI_t$, are called T-matrices of order t.

It has been shown by Cooper and Wallis [3] that if there exists a quadruple of T-matrices of order t, then a $BH[4t]$ exists. The same result was also used by Turyn [12].

Our construction relies on a result of Baumert and Hall [2] whose proved that if there exists a $BH[4t]$ and a quadruple of Williamson type matrices of order w, then there exists a Hadamard matrix of order $4 \cdot w \cdot t$.

For $w < 100$, Williamson type matrices are not yet known for the orders 35, 39, 47, 53, 65, 67, 71, 73, 83, 89, 94 (see Agayan and Sarukhanyan [1] for some recent constructions).

Baumert-Hall arrays exist (see Seberry Wallis [8, 10], Ono and Sawade [5], Sawade [6], Yang [15, 16, 17] for the following orders.

(i) $A = \{1 + 2^a \cdot 10^b \cdot 26^c,\ a, b, c$ nonnegative integers$\}$;

(ii) $B = \{1, 3, \ldots, 41, 45, \ldots, 69, 75, 77, 81, 85, 87, 91, 93, 95, 99, 101\}$;

(iii) $C = \{(2p+1) \cdot q,\ p = 3, 6$ or $p = 2^a \cdot 10^b \cdot 26^c$ and odd $q \leq 59$ or $q = 2^{a'} \cdot 10^{b'} \cdot 26^{c'} + 1$, where a, b, c, a', b', c' are non-negative integers$\}$;

(iv) $5b, 96$, where $b \in A \cup B \cup C$;

(v) $(p^r + 1) \cdot t,\ p^r \cdot (p^r + 1) \cdot t$, where t is the order of any $BH[4t]$ and $p^r \equiv 1 \pmod 4$ is a prime power;

(vi) $2n$, where $4n$ is the order of any Hadamard matrix.

In those cases where we cannot construct a Hadamard matrix of order $n = 4 \cdot q$ we try to construct Hadamard matrices of order $2^t \cdot q$, for some small values of $t (t > 2)$. In the present paper we construct an infinite class of T-matrices of order $k = 2^a \cdot 10^b \cdot 26^c + 2^{a'} \cdot 10^{b'} \cdot 26^{c'}$, a, b, c, a', b', c' non-negative integers and hence Hadamard matrices of order $n = 4 \cdot w \cdot k$ where w is the order of known Williamson type matrices.

2. Construction of T-matrices

DEFINITION. Take any sequence $\tilde{A} = \{a_1, a_2, \dots, a_k\}$.

(i) The non-periodic auto-correlation function $N_A(\theta)$ is defined as

$$N_A(\theta) = \begin{cases} \sum_{i=1}^{n-\theta} a_i a_{i+\theta}, & \theta = 1, 2, \dots, k-1, \\ 0, & \theta \geq k. \end{cases}$$

(ii) The periodic auto-correlation function $P_A(\theta)$ is defined as

$$P_A(\theta) = \begin{cases} N_A(\theta) + N_A(n-\theta), & \theta = 1, 2, \dots, k-1, \\ 0, & \theta \geq k. \end{cases}$$

DEFINITION. If $\tilde{X} = \{\{a_1, a_2, \dots, a_n\}, \{b_1, b_2, \dots, b_n\}\}$ are two sequences where $a_i, b_j \in \{+1, -1\}$ and $N_X(\theta) = N_A(\theta) + N_B(\theta) = 0$, for $\theta = 1, 2, \dots, n-1$, then the sequences in $\tilde{X}$ are called Golay sequences of length n.

It has been proved (see Turyn [12]) that Golay sequences of length n exist for $n = 2^a \cdot 10^b \cdot 26^c$, a, b, c, non-negative integers (see also Geramita and Seberry [4, page 139]).

Hence if the sequences in $\tilde{X}$ are Golay, A is the circulant matrix with first row $(a_1, a_2, \dots, a_n)$ and B is the circulant matrix with first row $(b_1, b_2, \dots, b_n)$, then

$$AA^T + BB^T = 2nI_n.$$

For convenience we denote by $\tilde{A} = \{a_1, a_2, \dots, a_n\}$ the sequence and by $A = (a_1, a_2, \dots, a_n)$ the corresponding circulant matrix; also we let $\tilde{O}_n$ denote the sequence of n zeros.

