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ABSOLUTES OF ALMOST REALCOMPACTIFICATIONS

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Abstract

Given Hausdorff spaces X and Y and a perfect irreducible and θ-continuous map f from X onto Y, a technique that carries open (ultra) filters on X to open (ultra) filters on Y back and forth in a natural way is introduced. It is proved that if f is a perfect irreducible and θ-continuous map from X onto Y, then X is almost realcompact if and only if Y is almost realcompact. Several commutativity relations between the 'absolutes of almost realcompactifications' and the 'almost realcompactifications of absolutes' of a space X are discussed.

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1. Preliminaries

All spaces considered in this paper are Hausdorff. Throughout, $\tau(X)$ will denote the topology on the space X. If X is a space and A is a subset of X, then $\operatorname{int}_X(A)$, $\operatorname{cl}_X(A)$ and $\operatorname{bd}_X(A)$ will denote the interior of A, closure of A, and the boundary of A in X, respectively. A subset A of a space X is called *regular open* (respectively, *regular closed*) if $A = \operatorname{int}_X \operatorname{cl}_X(A)$ (respectively, $A = \operatorname{cl}_X \operatorname{int}_X(A)$.) The family $RO(X)$ (respectively, $R(X)$) denotes the complete Boolean algebra of regular open (respectively, regular closed) subsets of X, and $CO(X)$ denotes the algebra of all clopen (= closed and open) subsets of X. A space X is called *extremally disconnected* if $\operatorname{cl}_X(U)$ is open in X for each $U \in \tau(X)$. A space X is called *zero-dimensional* if it has a basis consisting of clopen subsets. A *map* from a space X to a space Y is a (not necessarily continuous) function $f\colon X \to Y$. The

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notation $f\colon X \twoheadrightarrow Y$ will mean that f is a surjection. Let $f\colon X \to Y$. If $Z \subseteq X$, the *small image* of A (under f) is defined (see [3], [14]) by $f^{\#}(A) = \{y \in Y\colon f^{\leftarrow}(Y) \subseteq A\}$ $(\subseteq f(A))$. If f is onto, then $f^{\#}(A) = Y \setminus f(X \setminus A)$. For a family $\mathscr{A} \subseteq \mathbf{P}(X)$, $f^{\#}(\mathscr{A})$ will denote the family $\{f^{\#}(A)\colon A \in \mathscr{A}\}$. A map $f\colon X \to Y$ is called *compact* if, for each $y \in Y$, $f^{\leftarrow}(y)$ is a compact subset of X, *perfect* if f is both a compact and a closed mapping, and *irreducible* if f is onto and, for each proper closed subset A of X, $f(A) \neq Y$. A mapping $f\colon X \to Y$ is called ***θ*-continuous at a point** $x \in X$ if, for each open neighborhood G of $f(x)$ in Y, there is an open neighborhood U of x in X such that $f(\operatorname{cl}_X(U)) \subseteq \operatorname{cl}_Y(G)$. If f is θ-continuous at each $x \in X$, then f is called ***θ*-continuous** (see [4]). A bijection $f\colon X \twoheadrightarrow Y$ is called a ***θ*-homeomorphism** provided that both the maps f and $f^{\leftarrow}$ are θ-continuous. Certainly, every continuous function is θ-continuous. If $f\colon X \to Y$ is θ-continuous and Y is regular, then f is continuous. Also, the composition of two θ-continuous functions is θ-continuous. Note that the next two facts contain well known results about maps.

(1.1) FACTS (see [3], [14], [20]).

(a) Continuous maps with compact domain are perfect.

(b) If $f\colon X \to Y$ is a perfect map, and if C is a compact subset of Y, then $f^{\leftarrow}(C)$ is a compact subset of X.

(c) The composition of two perfect (respectively, irreducible and closed) maps is a perfect (respectively, irreducible and closed) map.

(d) If $f\colon X \to Y$ is a perfect map, then

(i) for each closed subset A of X, $f|_A$ is a perfect mapping (whether regarded as a function into Y or as a function onto $f(A)$), and

(ii) if $B \subseteq Y$, then $f|_{f^{\leftarrow}(B)}\colon f^{\leftarrow}(B) \to B$ is a perfect mapping.

(e) If $f\colon X \twoheadrightarrow Y$ is a closed and irreducible surjection, and if S is a dense subset of Y, then $f^{\leftarrow}(S)$ is a dense subset of X, and $f|_{f^{\leftarrow}(S)}$ is a closed and irreducible surjection from $f^{\leftarrow}(S)$ onto S.

(f) If $f\colon X \twoheadrightarrow Y$ is a perfect surjection, then there is a closed subset C of X such that $f|_C\colon C \twoheadrightarrow Y$ is irreducible.

(g) A surjection $f\colon X \twoheadrightarrow Y$ is closed and irreducible if and only if $f^{\#}(G)$ is a non-empty open subset of Y for each non-empty open subset G of X.

(h) If $f\colon X \to Y$ is θ-continuous, then for each $U \in \tau(Y)$, $f^{\leftarrow}(\operatorname{cl}_Y(U))$ is a neighborhood of $f^{\leftarrow}(U)$, and $f[\operatorname{cl}_X(f^{\leftarrow}(U))] \subseteq \operatorname{cl}_Y(U)$.

(1.2) FACTS (see [3], [14], [20]). Let $f\colon X \to Y$ be a closed, irreducible and θ-continuous mapping.

(a) If $U \in \tau(Y)$, then $f[\operatorname{cl}_X(f^{\leftarrow}(U))] = \operatorname{cl}_Y(U)$ and $\operatorname{int}_X[f^{\leftarrow}(\operatorname{cl}_Y(U))] = \operatorname{int}_X \operatorname{cl}_X(f^{\leftarrow}(U))$.

(b) If $G \in \tau(X)$, then $\operatorname{int}_X \operatorname{cl}_X(G) = \operatorname{int}_X[f^{\leftarrow}(\operatorname{cl}_Y(f^{\#}(G)))] \subseteq \operatorname{cl}_X(G) \subseteq f^{\leftarrow}[\operatorname{cl}_Y(f^{\#}(G))]$.

(c) If $G \in \tau(X)$, then $f(\operatorname{cl}_X(G)) = \operatorname{cl}_Y(f^{\#}(G))$.

(d) For each $U, V \in RO(Y)$, $\operatorname{int}_X \operatorname{cl}_X(f^{\leftarrow}(U \cap V)) = \operatorname{int}_X \operatorname{cl}_X(f^{\leftarrow}(U)) \cap \operatorname{int}_X \operatorname{cl}_X(f^{\leftarrow}(V))$.

(e) For each $W \in \tau(Y)$, $f^{\#}[\operatorname{int}_X \operatorname{cl}_X[f^{\leftarrow}(\operatorname{int}_Y \operatorname{cl}_Y(W)))] = \operatorname{int}_Y \operatorname{cl}_Y(W)$.

(f) If f is also perfect, then for each G in $RO(X)$, $f^{\#}(G) \in RO(Y)$.

(1.3) An *open filter* on a space X will always mean a filter in the lattice $\tau(X)$ of all open subsets of X. A subcollection $\mathscr{B}$ of $\tau(X)$ is said to be an *open filter base* for some open filter on X if for every $B_1, B_2 \in \mathscr{B}$, there exists a $B \in \mathscr{B}$ such that $B \subseteq B_1 \cap B_2$. If $\mathscr{A}$ is any non-empty family of open subsets of X with the finite intersection property, then $\langle \mathscr{A} \rangle$ will denote the open filter on X generated by $\mathscr{A}$. An *open ultrafilter* on X is an open filter on X which is a maximal element (with respect to set inclusion) in the family of all open filters on X. It is proved in [21, Example 12 G] that an open filter $\mathscr{U}$ on a space X is an open ultrafilter if and only if for each nonempty open subset U of X, either $U \in \mathscr{U}$ or $X \setminus \operatorname{cl}_X(U) \in \mathscr{U}$, if and only if for each $V \in \tau(X)$, if $V \cap U \neq \varnothing$ for every $U \in \mathscr{U}$, then $V \in \mathscr{U}$. This characterization of open ultrafilters will be used subsequently without reference. Note that every open ultrafilter $\mathscr{U}$ is prime (i.e. $A \cup B \in \mathscr{U}$ implies that $A \in \mathscr{U}$ or $B \in \mathscr{U}$ for each A, B in $\tau(X)$), and for a set $U \in \tau(X)$, $U \in \mathscr{U}$ if and only if $\operatorname{int}_X \operatorname{cl}_X(U) \in \mathscr{U}$. If $\mathscr{F}$ is a filterbase on X, then $\operatorname{ad}_X(\mathscr{F}) = \bigcap\{\operatorname{cl}_X(F)\colon F \in \mathscr{F}\}$ denotes the *adherence* of $\mathscr{F}$ in X. $\mathscr{F}$ is called *free* if $\operatorname{ad}_X(\mathscr{F}) = \varnothing$; otherwise, $\mathscr{F}$ is called *fixed*. For an open filter $\mathscr{F}$ on X, we shall denote by $\mathscr{F}_s$ the open filter on X generated by the filterbase $\{\operatorname{int}_X \operatorname{cl}_X(A)\colon A \in \mathscr{F}\}$, i.e.,

$$\mathscr{F}_s = \langle \{\operatorname{int}_X \operatorname{cl}_X(A)\colon A \in \mathscr{F}\} \rangle.$$

The following fact is needed in the sequel.

