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RESEARCH ARTICLE

Planar random-cluster model: scaling relations

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Abstract
This paper studies the critical and near-critical regimes of the planar random-cluster model on $\mathbb{Z}^2$ with cluster-weight $q \in [1, 4]$ using novel coupling techniques. More precisely, we derive the *scaling relations* between the critical exponents β, γ, δ, η, ν, ζ as well as α (when $\alpha \geq 0$). As a key input, we show the stability of crossing probabilities in the near-critical regime using new interpretations of the notion of the influence of an edge in terms of the rate of mixing. As a byproduct, we derive a generalisation of Kesten's classical scaling relation for Bernoulli percolation involving the 'mixing rate' critical exponent ι replacing the four-arm event exponent ξ_4.

Contents

1. Introduction

1.1. Motivation

Understanding the behaviour of physical systems undergoing a continuous phase transition at and near their critical point is one of the major challenges of modern statistical physics on both the physics and mathematical sides. In the first half of the twentieth century, the understanding relied essentially on exact computations, as exemplified by the analysis of mean-field systems and Onsager's revolutionary solution of the 2D Ising model [Ons44]. In the 1960s, a major advance was achieved by American chemist Benjamin Widom, who proposed in [Wid65] that many quantities at and near criticality follow power laws, translating the question of understanding the phase transition into the question of computing so-called *critical exponents*. Also, the arrival of the renormalisation group (RG) formalism (see [Fis98] for a historical exposition) led to a (nonrigorous) deep physical and geometrical understanding of continuous phase transitions. The RG formalism suggests that 'coarse-graining' renormalisation transformations correspond to appropriately changing the scale and parameters of the model under study. The large-scale limit of the critical regime then arises as the fixed point of the renormalisation transformations.

A striking consequence of the RG formalism is that because the critical fixed point is usually unique, the scaling limit at the critical point must satisfy translation, rotation, scale and even conformal invariance [Pol70]. In two dimensions, this prediction allowed for the computation of the critical exponents ruling the behaviour of thermodynamical quantities and the classification of models into *universality classes*, meaning classes of models undergoing the same critical behaviour.

Another observation related to the previous developments is that the critical exponents are related to each other: if the behaviours of the specific heat, order parameter, susceptibility, source-field, two-point function and correlation length are governed by the exponents α, β, γ, δ, η and ν, respectively, then the following *scaling relations* were predicted by physicists; see, for example, [EF63, Fis64, Wid65] for some early works on the subject (below, the dimension d of the lattice is assumed to be equal to 2, but we state the relations in this generality as they are predicted to extend to any dimension below the so-called upper critical dimension of the system):

$$\frac{2-\alpha}{d} = \nu = \frac{2\beta}{d-2+\eta}, \tag{1.1}$$

$$2-\eta = d\frac{\delta-1}{\delta+1} = \frac{\gamma}{\nu}. \tag{1.2}$$

A striking feature of these relations is that they hold *for different universality classes*, meaning the critical exponents may vary for different models, but they are always related via equations (1.1) and (1.2).

The aim of this paper is to provide rigorous proofs of these scaling relations for a family *of planar percolation models*. Percolation models are models of random subgraphs of a given lattice. Bernoulli percolation is perhaps the most-studied such model, and breakthroughs in the understanding of its phase transition have often served as milestones in the exciting history of statistical physics. The random-cluster model (also called Fortuin-Kasteleyn percolation) is another example of a percolation model. It was introduced by Fortuin and Kasteleyn around 1970 [For71, FK72] as a generalisation of Bernoulli percolation. It was found to be related to many other models of statistical mechanics, including the Ising, Potts and Ashkin-Teller models, and to exhibit a very rich critical behaviour. Of particular importance from the point of view of physics and relevant to our paper is the fact that the scaling limits of the random-cluster models at criticality are expected to fall into different universality classes and to be related to various 2D conformal field theories.

Let us conclude this section by reminding the reader that the theory of Bernoulli percolation is now well developed, with a deep qualitative understanding of the properties of the scaling limit [AB99] and crossing probabilities [Rus78, SW78], universal critical exponents [KSZ98], scaling relations [Kes87, Nol08], noise sensitivity, near-critical window [GPS10, GPS18] and so on. For a variant of the model (site percolation on the triangular lattice), the existence of the scaling limit and its conformal invariance was proved [Smi01] and critical exponents have been computed [SW01]; see [BD13] and references therein for an overview of two-dimensional Bernoulli percolation. Deriving all these properties for Bernoulli percolation relies on specific features, such as independence of the states of different edges and geometric interpretations of differential formulae using so-called pivotal events. These features are not satisfied for more general random-cluster models. Another more prosaic goal of this paper is therefore to develop robust tools enabling one to bypass these special characteristics of Bernoulli percolation to extend the results mentioned in the abstract to the whole regime of critical random-cluster models undergoing a continuous phase transition. As such, these tools may have a number of implications that are not mentioned in the present paper, in particular for the study of other planar-dependent percolation models.