Given the $(1, -1)$ sequences $\tilde{A} = \{a_1, a_2, \dots, a_n\}$, $\tilde{B} = \{b_1, b_2, \dots, b_n\}$, $\tilde{C} = \{c_1, c_2, \dots, c_m\}$ and $\tilde{D} = \{d_1, d_2, \dots, d_m\}$ we form the following $(0, 1, -1)$ sequences of length $n + m$:

$$\begin{aligned} \tilde{X} &= \{(\tilde{C} + \tilde{D})/2, \tilde{O}_n\}, \\ \tilde{Y} &= \{\{\tilde{C} - \tilde{D}\}/2, \tilde{O}_n\}, \\ \tilde{Z} &= \{\tilde{O}_m, \{\tilde{A} + \tilde{B}\}/2\}, \\ \tilde{W} &= \{\tilde{O}_m, \{\tilde{A} - \tilde{B}\}/2\}, \end{aligned} \tag{1}$$

and let X, Y, Z, W be the corresponding circulant matrices of order $n + m$.

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Theorem 1. *If A, B, C, D are as above, then the quadruple X, Y, Z, W are T-matrices of order $n+m$, $n \geq m$, if and only if*

(i) $N_A(\theta)+N_B(\theta)+N_C(\theta)+N_D(\theta)=0$, $\theta=1,2,\ldots,m-1$, *and for* $n>m$,

(ii) $N_A(\theta)+N_B(\theta)+N_A(n+m-\theta)+N_B(n+m-\theta)=0$, $\theta=m,\ldots,n-1$.

Proof. From the construction of X, Y, Z, W we have $X_1 * X_2 = 0$ for all $X_1, X_2 \in \{X, Y, Z, W\}$, with $X_1 \neq X_2$.

Now if $Q=(0,1,0,\ldots,0)$ is the circulant matrix of order $n+m$, then

$$X=\frac{1}{2}\sum_{i=1}^{m}(c_i+d_i)Q^{i-1}, \qquad Y=\frac{1}{2}\sum_{i=1}^{m}(c_i-d_i)Q^{i-1},$$

$$Z=\frac{1}{2}\sum_{i=1}^{n}(a_i+b_i)Q^{m+i-1}, \qquad W=\frac{1}{2}\sum_{i=1}^{n}(a_i-b_i)Q^{m+i-1}.$$

Hence

$$\begin{aligned}
XX^T+YY^T &= \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}(c_ic_j+d_id_j)Q^{i-j}\\
&= mI_{n+m}+\frac{1}{2}\sum_{\theta=1}^{m-1}\sum_{j=1}^{m-\theta}(c_jc_{j+\theta}+d_jd_{j+\theta})Q^{\theta}\\
&\quad+\frac{1}{2}\sum_{\theta=1-m}^{-1}\sum_{j=1-\theta}^{m}(c_jc_{j+\theta}+d_jd_{j+\theta})Q^{\theta}\\
&= mI_{n+m}+\frac{1}{2}\sum_{\theta=1}^{m-1}(N_C(\theta)+N_D(\theta))Q^{\theta}\\
&\quad+\frac{1}{2}\sum_{\theta=n+1}^{n+m-1}(N_C(n+m-\theta)+N_D(n+m-\theta))Q^{\theta},
\end{aligned}$$

because $c_i=0$, $d_i=0$ for $i=m+1,\ldots,n+m$. Similarly

$$\begin{aligned}
ZZ^T+WW^T &= nI_{n+m}+\frac{1}{2}\sum_{\theta=1}^{n-1}\sum_{i=1}^{n-\theta}(a_ja_{j+\theta}+b_jb_{j+\theta})Q^{\theta}\\
&\quad+\frac{1}{2}\sum_{\theta=-n+1}^{-1}\sum_{j=1-\theta}^{n}(a_ja_{j+\theta}+b_jb_{j+\theta})Q^{\theta}\\
&= nI_{n+m}+\frac{1}{2}\sum_{\theta=1}^{n-1}(N_A(\theta)+N_B(\theta))Q^{\theta}\\
&\quad+\frac{1}{2}\sum_{\theta=m+1}^{n+m-1}(N_A(n+m-\theta)+N_B(n+m-\theta))Q^{\theta}.
\end{aligned}$$