(1.4) Fact [16, 1.9]. Let $\mathscr{F}$ be an open filter on a space X.

(a) $\bigcap\{\mathscr{U}\colon \mathscr{U}$ is an open ultrafilter on X, $\mathscr{U} \supset \mathscr{F}\} = \{T \in \tau(X)\colon \operatorname{int}_X \operatorname{cl}_X(T) \in \mathscr{F}\}$, and $\operatorname{ad}_X\{T \in \tau(X)\colon \operatorname{int}_X \operatorname{cl}_X(T) \in \mathscr{F}\} = \operatorname{ad}_X(\mathscr{F})$.

(b) $\mathscr{F}$ is contained in a unique open ultrafilter $\mathscr{U}$ if and only if $\mathscr{U}_s \subseteq \mathscr{F} \subseteq \mathscr{U}$.

(c) $\operatorname{ad}_X(\mathscr{F}) = \operatorname{ad}_X(\mathscr{F}_s)$.

(d) If $\mathscr{U}$ is an open ultrafilter on X, then $\mathscr{U}_s$ is contained in a unique open ultrafilter on X, viz., $\mathscr{U}$ itself.

We now prove two results which will be used frequently in the coming sections. (See for instance 1.15, 1.17, 1.21, 1.22, 2.2, 2.3, 2.13, 2.14 and 2.15.)

(1.5) Definition. Let X and Y be spaces and f a map from X onto Y. If $\mathscr{F}$ is an open filter on X and $\mathscr{G}$ is an open filter on Y, we define

$$\mathscr{F}^{\rightarrow} = \{W \in \tau(Y)\colon \operatorname{int}_Y \operatorname{cl}_Y(W) \supseteq f^{\#}(A) \text{ for some } A \in \mathscr{F}\},$$

and

$$\mathscr{G}^{\leftarrow} = \{U \in \tau(X): f^{\#}(u) \in \mathscr{G}\}.$$

(1.6) PROPOSITION. *Let X and Y be Hausdorff spaces and f: $X \twoheadrightarrow Y$ a perfect, irreducible and θ-continuous surjection. If $\mathscr{F}$ is any open filter on X, then the following statements are true.*

(a) *$f^{\#}(\mathscr{F})$ is an open filter base on Y, and* $f(\mathrm{ad}_X(\mathscr{F})) = \mathrm{ad}_Y(f^{\#}(\mathscr{F})) = \mathrm{ad}_Y\langle f^{\#}(\mathscr{F})\rangle$.

(b) (i) *$\mathscr{F}^{\rightarrow}$ is an open filter on Y containing $f^{\#}(\mathscr{F})$, $\mathscr{F}^{\rightarrow} = \bigcap\{\mathscr{W}$: $\mathscr{W}$ is an open ultrafilter on Y such that $\mathscr{W} \supseteq f^{\#}(\mathscr{F})\}$, and* $\mathrm{ad}_Y(\mathscr{F}^{\rightarrow}) = \mathrm{ad}_Y(f^{\#}(\mathscr{F}))$.

(ii) *Furthermore, if $\mathscr{F}$ is an open ultrafilter on X, then $\mathscr{F}^{\rightarrow}$ is an open ultrafilter on Y.*

PROOF. Obviously, $f^{\#}(\mathscr{F})$ is an open filter base on Y. Also, $f(\mathrm{ad}_X(\mathscr{F})) = f[\bigcap\{\mathrm{cl}_X(A)\colon A \in \mathscr{F}\}] \subseteq \bigcap\{f(\mathrm{cl}_X(A))\colon A \in \mathscr{F}\} = \bigcap\{\mathrm{cl}_Y(f^{\#}(A)\colon A \in \mathscr{F})\}$ (by 1.2) $= \mathrm{ad}_Y(f^{\#}(\mathscr{F}))$. To prove the reverse inequality, let $p \in \mathrm{ad}_Y(f^{\#}(\mathscr{F}))$. It suffices to show that $f^{\leftarrow}(p) \cap \mathrm{ad}_X(\mathscr{F}) \neq \varnothing$. Assume that $f^{\leftarrow}(p) \cap \mathrm{ad}_X(\mathscr{F}) = \varnothing$. Then $f^{\leftarrow}(p) \subseteq \{X \setminus \mathrm{cl}_X(A)\colon A \in \mathscr{F}\}$ and, since $f^{\leftarrow}(p)$ is a compact subset of X, there exist finitely many $A_i \in \mathscr{F}$, $i = 1, 2, \ldots, n$ such that $f^{\leftarrow}(p) \subseteq \bigcup\{X \setminus \mathrm{cl}_X(A_i)\colon i = 1, 2, \ldots, n\} \subseteq X \setminus \mathrm{cl}_X(\bigcap\{A_i\colon i = 1, 2, \ldots, n\}) = X \setminus \mathrm{cl}_X(A)$, where $A = \bigcap\{A_i\colon i = 1, 2, \ldots, n\} \in \mathscr{F}$. Thus $f^{\leftarrow}(p) \cap \mathrm{cl}_X(A) = \varnothing$, whence $p \notin f(\mathrm{cl}_X(A)) = \mathrm{cl}_Y(f^{\#}(A))$, contradicting the fact that $p \in \mathrm{ad}_Y(f^{\#}(\mathscr{F}))$. Therefore, $p \in f(\mathrm{ad}_X(\mathscr{F}))$, and (a) follows. An easy verification shows that $\mathscr{F}^{\rightarrow}$ is an open filter on Y containing $f^{\#}(\mathscr{F})$, and the first equality in (b)(i) is a direct consequence of 1.4. The last part of (b)(i) follows from 1.4. To prove (b)(ii), let V be an open subset of Y such that $V \cap W \neq \varnothing$ for all $W \in \mathscr{F}^{\rightarrow}$. Then $V \cap f^{\#}(A) \neq \varnothing$ for each $A \in \mathscr{F}$. So, $(\mathrm{int}_Y \mathrm{cl}_Y(V)) \cap f^{\#}(A) \neq \varnothing$, whence $\mathrm{int}_X \mathrm{cl}_X(f^{\leftarrow}(\mathrm{int}_Y \mathrm{cl}_Y(V))) \cap A \neq \varnothing$ for each $A \in \mathscr{F}$. Using 1.2(e) and the fact that $\mathscr{F}$ is an open ultrafilter, we obtain $\mathrm{int}_Y \mathrm{cl}_Y(V) \in f^{\#}(\mathscr{F}) \subseteq \mathscr{F}^{\rightarrow}$, whence $V \in \mathscr{F}^{\rightarrow}$, and $\mathscr{F}^{\rightarrow}$ is an open ultrafilter on Y. The proof of the proposition is complete.

(1.7) PROPOSITION. *Let X and Y be Hausdorff spaces and f: $X \twoheadrightarrow Y$ a perfect, irreducible and θ-continuous surjection. If $\mathscr{G}$ is an open filter on Y, then the following statements are true.*

(a) *$\mathscr{G}^{\leftarrow}$ is an open filter on X such that for every $U \in \mathscr{G}$,* $\mathrm{int}_X \mathrm{cl}_X(f^{\leftarrow}(U)) \in \mathscr{G}^{\leftarrow}$, *and* $f[\mathrm{ad}_X(\mathscr{G}^{\leftarrow})] = \mathrm{ad}_Y(\mathscr{G})$.