1.2. Definition of the random-cluster model

As mentioned in the previous section, the model of interest in this paper is the random-cluster model, which we now define. For background, we direct the reader to the monograph [Gri06] and the lecture notes [Dum17] for an exposition of recent results.

Consider the square lattice $(\mathbb{Z}^2, \mathbb{E})$ that is the graph with vertex-set $\mathbb{Z}^2 = \{(n,m) : n,m \in \mathbb{Z}\}$ and edges between nearest neighbours. In a slight abuse of notation, we will write $\mathbb{Z}^2$ for the graph itself. Consider a finite subgraph $G = (V, E)$ of the square lattice (V denotes the vertex-set and E the edge-set), and let ∂G be the set of vertices in V incident to at most three edges in E. Write Λ_n for the subgraph of $\mathbb{Z}^2$ spanned by the vertex-set $\{-n, \dots, n\}^2$. For $1 \le r < R$, write $\mathrm{Ann}(r, R)$ for the annulus $\Lambda_R \setminus \Lambda_{r-1}$. We also write $\Lambda_n(x)$ and $\Lambda_n(e)$ for the boxes of size n recentred around x and the bottom left endpoint of the edge e, respectively.

To define the model, consider first a finite graph G. A percolation configuration ω on G is an element of $\{0,1\}^E$. An edge e is said to be *open* (in ω) if $\omega_e = 1$; otherwise it is *closed*. A configuration ω can be seen as a subgraph of G with vertex-set V and edge-set $\{e \in E : \omega_e = 1\}$. When speaking of connections

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in ω, we view ω as a graph. For sets of vertices A and B, we say that A is connected to B if there exists a path of edges of ω that connect a vertex of A to a vertex of B. This event is denoted by $A\longleftrightarrow B$. We also speak of connections in a set of vertices C if the endpoints of the edges of the path are all in C.

A *cluster* is a connected component of ω. The *boundary conditions* ξ on G are given by a partition of ∂G. We say that two vertices of G are *wired together* if they belong to the same element of the partition ξ.

Definition 1.1. The random-cluster measure on G with edge-weight p, cluster-weight $q > 0$ and boundary conditions ξ is given by

$$\phi^{\xi}_{G,p,q}[\omega] := \frac{1}{Z^{\xi}(G,p,q)}(\tfrac{p}{1-p})^{|\omega|}q^{k(\omega^{\xi})},$$

where $|\omega| := \sum_{e\in E}\omega_e$, $k(\omega)$ is the number of connected components of the graph, ω^{ξ} is the graph obtained from ω by identifying wired vertices together, and $Z^{\xi}(G,p,q)$ is a normalising constant called the *partition function* chosen in such a way that $\phi^{\xi}_{G,p,q}$ is a probability measure.

Two specific families of boundary conditions will be of special interest to us. On the one hand, the *free* boundary conditions, denoted 0, correspond to no wirings between boundary vertices. On the other hand, the *wired* boundary conditions, denoted 1, correspond to all boundary vertices being wired together.

The random-cluster model may be modified to accommodate an external magnetic field as follows. Add to the lattice $\mathbb{Z}^2$ a vertex $\mathfrak{g}$ called the *ghost vertex*, and connect it to each vertex v of $\mathbb{Z}^2$ with an edge $v\mathfrak{g}$. The random-cluster measure $\phi^{i}_{G,p,q,h}$ (for $i = 0$ or 1 and $h \geq 0$) is defined exactly as the random-cluster model on G, except that the boundary is now $\partial G \cup \{\mathfrak{g}\}$, and the edge-weight is p for the edges of G and $1 - e^{-h}$ for edges having $\mathfrak{g}$ as an endpoint: that is,

$$\phi^{i}_{G,p,q,h}[\omega] = \frac{1}{Z^{\xi}(G,p,q,h)}(\tfrac{p}{1-p})^{|\omega|}(e^{h}-1)^{\Delta(\omega)}q^{k(\omega^{i})},$$

where $\Delta(\omega) := \sum_{v\in V}\omega_{v\mathfrak{g}}$. The probability that $\mathfrak{g}$ is in the cluster of 0 has an interpretation in terms of spin models with a magnetic field: for $q = 2$, this probability is equal to the *spontaneous magnetisation* with an external field h for the Ising model on the square lattice. A similar interpretation holds for the 3-state and 4-state Potts models. For more details on this topic, see [BBCK00].