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Thus

$$XX^T + YY^T + ZZ^T + WW^T = (n+m)I_{n+m}$$
$$+ \frac{1}{2}\sum_{\theta=1}^{m-1}(N_A(\theta) + N_B(\theta) + N_C(\theta) + N_D(\theta))Q^\theta$$
$$+ \frac{1}{2}\sum_{\theta=m}^{n-1}(N_A(\theta) + N_B(\theta) + N_A(n+m-\theta) + N_B(n+m-\theta))Q^\theta$$
$$+ \frac{1}{2}\sum_{\theta=n}^{n+m-1}(N_A(n+m-\theta) + N_B(n+m-\theta)$$
$$+ N_C(n+m-\theta) + N_D(n+m-\theta))Q^\theta.$$

From the requirement

$$XX^T + YY^T + ZZ^T + WW^T = (n+m)I_{n+m}$$

we obtain the conditions (i) and (ii) given in Theorem 1.

COROLLARY 1. *If $\tilde{A}, \tilde{B}$ are Golay sequences of length n and $\tilde{C}, \tilde{D}$ are Golay sequences of length m, then X, Y, Z, W form a quadruple of T-matrices of order $n+m$.*

PROOF. Since $N_A(\theta)+N_B(\theta) = 0$, $\theta = 1, 2, \dots, n-1$, and $N_C(\theta)+N_D(\theta) = 0$, $\theta = 1, 2, \dots, m-1$, the conditions of Theorem 1 are satisfied.

COROLLARY 2. *There exist T-matrices of order $k = 2^a \cdot 10^b \cdot 26^c + 2^{a'} \cdot 10^{b'} \cdot 26^{c'}$, where a, b, c, a', b', c' are non-negative integers.*

PROOF. Since Golay sequences exist for $n = 2^a \cdot 10^b \cdot 26^c$ and $m = 2^{a'} \cdot 10^{b'} \cdot 26^{c'}$, then according to Corollary 1 there exist T-matrices of order $k = n+m$.

The case $m = 1$ has been treated by Turyn [12] who constructed T-matrices of order $2^a \cdot 10^b \cdot 26^c + 1$, a, b, c are non-negative integers. For the construction of Hadamard matrices of order $n = 4 \cdot w \cdot k$ we have taken $w < 100$ except 35,39,47,53,65,67,71,73,83,89,94 and $k = 2^a \cdot 10^b \cdot 26^c + 2^{a'} \cdot 10^{b'} \cdot 26^{c'} \leq 40000$ for all non-negative integers a, b, c, a', b', c', and then found t and odd q with $2^t \cdot q = 4 \cdot w \cdot k$, $q < 10000$. The new Hadamard matrices are listed in Table 1 and the corresponding values of a, b, c, a', b', c' have been tabulated and are available on request. We have also constructed T-matrices of order $2m+1$ by taking $n = m+1$, $m \geq 1$. Such matrices were also constructed by Turyn [12] by imposing some additional assumptions on skewness and symmetry for X, Y, Z, W, see also Geramita and Seberry [4, page 142]. The restrictions imposed by Turyn [12] are needed for the construction of T-sequences of order $4m+3$ from T-sequences of order $2m+1$ but are not necessary for the construction of T-matrices of a given order.

From Theorem 1 and for $n = m+1$ we have the necessary and sufficient conditions

$$(2)\quad \begin{aligned} &N_A(\theta)+N_B(\theta)+N_C(\theta)+N_D(\theta)=0, \qquad \theta = 1,2,\ldots,m-1, \text{ and}\\ &N_A(m)+N_B(m)=0. \end{aligned}$$

If P_A, P_B, P_C, P_D are respectively the number of -1's in each row (column) of A, B, C, D, and if

$$AA^T + BB^T = (2n, u_1, \ldots, u_{n-1}), \quad CC^T + DD^T = (2m, v_1, \ldots, v_{m-1})$$

then multiplying on the left by a row vector of ones and on the right by the column vector of ones we have

$$(3)\quad \begin{aligned} (n-2P_A)^2 + (n-2P_B)^2 &= 2n + (u_1 + \cdots + u_{n-1}),\\ (m-2P_C)^2 + (m-2P_D)^2 &= 2m + (v_1 + \cdots + v_{m-1}) \end{aligned}$$

but

$$u_i = P_A(i)+P_B(i) = N_A(i)+N_B(i)+N_A(n-i)+N_B(n-i), \qquad i = 1,\ldots,n-1,$$

and

$$v_j = P_C(j)+P_D(j) = N_C(j)+N_D(j)+N_C(m-j)+N_D(m-j), \qquad j = 1,\ldots,m-1.$$