(b) *If $\mathscr{G}$ is an open ultrafilter on Y, then $\mathscr{G}^{\leftarrow}$ is an open ultrafilter on X.*

PROOF. It is easy to see that $\mathscr{G}^{\leftarrow}$ is an open filter on X. Now, if $U \in \mathscr{G}$, then by 1.2 and the fact that $U \subseteq f^{\#}(\mathrm{int}_X(f^{\leftarrow}(\mathrm{cl}_Y(U)))$, it follows that

$f^{\#}(\operatorname{int}_X \operatorname{cl}_X(f^{\leftarrow}(U))) \in \mathscr{G}$. Hence $\operatorname{int}_X \operatorname{cl}_X(f^{\leftarrow}(U)) \in \mathscr{G}^{\leftarrow}$. Now, let $p \in f(\operatorname{ad}_X(\mathscr{G}^{\leftarrow}))$. If $p \notin \operatorname{ad}_Y(\mathscr{G})$, then there exists a regular open set $U \in \mathscr{G}$ and a regular open neighborhood V of p in Y such that $V \cap U = \varnothing$. By 1.2, $G = \operatorname{int}_X \operatorname{cl}_X f^{\leftarrow}(V)$ is an open neighborhood of $f^{\leftarrow}(p)$ in X such that $G \cap \operatorname{int}_X \operatorname{cl}_X(f^{\leftarrow}(U)) = \varnothing$. Since $\operatorname{int}_X \operatorname{cl}_X f^{\leftarrow}(U)) \in \mathscr{G}^{\leftarrow}$, it follows that $f^{\leftarrow}(p) \cap \operatorname{ad}_X(\mathscr{G}^{\leftarrow}) = \varnothing$, contradicting the fact that $p \in f(\operatorname{ad}_X(\mathscr{G}^{\leftarrow}))$. Thus, $f(\operatorname{ad}_X(\mathscr{G}^{\leftarrow})) \subseteq \operatorname{ad}_Y(\mathscr{G})$. To prove the reverse inequality, let $p \in \operatorname{ad}_Y(\mathscr{G})$. It suffices to show that $f^{\leftarrow}(p) \cap \operatorname{ad}_X(\mathscr{G}^{\leftarrow}) \neq \varnothing$. Assume that $f^{\leftarrow}(p) \cap \operatorname{ad}_X(\mathscr{G}^{\leftarrow}) = \varnothing$. Then $f^{\leftarrow}(p) \subseteq \bigcup\{X \setminus \operatorname{cl}_X(A)\colon A \in \mathscr{G}^{\leftarrow}\}$. Since $f^{\leftarrow}(p)$ is compact, there exist finitely many A_i, $i = 1, 2, \ldots, n$ in $\mathscr{G}^{\leftarrow}$ such that $f^{\leftarrow}(p) \subseteq \bigcup\{X \setminus \operatorname{cl}_X(A_i)\colon i = 1, 2, \ldots, n\} = B$ (say). Then, $A = \bigcap\{A_i\colon i = 1, 2, \ldots, n\} \in \mathscr{G}^{\leftarrow}$. Consequently, $f^{\#}(A) \in \mathscr{G}$, $f^{\#}(B)$ is an open neighborhood of p in Y and $f^{\#}(A) \cap f^{\#}(B) = \varnothing$, contradicting the fact that $p \in \operatorname{ad}_Y(\mathscr{G})$. Thus, $f^{\leftarrow}(p) \cap \operatorname{ad}_X(\mathscr{G}^{\leftarrow}) \neq \varnothing$, whence $p \in f(\operatorname{ad}_X(\mathscr{G}^{\leftarrow}))$. Therefore, $\operatorname{ad}_Y(\mathscr{G}) \subseteq f(\operatorname{ad}_X(\mathscr{G}^{\leftarrow}))$, and (a) follows. To prove (b) let $\mathscr{G}$ be an open ultrafilter on Y. If $B \in \tau(X)$ and $B \notin \mathscr{G}^{\leftarrow}$, then $f^{\#}(B) \notin \mathscr{G}$. Since $\mathscr{G}$ is an open ultrafilter, it follows that $f^{\#}(X \setminus \operatorname{cl}_X(B)) = Y \setminus f(\operatorname{cl}_X(B)) = Y \setminus \operatorname{cl}_Y(f^{\#}(B)) \in \mathscr{G}$. So $X \setminus \operatorname{cl}_X(B) \in \mathscr{G}^{\leftarrow}$. Hence, $\mathscr{G}^{\leftarrow}$ is an open ultrafilter on X.

(1.8) Remark. A straightforward verification shows that if $X \twoheadrightarrow Y$ is a perfect, irreducible and θ-continuous surjection, if $\mathscr{F}$ is an open ultrafilter on X, and if $\mathscr{G}$ is an open ultrafilter on Y, then

(a) $\mathscr{F} = (\mathscr{F}^{\rightarrow})^{\leftarrow}$, and

(b) $\mathscr{G} = (\mathscr{G}^{\leftarrow})^{\rightarrow}$.

(1.9) Extensions of spaces. An *extension* of a space X is a pair $[Y, j]$, where Y is a Hausdorff space and j is a topological dense embedding of X into Y. However, as is customary, we shall identify $j(x)$ with x for each $x \in X$ and regard X as a dense subspace of Y. Two extensions Y_1 and Y_2 of a space X are called *equivalent* if there is a homeomorphism from one onto the other whose restriction to X is the identity map ι_X on X. We identify two equivalent extensions of a space X. With this convention, if $\mathbf{E}(X)$ is the class of all the extensions of a space X, then $\mathbf{E}(X)$ is a set and $|\mathbf{E}(X)| \leqslant |\mathbf{P}(\mathbf{P}(\mathbf{P}(\mathbf{P}(X))))|$. If Y, $Z \in \mathbf{E}(X)$, then Y is said to be *projectively larger* than Z (written hereafter $Y \geqslant Z$) if there is a continuous mapping $\phi\colon Y \to Z$ such that $\phi|_X = \iota_X$. If $\mathbb{P}$ is a topological property and $\mathbb{P}(X) = \{Y \in \mathbf{E}(X)\colon Y$ has the property $\mathbb{P}\}$, then an element Y of $\mathbb{P}(X)$ is called a *projective maximum* for $\mathbb{P}(X)$ if $Y \geqslant Z$ for each $Z \in \mathbb{P}(X)$. A projective maximum in $\mathbb{P}(X)$, if it exists, is unique.

(1.10) (See [12].) Let Y be an extension of a space X.

(a) If $\mathscr{U}$ is an open (ultra) filter on X, then $\mathscr{U}^* = \{U \in \tau(Y)\colon U \cap X \in \mathscr{U}\}$ is an open (ultra) filter on Y which converges in Y if and only if $\mathscr{U}$ converges in Y.

(b) If $\mathscr{W}$ is an open (ultra) filter on Y, then $\mathscr{W}_* = \{W \cap X: W \in \mathscr{W}\}$ is an open (ultra) filter on X which converges in Y if and only if $\mathscr{W}$ converges in Y.

If more than one extension is involved, the meanings of $\mathscr{U}^*$ and $\mathscr{W}_*$ will be clear from the context.

(1.11) Let $Y \in \mathbf{E}(X)$. For a point $y \in Y$, let

(a) $$\mathscr{O}_Y^y = (\mathscr{N}_y)_* = \{U \cap X: U \in \mathscr{N}_y\}$$

where $\mathscr{N}_y$ is the open neighborhood filter of y in Y. For an open subset U of X, let

(b) $$o_Y(U) = \{y \in Y: U \in \mathscr{O}_Y^y\}.$$

The family $\{o_Y(U): U \in \tau(X)\}$ forms an open base for a coarser Hausdorff topology $\tau^\#$ on Y. The space $(Y, \tau^\#)$, denoted by $Y^\#$ is an extension of X, and $(Y^\#)^\# = Y^\#$ (see [1], [15]).

Now, let Y^+ be the space with the underlying set of Y and the topology generated by the family $\mathscr{B} = \{U \cup \{y\}: y \in Y, U \in \mathscr{O}_Y^y\} \cup \tau(X)$. The space Y^+ is a Hausdorff extension of X, $\tau(Y) \subseteq \tau(Y^+)$, X is an open-dense subspace of Y^+, and $Y^+ \setminus X$ is closed and discrete. Moreover, $(Y^+)^+ = Y^+$. It is well known and easy to prove that for each $U \in \tau(X)$, $\mathrm{cl}_Y(U) = \mathrm{cl}_{Y^+}(U) = \mathrm{cl}_{Y^\#}(U)$ [15]. The following facts are used in the sequel.

(1.12) Facts (see [15]). Let Y be an extension of a space X.

(a) For each $U \in \tau(X)$, $U \subseteq o_Y(U) = Y \setminus \mathrm{cl}_Y(X \setminus U) = \cup\{W \in \tau(Y): W \cap X \subseteq U\}$.