For $q \geq 1$ and $i = 0, 1$, the family of measures $\phi^{i}_{G,p,q,h}$ converges weakly as G tends to the whole square lattice [Gri06, Theorem 4.19]. The limiting measures on $\{0,1\}^{\mathbb{E}}$ are denoted by $\phi^{i}_{p,q,h}$ and are called *infinite-volume* random-cluster measures with free and wired boundary conditions. They are invariant under translations and ergodic. When $h = 0$, we simply drop it from the notation.

The random-cluster model undergoes a phase transition at $h = 0$ and a critical parameter $p_c = p_c(q)$ in the following sense: if $p > p_c(q)$, the probability

$$\theta(p) := \phi^{1}_{p,q}[0 \text{ is in an infinite cluster}]$$

is strictly positive, while for $p < p_c(q)$, it is equal to 0. In the past 10 years, considerable progress has been made in the understanding of this phase transition: the critical point was proved in [BD12] (see also [DM16, DRT18]) to be equal to

$$p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

It was also shown in these papers that the *correlation length*

$$\xi(p) := \lim_{n\to\infty} -n/\log[\pi_1(p;n) - \theta(p)] \in [0,\infty] \tag{1.3}$$

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is finite as soon as $p \neq p_c$, where

$$\pi_1(p;n) := \phi^1_{p,q}[0 \longleftrightarrow \partial\Lambda_n] \tag{1.4}$$

(when $p = p_c$, we drop p_c from this notation).

For $q \geq 1$, it was proved in [DST17, DGHMT16] (see also [RS20] when $q > 4$) that the correlation length at p_c is infinite if and only if $q \leq 4$. As the divergence of the correlation length is one of the characterisations of a continuous phase transition, and as we are interested in this type of phase transition only, in the whole paper we will restrict our attention to the range $q \in [1, 4]$. Also, since the $q = 1$ case was already treated by Kesten in [Kes87] and later solved in numerous other places (see references below), we will often assume that $q > 1$.

Two notational conventions

Since $q \in [1, 4]$ will always be fixed, we drop it from notation. For $q \in [1, 4]$, there is a unique infinite-volume random-cluster measure, so we omit the superscript corresponding to the boundary condition and denote it simply by ϕ_p.

For two families $(f_i)_{i\in I}$ and $(g_i)_{i\in I}$, introduce $f \asymp g$ (respectively, $f \lesssim g$and $f \gtrsim g$) to refer to the existence of constants $c, C \in (0, \infty)$ such that for every $i \in I$, $cg_i \leq f_i \leq Cg_i$ (respectively, $f_i \leq Cg_i$ and $f_i \geq cg_i$). In most cases, the family I will be obvious from context and omitted. In the special case where I contains (implicitly or explicitly) the edge-parameter $p \in (0, 1)$, we further require that p is not close to 0 or 1 (which is justified for every application we have in mind since we are interested in properties for p close to p_c).

1.3. Stability below the characteristic length

When studying a noncritical system, a natural length-scale is provided by the *characteristic length*, which appeared in a simplified context of Bernoulli percolation in the work of Kesten [Kes87] (see also [BCKS99]) and was explicit for the random-cluster model in [DGP14]. To define the characteristic length, we first introduce the notion of crossing probability.

A *quad* $(\mathcal{D}; a, b, c, d)$ is a finite subgraph of $\mathbb{Z}^2$ whose boundary $\partial\mathcal{D}$ is a simple path of edges of $\mathbb{Z}^2$, along with four points a, b, c, d found on $\partial\mathcal{D}$ in counterclockwise order. These points define four arcs (ab), (bc), (cd) and (da) corresponding to the parts of the boundary between them. We also see the quad as a domain of $\mathbb{R}^2$ with marked points on its boundary by taking the union of the faces enclosed by $\partial\mathcal{D}$. The typical example is the case of rectangles $[0, n] \times [0, m]$ or Λ_n with a, b, c, d being the corners of the rectangle, oriented in counterclockwise order, starting from the bottom-right corner. In this case, we omit a, b, c, d from the notation. We say that the quad $(\mathcal{D}, a, b, c, d)$ is *crossed* if (ab) is connected to (cd) in $\mathcal{D}$. The event is denoted by $\mathcal{C}(\mathcal{D})$.

We say that a quad $(\mathcal{D}; a, b, c, d)$ is η-regular at scale R for some $\eta > 0$ if $\mathcal{D}$ is contained in Λ_R and is the union of a finite number of translates of $\Lambda_{\eta R}$ by points of $\eta R\mathbb{Z}^2$, and $a, b, c, d \in \eta R\mathbb{Z}^2$.