Hence

$$\sum_{i=1}^{n-1} u_i + \sum_{j=1}^{m-1} v_j = 2\sum_{i=1}^{n-1}(N_A(i)+N_B(i)) + 2\sum_{j=1}^{m-1}(N_C(j)+N_D(j))$$

or

$$\sum_{i=1}^{n-1} u_i + \sum_{j=1}^{m-1} v_j = 2\sum_{i=1}^{m-1}(N_A(i)+N_B(i)+N_C(i)+N_D(i)) + 2\sum_{i=m}^{n-1}(N_A(i)+N_B(i)).$$

If $n = m+1$ and relations (2) hold we have

$$(4)\quad \sum_{i=1}^{n-1} u_i + \sum_{j=1}^{m-1} v_j = 0$$

and $(n-2P_A)^2 + (n-2P_B)^2 + (m-2P_C)^2 + (m-2P_D)^2 = 2(n+m)$, or

$$(5)\quad \begin{aligned} &(m+1-P_A-P_B)^2 + (P_A-P_B)^2\\ &\qquad + (m-P_C-P_D)^2 + (P_C-P_D)^2 = 2m+1. \end{aligned}$$

Relation (5) is necessary in order to have

$$XX^T + YY^T + ZZ^T + WW^T = (n+m)I_{n+m}. \tag{6}$$

For a given m we find P_A, P_B, P_C, P_D satisfying (5) and then on the computer examine which particular A, B, C, D satisfy (2). In Table 2 are listed A, B, C, D found in this way and in Table 3 are given T-matrices constructed from A, B, C, D as described in Theorem 1.

Note that the corresponding Turyn sequences of length $m+1, m+1, m, m$ for $m = 8, 9, 11, 13, 15, 17$ do not exist; see Geramita and Seberry [4, pages 142–143], whereas in our case we found T-matrices for $m = 8, 9, 11$.

Another distinction should be made between T-sequences and T-matrices, because, by Geramita and Seberry [4, page 145], T-sequences generate T-matrices but not vice-versa.

DEFINITION. The four $(0, 1, -1)$ sequences $\tilde{X} = \{x_1, \ldots, x_t\}$, $\tilde{Y} = \{y_1, \ldots, y_t\}$, $\tilde{Z} = \{z_1, \ldots, z_t\}$, $\tilde{W} = \{w_1, \ldots, w_t\}$ are called T-sequences of length t if

(i) $M * N = \tilde{O}_t$ for $M, N \in \{\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}\}$, $M \neq N$,

(ii) $N_X(\theta) + N_Y(\theta) + N_Z(\theta) + N_W(\theta) = 0$, $\theta = 1, \ldots, t-1$,

(iii) $\tilde{X} + \tilde{Y} + \tilde{Z} + \tilde{W}$ is a $(1, -1)$ sequence of length t.

If the conditions (i), (ii) of Theorem 1 are satisfied it does not necessarily, mean that $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}$ are T-sequences, although they generate T-matrices.

TABLE 1

New Hadamard matrices of order $2^s \cdot q, t \leq s < t'$

q	t	t'	q	t	t'	q	t	t'	q	t	t'
			1969	4	9	4087	4	5			
179	4	8	2053	3	7	4101	3	5	6667	3	4
251	4	6	2327	4	6	4267	4	7	6981	4	6
269	3	8	2357	5	6	4285	4	5	7183	3	7
			2427	4	6	4571	3	11	7223	4	5
537	4	5	2513	4	5	4757	5	7	7697	4	8
653	3	10	2563	4	5	5001	3	5	7729	3	4
737	4	6	2571	4	5	5191	4	9	7961	4	5
						5293	4	6	8187	4	5
1169	4	5	2837	5	10	5359	4	5	8257	4	5
1257	4	5	2881	4	6	5413	3	7	8339	3	9
						5549	4	9	8393	3	4
1301	3	14	3127	3	4	5959	3	4	8489	3	8
1631	4	5	3263	4	7	5965	4	5	9107	3	8
1753	6	11	3401	4	9	5999	4	5	9287	4	6
1883	3	6	3497	3	8	6159	3	4	9427	4	5
1943	4	6	4013	3	6	6431	3	4	9953	3	8

(t' is given in Geramita and Seberry [4, pages 415–425].)

TABLE 2

Giving A, B of order $n = m + 1$ and C, D of order m satisfying the conditions of Theorem 1.