(b) If $W \in \tau(Y)$, then

(i) $X \cap o_Y(W \cap X) = W \cap X$,

(ii) $W \subseteq o_Y(W \cap X) \subseteq \mathrm{cl}_Y(W \cap X) = \mathrm{cl}_Y(W) = \mathrm{cl}_Y[o_Y(W \cap X)] = \mathrm{cl}_{Y^\#}[o_Y(W \cap X)]$,

(iii) $\mathrm{int}_Y \mathrm{cl}_Y(W) = \mathrm{int}_Y \mathrm{cl}_Y(W \cap X) = o_Y(\mathrm{int}_X \mathrm{cl}_X(W \cap X))$.

(c) For each $y \in Y$, $\{\mathscr{O}_{Y^\#}^y: y \in Y^\#\} = \{\mathscr{O}_Y^y: y \in Y\} = \{\mathscr{O}_{Y^+}^y: y \in Y^+\}$.

(d) If Z is an extension of X such that Z and Y have the same underlying set, then $Y^\# \leqslant Z \leqslant Y^+$ if and only if $\{\mathscr{O}_Y^y: y \in Y\} = \{\mathscr{O}_Z^y: y \in Z\}$ and, in both cases, $Z^\# = Y^\#$, and $Z^+ = Y^+$.

(e) For any space $Y \in \mathbf{E}(X)$, $(Y^\#)^+ = Y^+$ and $(Y^+)^\# = Y^\#$.

An extension Y of a space X is called a *simple* (respectively *strict*) *extension* if $Y = Y^+$ (respectively $Y = Y^\#$). The extensions Y^+ and $Y^\#$ were introduced by Banaschewski [1].

Remark. If X is a space, then $RO(X)$ forms an open base for a Hausdorff topology τ_s on X. The space (X, τ_s) is denoted by X_s and is called the *semiregularization* of X (see [15]). A space X is *semiregular* if and only if $X = X_s$. If $Y \in \mathbf{E}(X)$, then $Y_s \in \mathbf{E}(X_s)$. Moreover if $Y \in \mathbf{E}(X)$ and Y is semiregular, then $Y = Y^\#$.

(1.13) THE KATĚTOV AND FOMIN EXTENSIONS. Recall that a space X is *H-closed* provided that X is closed in every Hausdorff space Y in which X is embedded. Let X be a space and let $\mathbf{F}(X) = \{\mathscr{U}\colon \mathscr{U}$ is a free open ultrafilter on $X\}$. Let $\tilde{X} = X \cup \mathbf{F}(X)$. Define a topology τ^+ on $\tilde{X}$ as follows: X is open in $\tilde{X}$, and for $\mathscr{U} \in \tilde{X} \setminus X$ a basic open neighborhood of $\mathscr{U}$ is $U \cup \{\mathscr{U}\}$ where U is open in X and $U \in \mathscr{U}$. Then $(\tilde{X}, \tau^+)$ is a simple H-closed extension of X, denoted by κX, and is called the *Katětov extension* of X (see [9]). Now let $\tau^{\#}$ be another topology on $\tilde{X}$ generated by the (open basis) $\{o(U)\colon U \in \tau(X)\}$, where $o(U) = U \cup \{\mathscr{U} \in \tilde{X} \setminus X\colon U \in \mathscr{U}\}$. Then $(\tilde{X}, \tau^{\#})$ is a strict H-closed extension of X, denoted by σX, and is called the *Fomin extension* of X (see [4]). For more details see [9], [15], [16].

(1.14) THE ABSOLUTE EX OF A SPACE X. A complete description and historical development of the theory of absolutes occurs in the survey paper of Woods [23]. Here we shall briefly summarize those properties of the absolutes which will be used subsequently in this paper. In what follows βX (resp. υX) will denote the Stone-Čech compactifications (resp. Hewitt realcompactifications) of a Tychonoff space X.

Let X be a Hausdorff space and let EX be the set of all the convergent open ultrafilters on X. For each $U \in \tau(X)$, let

(a) $O_X U = \{\mathscr{F}\colon \mathscr{F} \in EX, U \in \mathscr{F}\}$.

The family $\{O_X U\colon U \in \tau(X)\}$ forms an open base for a Hausdorff topology $\tau(EX)$ on EX. The space $(EX, \tau(EX))$, denoted by EX henceforth, is extremally disconnected and zero-dimensional, and it has the following properties (see [8], [13], [14], [19], [20] and [23]).

(b) For each $U \in \tau(X)$ and $\{U_i\colon i \in \Lambda\} \subseteq \tau(X)$, $O_X(X \setminus \mathrm{cl}_X(U)) = EX \setminus O_X(U)$, and $\mathrm{cl}_{EX}[\bigcup\{O_X U_i\colon i \in \Lambda\}] = O_X[\bigcup_{i \in \Lambda} U_i]$.

(c) The map $k_X\colon EX \twoheadrightarrow X$ given by $k_X(\mathscr{F}) = ad_X(\mathscr{F})$ is well defined and is a perfect irreducible and θ-continuous surjection such that $k_X(O_X U) = \mathrm{cl}_X(U)$ for all $U \in \tau(X)$.

(d) The space EX is unique in the sense that if Z is any extremally disconnected and zero-dimensional space, and $\phi\colon Z \twoheadrightarrow X$ is a perfect, irreducible and θ-continuous surjection, then there is a homeomorphism $f\colon EX \twoheadrightarrow Z$ such that $\phi \circ f = k_X$. In this case we write $[Z, \phi] \equiv_X [EX, k_X]$.

(e) The space X is H-closed if and only if EX is compact.

(f) $\beta EX = \mathrm{S}(R(X))$, the Stone space of the complete Boolean algebra $R(X)$. Moreover if hX is any H-closed extension of X, then there exists a perfect, irreducible and θ-continuous mapping from βEX onto hX.

(g) $\beta EX \setminus EX \simeq \sigma X \setminus X$. In particular, $\sigma X \setminus X$ is zero-dimensional.

(h) (i) If $U \in \tau(X)$, then $k_X^{\#}(O_X U) = \mathrm{int}_X \mathrm{cl}_X U$.

(ii) If $W \in \tau(EX)$, then $\mathrm{cl}_X(k_X(W)) = \mathrm{cl}_X(k_X^{\#}(W))$, and there is a set $V \in RO(X)$ such that $\mathrm{cl}_{EX}(W) = O_X V$. Furthermore, $\mathrm{cl}_X(k_X^{\#}(W)) = \mathrm{cl}_X(V)$ and $O_X(k_X^{\#}(W)) = O_X V$.

(i) Let U be an open subset of the space X. Let $k_U\colon O_X U \twoheadrightarrow \mathrm{cl}_X U$ be defined by $k_U = k_X|_{O_X U}$. Then k_U is a perfect, irreducible and θ-continuous mapping from $O_X U$ onto $\mathrm{cl}_X(U)$.

An immediate application of 1.6, 1.7, and 1.8 leads to a short proof of a well known theorem proved by Iliadis and Fomin [8].

(1.15) THEOREM. *Let X and Y be Hausdorff spaces such that there exists a perfect, irreducible and θ-continuous mapping $f\colon X \twoheadrightarrow Y$. Then there exists a homeomorphism $\phi\colon EX \twoheadrightarrow EY$ such that $k_Y \circ \phi = f \circ k_X$.*

PROOF. Let $\mathscr{U}$, $\mathscr{V}$, be two distinct members of EX. Then there exist open sets $U \in \mathscr{U}$ and $V \in \mathscr{V}$ such that $U \cap V = \varnothing$. Now $f^{\#}(U) \in \mathscr{U}^{\rightarrow}$ (see 1.6), $f^{\#}(V) \in \mathscr{V}^{\rightarrow}$ and $f^{\#}(U) \cap f^{\#}(V) = \varnothing$. Hence, $\mathscr{U}^{\rightarrow} \neq \mathscr{V}^{\rightarrow}$. Moreover, by 1.6, $\mathscr{U}^{\rightarrow} \in EY$ for each $\mathscr{U} \in EX$. Thus, $\phi\colon EX \to EY$, defined by $\phi(\mathscr{U}) = \mathscr{U}^{\rightarrow}$, is a one-to-one mapping from EX into EY. By 1.8, ϕ is onto, and $k_Y \circ \phi = f \circ k_X$. Let $O_X U$ be a basic open set in EX. Then $\mathscr{U} \in O_X U$ if and only if $U \in \mathscr{U}$, if and only if $\mathrm{int}_X \mathrm{cl}_X(U) \in \mathscr{U} = (\mathscr{U}^{\leftarrow})^{\rightarrow}$, if and only if $f^{\#}(\mathrm{int}_X \mathrm{cl}_X(U)) \in \mathscr{U}^{\leftarrow}$, if and only if $\phi(\mathscr{U}) = \mathscr{U}^{\rightarrow} \in O_Y(f^{\#}(\mathrm{int}_X \mathrm{cl}_X(U)))$, which is open in EY. Thus, $\phi(O_X U) = O_Y(f^{\#}(\mathrm{int}_X \mathrm{cl}_X(U))$, and since ϕ is a bijection it follows that ϕ is an open mapping. To show that ϕ is continuous, let $O_Y V$ be a basic open set in EY. Note that $O_Y V = O_Y(\mathrm{int}_X \mathrm{cl}_X(V)) = O_Y[f^{\#}(\mathrm{int}_X \mathrm{cl}_X(f^{\leftarrow}(\mathrm{int}_Y \mathrm{cl}_Y(V))))] = \phi[O_X[\mathrm{int}_X \mathrm{cl}_X(f^{\leftarrow}(\mathrm{int}_Y \mathrm{cl}_Y(V)))]]$. Since ϕ is a bijection, the set $\phi^{\leftarrow}(O_Y(V)) = O_X[\mathrm{int}_X \mathrm{cl}_X(f^{\leftarrow}(\mathrm{int}_Y \mathrm{cl}_Y(V)))]$ is open. Hence ϕ is continuous, and the proof of the theorem is complete.

Two spaces X and Y are called *co-absolute* if $EX \simeq EY$.

(1.16) THE ABSOLUTE PX. Banaschewski [2] and Mioduschewski and Rudolf [13] constructed another absolute PX where the underlying set is EX but the topology $\tau(PX)$ is generated by the family $\{O_X U \cap k_X^{\leftarrow}(V)\colon U, V \in \tau(X)\}$. It turns out that the space $(PX, \tau(PX))$, written as PX henceforth, is extremally disconnected and Hausdorff, but is no longer Tychonoff in general. However, the map k_X (which we now denote by $\pi_X\colon PX \twoheadrightarrow X$) is continuous as well as perfect and irreducible. Clearly X is H-closed if and only if PX is H-closed, and $[PX, \pi_X] \equiv {}_X[EX, k_X]$ if and only if X is regular. Also, $EX = (PX)_s$, and $RO(EX) = RO(PX) = CO(EX) = \{O_X U\colon\ U \in \tau(X)\}$ (see [19]). Moreover, $[PX, \pi_X]$ is unique in the sense that if Z is any extremally disconnected space and $g\colon Z \twoheadrightarrow X$ is a perfect, irreducible, continuous surjection, then there is homeomorphism $h\colon PX \twoheadrightarrow Z$ such that $g \circ h = \pi_X$.

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The following results will be used later.

(1.17). LEMMA. *Let α be an open ultrafilter on EX (respectively, PX). Then $\alpha^{\rightarrow} = \{W \in \tau(X) \colon O_X W \in \alpha\}$.*

PROOF. Let $\mathscr{W} = \{W \in \tau(X) \colon O_X W \in \alpha\}$. To show that $\alpha^{\rightarrow} = \mathscr{W}$, let $W \in \mathscr{W}$. Then $O_X W \in \alpha$. Since $\operatorname{int}_X \operatorname{cl}_X(W) = k_X^{\#}(O_X W) \in k_X^{\#}(\alpha) \subseteq \alpha^{\rightarrow}$ (see 1.6 and 1.14(f)), and since $\alpha^{\rightarrow}$ is an open ultrafilter, it follows that $W \in \alpha^{\rightarrow}$. Hence $\mathscr{W} \subseteq \alpha^{\rightarrow}$. To prove the reverse inequality, let $W \in \alpha^{\rightarrow}$. Then there is a set $A \in \alpha$ such that $k_X^{\#}(A) \subseteq \operatorname{int}_X \operatorname{cl}_X(W)$. Now $O_X(k_X^{\#}(A)) \subseteq O_X(\operatorname{int}_X \operatorname{cl}_X(W)) = O_X W$, and $O_X(k_X^{\#}(A)) = \operatorname{cl}_{EX}(A)$ (see 1.14(f)). Since EX is extremally disconnected, $\operatorname{cl}_{EX}(A) \in \alpha$. Hence, $O_X W \in \alpha$, and therefore $W \in \mathscr{W}$. Thus $\alpha^{\rightarrow} \in \mathscr{W}$. The same argument holds for PX, and the result follows.

(1.18) LEMMA. *Let X be a space and $\mathscr{U} \in EX$ (respectively $\mathscr{U} \in PX$). If G is open in EX (respectively PX), and if $\mathscr{U} \in G$, then $k_X^{\#}(G) \in \mathscr{U}$ (respectively $\pi_X(G) \in \mathscr{U}$).*

PROOF. Suppose $k_X^{\#}(G) \notin \mathscr{U}$. Since $k_X^{\#}(G) \in \tau(X)$, and since $\mathscr{U}$ is an open ultrafilter on X, there exists a set $A \in \mathscr{U}$ such that $k_X^{\#}(G) \cap A = \varnothing$. Therefore, by 1.14(f), $\operatorname{cl}_{EX} G \cap O_X A = \varnothing$. Hence $G \cap O_X A = \varnothing$, which is impossible since $\alpha \in G$ and $\alpha \in O_X A$. This proves the result for EX. For PX, the result follows by the same proof using 1.16 and the fact that for any space Z, and any open subset U of Z, $\operatorname{cl}_Z(U) = \operatorname{cl}_{Z_s}(U)$.

(1.19) LEMMA. *Let X be a space such that every closed and nowhere dense subset of EX is compact. Let G be a nonempty open subset of EX and $\alpha \in o_{\sigma EX}(G)$. Then there exists an open subset U of X such that $\alpha \in o_{\sigma EX}(O_X U) \subseteq o_{\sigma EX}(G)$.*

PROOF. If $\alpha \in o_{\sigma EX}(O_X G) \cap EX = G$, there is an open subset U of X such that $\alpha \in O_X U \subseteq G$. Hence $\alpha \in o_{\sigma EX}(O_X U) \subseteq o_{\sigma EX}(G)$. If $\alpha \in o_{\sigma EX}(G) \setminus EX$, then $G \in \alpha$. By 1.14(f), there is a regular open subset U_0 of X such that $\operatorname{cl}_{EX}(G) = O_X U_0$. Also, $\operatorname{cl}_{EX}(G) \setminus G$ is compact by hypothesis. Let $p \in \operatorname{cl}_{EX}(G) \setminus G$. Since α is a free open ultrafilter on EX, there is a set $A_p \in \alpha$ such that $p \notin \operatorname{cl}_{EX}(A_p) = O_X U_p$ for some $U_p \in RO(X)$. So $p \in EX \setminus O_X U_p$. Hence $\{EX \setminus O_X U_p \colon p \in \operatorname{cl}_{EX}(G) \setminus G\}$ is an open covering of the compact set $\operatorname{cl}_{EX}(G) \setminus EX$. Therefore there exist finitely many indices $i = 1, 2, \ldots, n$ with $O_X U_{p_i} \in \alpha$ for all $i = 1, 2, \ldots, n$ and $\operatorname{cl}_{EX}(G) \setminus G \subseteq \bigcup_{i=1}^{n}(EX \setminus O_X U_{p_i}) = EX \setminus \bigcap_{i=1}^{n} O_X U_{p_i} = EX \setminus O_X(\bigcap_{i=1}^{n} U_{p_i}) = EX \setminus O_X U_1$, where $U_1 = \bigcap_{i=1}^{n} U_{p_i}$ is open in X. Obviously, $O_X(U_0 \cap U_1) = O_X U_0 \cap O_X U_1 \in \alpha$. Also, $O_X U_1 \cap \operatorname{cl}_{EX}(G) = (O_X U_1 \cap G) \cup (O_X U_1 \cap (\operatorname{cl}_{EX}(G) \setminus G)) = O_X U_1 \cap G$. So $O_X(U_0 \cap U_1) \subseteq G$. Take $U = U_0 \cap U_1$.

Then U is open in X (in fact regular open in X), and $O_X U \subseteq G \in \alpha$. Thus, again, $\alpha \in o_{\sigma EX}(O_X U) \subseteq o_{\sigma EX}(G)$, and the result follows.

(1.20) REMARK. Iliadis and Fomin [8] shows that $EhX = \beta EX$ for every H-closed extension hX of a space X. Porter, Vermeer and Woods [17] showed that $E\kappa X = \kappa EX$ if and only if X is H-closed. Porter and Votaw [16] proved that $\sigma EX = E\sigma X$ if and only if the set of nonisolated points of EX is compact. Katětov [9] proved that every closed nowhere dense subset of a space is compact if and only if the set of nonisolated points of X is compact. From [10] and [16] it follows that $\sigma EX = E\sigma X$ if and only if every closed and nowhere dense subset of EX is compact. In [13] it is proved that $P\kappa X = \kappa PX$ for every space X. Below, in 1.21 and 1.22, we shall briefly describe these homeomorphisms explicitly. These homeomorphisms will be used in the sequel (see 2.13, 2.15).