(Here + stands for 1 and − for −1, $2m + 1 = f_1^2 + f_2^2 + f_3^2 + f_4^2 = (m + 1 - P_A - P_B)^2 + (P_A - P_B)^2 + (m - P_C - P_D)^2 + (P_C - P_D)^2$ and P_A, P_B, P_C, P_D are respectively the number of −1's in each row (column) of A, B, C, D.)

n	m	f_1	f_2	f_3	f_4	P_A	P_B	P_C	P_D	A	B	C	D
2	1	1	1	1	0	1	0	0	0	(−+)	(++)	(+)	(+)
3	2	2	1	0	0	1	0	1	1	(−++)	(+++)	(−+)	(−+)
		1	0	2	0	1	1	0	0	(−++)	(+−+)	(++)	(++)
4	3	1	1	2	1	2	1	1	0	(−++−)	(−+++)	(+−+)	(+++)
										(+−−+)	(+++−)	(+−+)	(+++)
5	4	2	1	2	0	2	1	1	1	(−−+++)	(+++−+)	(+−++)	(++−+)
										(−−+++)	(+++−+)	(++−+)	(++−+)
										(−+−++)	(++−++)	(−+++)	(−+++)
										(−−+++)	(+−+++)	(+−++)	(+−++)
										(−+++−)	(−++++)	(+−++)	(+−++)
										(++−+−)	(++−++)	(+++−)	(+++−)
6	5	3	1	1	0	2	1	2	2	(++−−++)	(−+++++)	(−+−++)	(−+−++)
		1	1	3	0	3	2	1	1	(++−−+−)	(−++++−)	(+−+++)	(+−+++)
7	6	3	0	2	0	2	2	2	2	(−−+++++)	(+−+−+++)	(++−−++)	(+−++−+)
		3	2	0	0	3	1	3	3	(−−+++−+)	(+++−+++)	(++−+−−)	(++−+−−)
8	7	3	1	2	1	3	2	3	2	(+++−−+−+)	(−+++++−+)	(+−+++−−)	(−++−+++)
										(+++−++−−)	(++++−+−+)	(++−−+−+)	(++++−−+)
9	8	4	1	0	0	3	2	4	4	(+−−+++++−)	(+−−++++++)	(−−++−+−+)	(−−++−+−+)
		1	0	4	0	4	4	2	2	(+++−++−−−)	(+++−−+−−+)	(+++−+−++)	(+++−+−++)
10	9	1	1	4	1	5	4	3	2	(−−−++++−+−)	(+++−++−+−−)	(+−++−−+++)	+−+++−+++)
		3	1	3	0	4	3	3	3	(+++−−+−+−+)	(+++++−++−−)	(+++−−++−+)	(++++−−+−+)
11	10	4	1	2	0	4	3	4	4	(−+−−+−+++++)	(+++++−+−−++)	(++−−+++−+−)	(++−−+++−+−)
										(++++−++−−+−)	(++−−−++++++)	(−+−++−−+++)	(++−+−+−++−)
		2	1	4	0	5	4	3	3	(+++−−+−−+−+)	(+++−+++−+−−)	(+++−−+++−+)	(+++++−−+−+)
12	11	3	3	2	1	6	3	5	4	(+−−+−−+++−−+)	(−+++−+−+++++)	(−++++−+−+−−)	(−−+++++−++−)
13	12	3	0	4	0	5	5	4	4	(++−+−++−−+−++)	(−+++−+−−−++++)	(−+−−+++−++++)	(++++−+++−−+−)
		2	1	4	2	6	5	5	3	(++++−+−−++−−−)	(++++−+−−++−−+)	(+++−−+−+−−++)	(+++++−+−+−++)

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TABLE 3

T-matrices of order $g = 2m + 1$.

(The numbers (without the sign) inside the parentheses indicate the position of ±1's, the negative sign meaning that it is a -1; the remaining elements are 0.)