(1.21) Suppose that every closed and nowhere dense subset of EX is compact. We construct a homeomorphism ϕ from σEX onto $E\sigma X$ as follows. Let $\alpha \in EX$. Then $\alpha^* = \{G \in \tau(\sigma X)\colon\ G \cap X \in \alpha\}$ is a convergent open ultrafilter on σX such that $\operatorname{ad}_X(\alpha) = \operatorname{ad}_{\sigma X}(\alpha^*)$. If $\alpha \in \sigma EX \setminus EX$, then $(\alpha^{\rightarrow})^*$ is an open ultrafilter on σX which converges to a point in $\sigma X \setminus X$. An easy verification (using 1.6, 1.7, 1.8, 1.10, and 1.17) shows that the mapping $\phi\colon \sigma EX \to E\sigma X$ given by

$$\phi(\alpha) = \begin{cases} \alpha^*, & \alpha \in EX, \\ \alpha^{\rightarrow *}, & \alpha \in \sigma EX \setminus EX, \end{cases}$$

is well defined, one-to-one, and onto. Now let iU $\in \tau(X)$, and let $V \in \tau(\sigma X)$. Then, using 1.12 and 1.14, we obtain

(a) $\phi[O_X(V \cap X)] = \phi(EX) \cap O_{\sigma X} V$,

(b) $\phi[O_X U] = \phi(EX) \cap O_{\sigma X}(o_{\sigma X}(U))$, and

(c) $\phi(EX) = k_{\sigma X}^{\leftarrow}(X)$.

From (a), (b), and (c) it follows that $\phi|_{EX}\colon EX \to k_{\sigma X}^{\leftarrow}(X)$ is a homeomorphism, and that $\phi(EX)$ is dense in $E\sigma X$. Now suppose that $G \in \tau(EX)$, $V \in \tau(\sigma X)$, and $U \in \tau(X)$. A routine verification using 1.18 shows that

(d) $\phi[o_{\sigma EX}(G)] \subseteq O_{\sigma X}[o_{\sigma X}(k_X^{\#}(G))]$,

(e) $\phi[o_{\sigma EX}(O_X(V \cap X))] = O_{\sigma X} V$, and

(f) $\phi[o_{\sigma EX}(O_X U)] = O_{\sigma X}(o_{\sigma X}(U))$.

In particular, it will follow that ϕ is continuous. From 1.19, it follows that the family $\{o_{\sigma EX}(O_X U)\colon\ U \in \tau(X)\}$ forms an open base for σEX. This fact, combined with (f), now shows that ϕ is an open mapping, and hence a homeomorphism.

(1.22) Let X be a Hausdorff space. We describe explicitly a homeomorphism ψ from κPX onto $P\kappa X$. For $\alpha \in PX$, let $\alpha^* = \{G \in \tau(\kappa X)\colon\ G \cap X \in \alpha\}$. Then from 1.6, 1.7, 1.8, 1.10, and 1.17 it follows that the map ψ: $\kappa PX \to P\kappa X$ given by

$$\psi(\alpha) = \begin{cases} \alpha^*, & \alpha \in PX, \\ \alpha^{\to *}, & \alpha \in \kappa PX \setminus PX, \end{cases}$$

is a well defined bijection. Further, for each A, $U \in \tau(X)$ and V, $B \in \tau(\kappa X)$ one can show (using 1.16 and 1.17) that

(a) $\psi[\pi_X^{\leftarrow}(A)] = \pi_{\kappa X}^{\leftarrow}(A)$,
(b) $\psi[O_X U] = (O_{\kappa X} U) \cap \psi(PX)$,
(c) $\psi(PX) = \pi_{\kappa X}^{\leftarrow}(X)$,
(d) $(O_{\kappa X} V) \cap \phi(PX) = \psi(O_X(V \cap X))$, and
(e) $\pi_{\kappa X}^{\leftarrow}(B) \cap \psi(PX) = \psi(\pi_X^{\leftarrow}(B \cap X))$.

These facts show that $\psi|_{PX}$: $PX \twoheadrightarrow \psi(PX)$ is a homeomorphism, and that $\psi(PX)$ is open and dense in $P\kappa X$. The continuity of ψ follows from the fact that if $\alpha \in \kappa PX$, and if $(O_{\kappa X} V) \cap \pi_{\kappa X}^{\leftarrow}(B)$ is any basic open neighborhood of $\psi(\alpha)$ in $P\kappa X$, where V, B are open subsets of κX, then $\psi[o_{\kappa PX}(O_X(V \cap X))] = O_{\kappa X} V$, and, for $\alpha \in \kappa PX \setminus PX$, the set $\pi_X^{\leftarrow}(B \cap X) \cup \{\alpha\}$ is an open neighborhood of α in κPX such that $\psi[(\pi_X^{\leftarrow}(B \cap X)) \cup \{\alpha\}] \subseteq \pi_{\kappa X}^{\leftarrow}(B)$. To show that ψ is open, it suffices to show that $\psi(U \cup \{\alpha\})$ is open in $P\kappa X$ for each open subset $U \subseteq PX$, and for each $\alpha \in \kappa PX \setminus PX$ such that $U \in \alpha$. If $\lambda \in \psi(U \cup \{\alpha\}) \cap \pi_{\kappa X}^{\leftarrow}(X) = \psi(U)$, then $\psi(U)$ is an open neighborhood of λ in $P\kappa X$ contained in $\psi(U \cup \{\alpha\})$. Also, since $U \in \alpha$, we have $\pi_X^{\#}(U) \in \alpha^{\to}$, and $\pi_{\kappa X}^{\to}(\pi_X^{\#}(U)) = \psi[\pi_X^{\leftarrow}(\pi_X^{\#}(U))] = \psi(U)$. Also, $\pi_X^{\#}(U) \cup \{\alpha^{\to}\}$ is an open neighborhood of $\alpha^{\to}$ in κX and, moreover, $\pi_{\kappa X}^{\leftarrow}(\alpha^{\to}) = \alpha^{\to *} = \psi(\alpha)$. Thus, $\pi_{\kappa X}^{\leftarrow}[\pi_X^{\#}(U) \cup \{\alpha^{\to}\}]$ is an open neighborhood of $\alpha^{\to *}$ in $P\kappa X$ contained in $\psi(U \cup \{\alpha\})$. Hence each point of $\psi(U \cup \{\alpha\})$ is an interior point in $P\kappa X$. Hence $\psi(U \cup \{\alpha\})$ is open in $P\kappa X$. Thus ψ is an homeomorphism.

2. Almost realcompact spaces

Almost realcompact spaces were introduced and investigated by Frolík [5], and almost realcompactifications were investigated by Liu and Strecker [12]. Properties of almost realcompact spaces can be found in [5], [6], [7], [12], and [22].

(2.1) Definition. (a) An open filter $\mathscr{F}$ on a space X is said to have the *countable closure intersection property* (abbreviated c.c.i.p.) in X if for each countable subcollection $\mathscr{A} \subseteq \mathscr{F}$, $\bigcap\{\mathrm{cl}_X(A): A \in \mathscr{A}\} \neq \varnothing$.

(b) [5] A Hausdorff space X is called *almost realcompact* if every open ultrafilter on X with c.c.i.p. in X converges.

(c) [12] An *almost realcompactification* of a space X is a space Y such that Y is almost realcompact and X is a dense subspace of Y.

(d) [12] An extension Y of a space X is called a *ρ-extension* of X provided that Y is almost realcompact, and that, if Z is any almost realcompactification of X, then there is a continuous mapping $g\colon Y \to Z$ such that $g|_X = \iota_X$.

(2.2) LEMMA. *Let X and Y be spaces and let $f\colon X \to Y$ be a perfect, irreducible and θ-continuous mapping from X onto Y. Then*:

(a) *If $\mathcal{U}$ is an open ultrafilter on X with c.c.i.p. in X, then $\mathcal{U}^{\to}$ is an open ultrafilter on Y with c.c.i.p. in Y, and*

(b) *If $\mathcal{V}$ is an open ultrafilter on Y with c.c.i.p. in Y, then $\mathcal{V}^{\leftarrow}$ is an open ultrafilter on X with c.c.i.p. in X.*

PROOF. The proof of (a) is easy and is left as an exercise to the reader. To prove (b), observe that by 1.7, $\mathcal{V}^{\leftarrow}$ is an open ultrafilter on X. Let $\{V_i\colon i = 1,2,3,\dots\}$ be a countable subfamily of $\mathcal{V}^{\leftarrow}$. Without loss of generality, assume that $V_n \supseteq V_{n+1}$ for all $= 1,2,3,\dots$. Then $\{f^{\#}(V_i)\colon i = 1,2,\dots\}$ is a countable subfamily of $\mathcal{V}$ and hence, by hypothesis, $\bigcap\{\mathrm{cl}_Y[f^{\#}(V_i)]\colon i = 1,2,3,\dots\} \neq \varnothing$. Let $p \in \bigcap\{\mathrm{cl}_Y[f^{\#}(V_i)]\colon i = 1,2,3,\dots\}$. Then $f^{\leftarrow}(p)$ is a nonempty compact subset of X. If $f^{\leftarrow}(p) \cap \bigcap\{\mathrm{cl}_X(V_i)\colon i = 1,2,3,\dots\} = \varnothing$, then $f^{\leftarrow}(p) \subseteq \bigcup\{X \setminus \mathrm{cl}_X(V_i)\colon i = 1,2,3,\dots\}$. Since the V_n's are decreasing, and since $f^{\leftarrow}(p)$ is compact, there is a positive integer n_0 such that $f^{\leftarrow}(p) \subseteq X \setminus \mathrm{cl}_X(V_{n_0})$, and $V_{n_0} \in \mathcal{V}^{\leftarrow}$. But then $p \in \mathrm{cl}_Y(V_i)\colon i = 1,2,3,\dots\} \neq \varnothing$, and, hence, $\mathcal{V}^{\leftarrow}$ has c.c.i.p. in X. The proof of the proposition is now complete.

(2.3) THEOREM. *Let X and Y be spaces and f a perfect, irreducible and θ-continuous mapping from X onto Y. Then X is almost realcompact if and only if Y is almost realcompact.*

PROOF. The proof follows directly from 1.6, 1.7, and 2.2.

(2.4) COROLLARY. (a) *A space X is almost realcompact if and only if X_s is almost realcompact.*

(b) *A space X is almost realcompact if and only if PX is almost realcompact, if and only if EX is almost realcompact.*

(2.5) COROLLARY. *Let Y be an almost realcompactification of X, and let Z be a space with the same underlying set as Y such that $\tau(Y^{\#}) \subseteq \tau(Z) \subseteq \tau(Y^{+})$. Then* (*by* 1.12(d)) *$Z$ is almost realcompact if and only if Y is almost realcompact.*

In [12] Liu and Strecker constructed an almost realcompactification ρX of a space X. In fact, $\rho X = X \cup \{\mathcal{U} \in \mathbf{F}(X)\colon \mathcal{U}$ has c.c.i.p. in $X\}$ as a set with subspace topology induced by $\tau(\kappa X)$. The following results are proved in [12].

(2.6) ρX is a projective maximum in the class of realcompactifications of X.

(2.7) ρX is the largest ρ-extension of X.

(2.8) ρx is the smallest almost realcompact space between X and κX.

In what follows, we discuss the interplay between the absolutes and almost realcompactifications. We first state the following corollary.

(2.9) Corollary. *$E\rho X$ and $P\rho X$ are almost realcompactifications of EX and PX respectively. Moreover, $E\rho X$ is realcompact (see* [22]).

(2.10) The space βEX can also be realized as the space $\Theta X = \{\mathscr{U}\colon \mathscr{U}$ is an open ultrafilter on $X\}$ whose topology has the open base $\{O_X U\colon U \in \tau(X)\}$, where $O_X U = \{\mathscr{U} \in \Theta X\colon U \in \mathscr{U}\}$ (see [19].) Now, let ρX be the almost realcompactification of X (see [12].) Let $\mathscr{U} \in \Theta X$, and let $\mathscr{U}^* = \{G \in \tau(\kappa X)\colon G \cap X \in \mathscr{U}\}$. Then $\mathscr{U}^*$ is an open ultrafilter on κX and $\mathrm{ad}_{\kappa X}(\mathscr{U}^*) = \mathrm{ad}_{\kappa X}(\mathscr{U})$ consists of exactly one point. Define $\psi_{\kappa X}\colon\ \Theta X \to \kappa X$ by $\psi_{\kappa X}(\mathscr{U}) = \mathrm{ad}_{\kappa X}(\mathscr{U}^*)$. If $\mathscr{U} \in \Theta X \setminus EX$, then $\mathscr{U} \in \kappa X \setminus X$ and $\mathscr{O}^{\mathscr{U}}_{\kappa X} = \mathscr{U}$. Since $\psi_{\kappa X}(\mathscr{U}) = p$ if and only if $\mathscr{O}^{p}_{\kappa X} \subseteq \mathscr{U}$ (for each $\mathscr{U} \in \Theta X \setminus EX$), it follows that the map $\psi_{\kappa X}$ can be equivalently defined as

$$\psi_{\kappa X}(\mathscr{U}) = \begin{cases} k_X(\mathscr{U}), & \mathscr{U} \in EX, \\ \mathscr{U}, & \mathscr{U} \in \Theta x \setminus EX, \end{cases}$$

where k_X is the usual map from EX onto X. Then $\psi_{\kappa X}$ is a perfect, irreducible and θ-continuous mapping from ΘX onto κX. Now, let

$$E_\rho X = \{\mathscr{U} \in \Theta X\colon \mathscr{U} \text{ has c.c.i.p. relative to } X\}.$$

Obviously, $EX \subseteq E_\rho X$, $E_\rho X$ is a dense of ΘX, and EX is dense in $E_\rho X$.

(2.11) Proposition. *For every space X, $E_\rho X$ is a realcompactification of EX, and $E_\rho X = E\rho X$.*

Proof.

$$\begin{tikzcd}
& E_\rho X \arrow[r, hook] \arrow[dd, dashed, "\psi_{\rho X}"] & \Theta X \arrow[dd, "\psi_{\kappa X}"] \\
EX \arrow[ur, hook] \arrow[d, "k_X"] & & \\
X \arrow[r, hook] & \rho X \arrow[r, hook] & \kappa X
\end{tikzcd}$$

Define a map $\psi_{\rho X}\colon E_\rho X \to \rho X$ by $\psi_{\rho X} = \psi_{\kappa X|_{E_\rho X}}$. Now ρX is a dense subspace of κX, $E_\rho X$ is a dense subspace of ΘX, and $E_\rho X = \psi^{\leftarrow}_{\kappa X}(\rho X)$. Hence, by 1.1 and 1.2, $\psi_{\rho X}$ is a perfect, irreducible, and θ-continuous mapping from $E_\rho X$ onto ρX. Hence, by 1.15, $E_\rho X = E\rho X$. Since ρX is almost realcompact, $E_\rho X$ is realcompact by 2.4.

(2.12) NOTE. By 2.6, ρEX is a projective maximum in the class of all the almost realcompactifications of EX. Hence there exists a continuous mapping g: $\rho EX \to E_\rho X$ $(= E\rho X)$ that fixes the points of EX. We note that $E\rho X$ carries the subspace topology from βEX, whereas ρEX carries the subspace topology from κEX. Also, $EX \hookrightarrow vEX \hookrightarrow E\rho X = E_\rho X \hookrightarrow \beta EX$, where each embedding is dense and where, for a Tychonoff space Z, βZ and vZ have their usual meaning. We now regard ρX as the subspace of σX given by the points of ρX and denote the space $(\rho X, \tau(\sigma X)|_{\rho X})$ by ζX.