g	X	Y	Z	W
3	(1)	0	(3)	(-2)
5	(-1, 2)	0	(4, 5)	(-3)
	(1, 2)	0	(5)	(-3, 4)
7	(1, 3)	(-2)	(-4, 5, 6)	(-7)
	(1, 3)	(-2)	(4)	(-5, -6, 7)
9	(1, 4)	(-2, 3)	(7, 9)	(-5, -6, 8)
	(1, 2, -3, 4)	0	(7, 9)	(-5, -6, 8)
	(-1, 2, 3, 4)	0	(6, -7, 8, 9)	(-5)
	(1, -2, 3, 4)	0	(-6, 7, 8, 9)	(-5)
	(1, -2, 3, 4)	0	(-5, 6, 7, 8)	(-9)
	(1, 2, 3, -4)	0	(5, 6, -7, 8)	(-9)
11	(-1, 2, -3, 4, 5)	0	(7, 10, 11)	(6, -8, -9)
	(1, -2, 3, 4, 5)	0	(7, 10, -11)	(6, -8, -9)
13	(1, 6)	(2, -3, -4, 5)	(-8, 9, 11, 12, 13)	(-7, 10)
	(1, 2, -3, 4, -5, -6)	0	(9, 11, 13)	(-7, -8, 10, -12)
15	(3, 5)	(1, -2, 4, -6, -7)	(9, 10, 13, -14, 15)	(8, -11, -12)
	(1, 2, -6, 7)	(-3, -4, 5)	(8, 9, 10, 13, -14)	(-11, 12, -15)
17	(-1, -2, 3, 4, -5, 6, -7, 8)	0	(9, -10, 11, 12, 13, 14, 15, 16)	(-17)
	(1, 2, 3, -4, 5, -6, 7, 8)	0	(9, 10, -11, -12, 14, -15, -16)	(13, -17)
19	(1, -2, 3, 4, -6, 7, 8, 9)	(-5)	(14, 15, -19)	(-10, -11, -12, 13, 16, -17, 18)
	(1, 2, 3, -5, 7, -8, 9)	(-4, 6)	(10, 11, 12, 17, -18)	(-13, -14, 15, -16, 19)
21	(1, 2, -3, -4, 5, 6, 7, -8, 9, -10)	0	(12, 15 - 16, 17, 20, 21)	(-11, -13, -14, 18, 19)
	(2, -3, 4, -7, 8, 9)	(-1, 5, -6, 10)	(11, 12, -15, 16, 17, 20)	(13, 14, -18, -19, -21)
	(1, 2, 3, 8, -9, 10)	(-4, -5, 6, 7)	(11, 12, 13, -14, 16, -18, 19, -20)	(-15, -17, 21)
23	(-1, 3, 4, 5, 7, -8, 9, -11)	(2, -6, -10)	(15, -16, 19, 20, 23)	(12, -13, -14, -17, 18, -21, -22)
25	(2, 6, 7, 11)	(-1, -3, -4, 5, -8, 9, 10, 12)	(14, 16, -17, 18, -20, -21, 22, 24, 25)	(13, -15, 19, -23)
	(1, 2, 3, -10, 11, 12)	(-4, -5, 6, -7, 8, -9)	(13, 14, 15, 16, -17, 18, -19, -20, 21, 22, -23, -24)	(-25)

To see this we form the four $(0, 1, -1)$ sequences given in (1); then (i) and (iii) of the above definition are satisfied but not necessarily (ii) because we easily see that

$$N_X(\theta) + N_Y(\theta) = \begin{cases} \frac{1}{2}(N_C(\theta) + N_D(\theta)), & \theta = 1, \ldots, m-1, \\ 0, & \theta = m, \ldots, n+m-1, \end{cases}$$

and

$$N_Z(\theta) + N_W(\theta) = \begin{cases} \frac{1}{2}(N_A(\theta) + N_B(\theta)), & \theta = 1, \ldots, n-1, \\ 0, & \theta = n, \ldots, n+m-1. \end{cases}$$

Therefore $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}$ are T-sequences of order $n+m$ only if

(a) for $n \geq m+1, m \geq 2$,
$N_A(\theta) + N_B(\theta) + N_C(\theta) + N_D(\theta) = 0, \quad \theta = m, \ldots, n-1,$
$N_A(\theta) + N_B(\theta) = 0, \quad \theta = m, \ldots, n-1,$

(b) for $n = m, m \geq 2$,
$N_A(\theta) + N_B(\theta) + N_C(\theta) + N_D(\theta) = 0, \quad \theta = 1, \ldots, m-1,$

and

(c) for $n \geq 2, m = 1$, c)
$N_A(\theta) + N_B(\theta) = 0, \quad \theta = 1, \ldots, n-1.$

These conditions are stronger than those given in Theorem 1 except for $m = 1$, $n = 2$; $m \geq 2$, $n = m$; $m \geq 2$, $n = m+1$; where they coincide.

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