(2.13) THEOREM. *Let X be a Hausdorff space such that every closed and nowhere dense subset of EX is compact. Then $E\zeta X = \zeta EX$.*

PROOF. We consider the homeomorphism ϕ from $\sigma EX \to E\sigma X$ given by 1.21. Let $\delta \in \zeta EX \setminus EX$. Then $\delta \in \sigma EX \setminus EX$. Hence $\phi(\delta) = \delta^{\to *} \in E\sigma X \setminus \phi(EX)$. Suppose $\delta^{\to *}$ does not converge to a point in ζX. Then $\alpha = \mathrm{ad}_{\sigma X}(\delta^{\to *}) = k_{\sigma X}(\delta^{\to *}) \in \sigma X \setminus \zeta X$. Let $G \in \alpha$. Then $\alpha \in o_{\sigma X}(G)$. So, $o_{\sigma X}(G) \cap U \neq \varnothing$ for all $U \in \delta^{\to *}$ and, since $\delta^{\to *}$ is an open ultrafilter on σX, it follows that $o_{\sigma X}(G) \in \delta^{\to *}$. Hence, $G \in (\delta^{\to *})_* = \delta^{\to}$. Thus, $\alpha \subseteq \delta^{\to}$. Since both α and $\delta^{\to}$ are open ultrafilters on X, it follows that $\alpha = \delta^{\to}$. Now δ has the c.c.i.p. in EX. So, by (2.2), α has c.c.i.p. in X, whence $\alpha \in \zeta X$, a contradiction. Thus $\delta^{\to *}$ converges to a point in ζX. Consequently, $\hat{\delta} = \{G \cap \zeta X\colon G \in \delta^{\to *}\} \in E\zeta X$. Let $\phi_0 = \phi|_{\zeta EX}$ be given by

$$\phi_0(\delta) = \begin{cases} \delta, & \delta \in EX, \\ \hat{\delta}, & \delta \in \zeta EX \setminus EX. \end{cases}$$

Then ϕ_0 is a one-to-one mapping from ζEX into $E\zeta X$. To show that ϕ_0 is onto, let $\lambda \in E\zeta X \setminus EX$. Then $\lambda^0 = \{G \in \tau(\sigma X)\colon G \cap \zeta X \in \lambda\}$ is an open ultrafilter on σX. Hence $\lambda^0 = \alpha^{\to *}$ for some $\alpha \in \sigma EX$, and $\lambda^0_* = (\alpha^{\to *})_* = \alpha^{\to}$ is a free open ultrafilter on X, since $\alpha^{\to} \in \zeta X$. So $\alpha^{\to}$ has c.c.i.p. in X. Hence $\alpha = (\alpha^{\to})^{\leftarrow}$ has c.c.i.p. in EX. Therefore $\alpha \in \zeta EX \setminus EX$ and $\lambda = \hat{\alpha}$. Thus ϕ_0 is onto. Since ζEX is dense in σEX, $E\zeta X$ is dense in $E\sigma X$, ϕ_0 is a bijection, and ϕ is a homeomorphism, and so the theorem follows.

(2.14). THEOREM. *For every space X, $\zeta PX = P\zeta X$.*

PROOF.

$$\begin{tikzcd}
\zeta PX \arrow[rr, hook] \arrow[d] \arrow[dr, dashed, "\phi_0"] & & \sigma PX \arrow[d, "F"] \\
PX \arrow[r, hook] \arrow[d, "\pi_X"] & P\zeta X \arrow[r, hook] \arrow[d, "\pi_{\zeta X}"] & P\sigma X \arrow[d, "\pi_{\sigma X}"] \\
X \arrow[r, hook] & \zeta X \arrow[r, hook] & \sigma X
\end{tikzcd}$$

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We consider the homeomorphism F from σPX onto $P\sigma X$ given by

$$F(\delta) = \begin{cases} \delta, & \delta \in PX, \\ \delta^{\#}, & \delta \in \sigma PX \setminus PX, \end{cases}$$

where $\delta^{\#} = \{U \subseteq \sigma X\colon \text{is open}, O_{\sigma X}U \cap PX \in \delta\}$ (see [17, Theorem 4.2]). Let $\delta \in \zeta PX \setminus PX$. Then $\delta \in \sigma PX$, and $F(\delta) = \delta^{\#} \in P\sigma X \setminus PX$. Let $\alpha = \mathrm{ad}_{\sigma X}(\delta^{\#})$. Then $\alpha = \{G \in \tau(X)\colon$ there is a $U \in \delta^{\#}$ such that $G = U \cap X\}$ is an open ultrafilter on X. We will show that $\alpha \in \zeta X$. Suppose $\alpha \notin \zeta X$. Then there exists a countable subfamily $\{G_i\colon i \in \omega\} \subseteq \alpha$ such that

$$\varnothing = \bigcap_{i \in \omega} \mathrm{cl}_X(G_i) = \bigcap_{i \in \omega} [X \cap \mathrm{cl}_{\sigma X}(G_i)].$$

On the other hand, $PX \cap O_{\sigma X}(o_{\sigma X}(G_i)) \in \delta$ for all $i \in \omega$, and δ has c.c.i.p. in PX. Hence,

$$\varnothing \neq \pi_{\sigma X}\left[PX \cap \bigcap_{i \in \omega} O_{\sigma X}(o_{\sigma X}(G_i))\right] \subseteq \pi_{\sigma X}(PX) \cap \bigcap_{i \in \omega} \pi_{\sigma X}(O_{\sigma X}(o_{\sigma X}(G_i)))$$
$$= \bigcap_{i \in \omega} [X \cap \mathrm{cl}_{\sigma X}(o_{\sigma X}(G_i))] = \bigcap_{i \in \omega} \mathrm{cl}_X(G_i),$$

a contradiction. Thus $\alpha \in \zeta X$. In particular, it follows that $\hat{\delta} = \delta^{\#}|_{\zeta X}$ is a convergent open ultrafilter on ζZ and hence $\hat{\delta} \in P\zeta X$. So, define F_0 by

$$F_0(\delta) = \begin{cases} \delta, & \delta \in PX, \\ \hat{\delta}, & \delta \in \zeta PX \setminus PX. \end{cases}$$

Then F_0 is a one-to-one mapping from ζPX to $P\zeta X$. We now show that F_0 is onto. Let $\lambda \in P\zeta X \setminus PX$. Then $\alpha = \pi_{\zeta X}(\lambda) \in \zeta X \setminus X$, $\alpha = \{U \cap X\colon U \in \lambda\}$, and $\alpha^* = \lambda^* = \delta^{\#}$ (where $*$ is taken with respect to σX) for some $\delta \in \sigma PX$. To show that $\delta \in \zeta PX$, we show that $\delta = \alpha^{\leftarrow}$. It suffices to show that $\delta^{\#} = (\alpha^{\leftarrow})^{\#}$, or, alternatively, that $(\delta^{\#})_* = ((\alpha^{\leftarrow})^{\#})_* = \alpha$. Obviously, $(\delta^{\#})_* = (\alpha^*)_* = \alpha$. Also, $((\alpha^{\leftarrow})^{\#})_* = \alpha$. Obviously, $(\delta^{\#})_* = (\alpha^*)_* = \alpha$. Also, $((\alpha^{\leftarrow})^{\#})_* = \{W \subseteq X\colon$ W is open, $W = U \cap X$ for some open $U \subseteq \sigma X$ and $O_{\sigma X}U \cap PX \in \alpha^{\leftarrow}\}$. Since for every $U \in \tau(\sigma X)$, $O_{\sigma X}U = O_{\sigma X}[o_{\sigma X}(\mathrm{int}_X \mathrm{cl}_X(U \cap X))]$, it follows that $PX \cap O_{\sigma X}U = O_X(\mathrm{int}_X \mathrm{cl}_X W) = O_X W$ and hence, by 1.17, that $((\alpha^{\leftarrow})^{\#})_* = (\alpha^{\leftarrow})^{\rightarrow} = \alpha$. Since $\hat{\delta} = \lambda$, it follows that F_0 is onto. Now ζPX is dense in σPX, $P\zeta X$ is dense in $P\sigma X$, $F_0 = F|_{\zeta PX}$, and F_0 is a bijection, so that F_0 is a homeomorphism, and the theorem follows.

(2.15) THEOREM. *For every space X, $\rho PX = P\rho X$.*

PROOF. We consider the homeomorphism ψ from κPX onto $P\kappa X$ described in 1.22. If $\delta \in \rho PX \setminus PX$, then $\delta \in \kappa PX$, and $\psi(\delta) = \delta^{\rightarrow *} \in P\kappa X \setminus PX$. If $\alpha = \pi_{\kappa X}(\delta^{\rightarrow *})$, then $\alpha \in \kappa X \setminus X$. Moreover, $\alpha = \delta^{\rightarrow}$, and hence, by 2.2, α has c.c.i.p. in X; and so $\alpha \in \rho X \setminus X$. Let $\hat{\delta} = \{G \subseteq \rho X\colon G \cap X \in \alpha\}$. Then $\hat{\delta}$ is an open ultrafilter on ρX with c.c.i.p. in ρX. Since ρX is almost realcompact, $\hat{\delta}$ converges in ρX, and hence $\hat{\delta} \in P\rho X$. Define a map ψ_0: $\rho PX \to P\rho X$ by

$$\psi_0(\delta) = \begin{cases} \delta, & \delta \in PX, \\ \hat{\delta}, & \delta \in \rho PX \setminus PX. \end{cases}$$

Reasoning similar to that used in (2.13) shows that ψ_0 is a one-to-one mapping from ρPX onto $P\rho X$. The result now follows from the fact that ρPX is dense in κPX, $P\rho X$ is dense in $P\kappa X$, ψ is a homeomorphism, and $\psi_0 = \psi|_{\rho PX}$.

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