Now, consider $\delta_L > 0$ small enough. How small δ_L should be is dictated by the proof of equation (RSW), and we simply wish to mention here that δ_L can be taken independent of $q \in [1, 4]$ and can easily be estimated (even though the value is irrelevant for our study); see also Remark 2.14.

Definition 1.2 (Characteristic length). For each $q \geq 1$ and $p \in (0, 1)$, let

$$L(p) = L(p, q) := \inf\{R \geq 1 : \phi_p[\mathcal{C}(\Lambda_R)] \notin [\delta_L, 1 - \delta_L]\} \in [1, \infty]. \tag{1.5}$$

We insist on the fact that we consider the unique infinite-volume measure ϕ_p for the definition.

Note that $L(p) < +\infty$ for every $p \neq p_c$ by [BD12]; by duality, $L(p_c) = +\infty$ as long as $\delta_L < 1/2$, which we will always assume. The interest of the characteristic length lies in its connection with the *scaling window*: that is, the regime of parameters (R, p) for which one expects typical properties of the random-cluster model in Λ_R with parameters p to be similar to the critical ones. In physics, the

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statement that the system looks critical is usually related to another length-scale: namely, the correlation length $\xi(p)$ defined in equation (1.3). The correlation length encodes the rate of exponential decay of the probability of being connected to distance n but not to infinity as n tends to infinity; it is not a priori directly related to $L(p)$. Nevertheless, the following result reunites the two notions of correlation and characteristic lengths, thus affirming that the characteristic length is simply the correlation length in disguise.

Theorem 1.3 (Equivalence correlation/characteristic lengths). *Fix* $1 \le q \le 4$. *We have that for* $p \in (0,1)$,

$$L(p) \asymp \xi(p). \tag{1.6}$$

The proof is based on a coarse-grained procedure. We wish to highlight that the result is new for every $1 < q \le 4$ – even for $q = 2$, for which [DGP14, Theorem 1.2] proves almost the same statement, but with a logarithmic control over the ratio of $L(p)/\xi(p)$ rather than a constant one. Combined with [BD12b], for instance, this enables us to estimate Ising quantities; see Theorem 1.13 below.

One of the main results of [Kes87] is that the scaling window is simply the set of $(p;R)$ such that $R = O(L(p))$. This result is sometimes referred to as *stability below the characteristic length*; it is the subject of the following theorem in the context of the random-cluster model. Together with Theorem 1.3, the stability result legitimates the two physical interpretations of the correlation length: in terms of rate of decay and in terms of scaling window. We state the result for $q \neq 1$ as the case $q = 1$ was already treated in [Kes87]. Recall the definition of $\pi_1(p;R)$ and the fact that we omit p when $p = p_c$.

Theorem 1.4 (Stability below the characteristic length). *Fix* $1 < q \le 4$.
(Stability of crossing probabilities) *There exists* $\varepsilon > 0$ *such that for every* η*-regular discrete quad* $(\mathcal{D}, a, b, c, d)$ *at scale* $R \ge 1$ *and every* $p \in (0,1)$ *(the constant in* $\lesssim$ *depends on* η *but* ε *does not),*

$$|\phi_p[\mathcal{C}(\mathcal{D})] - \phi_{p_c}[\mathcal{C}(\mathcal{D})]| \lesssim \left(\tfrac{R}{L(p)}\right)^{\varepsilon}. \tag{1.7}$$

(Stability of the one-arm event) *For every* $p \in (0,1)$ *and every* $R \le L(p)$,

$$\pi_1(R) \asymp \pi_1(p;R). \tag{1.8}$$

The stability of arm event probabilities given by equation (1.8) extends to more general arm events. Moreover, an improved version may be formulated; see Remark 7.2 for details.

The strategy for proving Theorem 1.4 is related to Kesten's original one in that it uses Theorem 1.6 below to study the behaviour of derivatives of crossing events. Nevertheless, several additional difficulties occur, mainly due to the replacement of pivotality by influence in the differential formulas for probabilities of events: recall [Gri06, Theorem 2.46] the general formula, valid for every $q > 0$,

$$\tfrac{\mathrm{d}}{\mathrm{d}p}\phi_p[\mathcal{C}(\mathcal{D})] = \tfrac{1}{p(1-p)} \sum_{e \in \mathbb{E}} \mathrm{Cov}_p(\omega_e, \mathcal{C}(\mathcal{D})), \tag{1.9}$$

where Cov_p denotes the covariance under ϕ_p. For $q = 1$, the sum of covariances gets nicely rewritten in terms of *pivotal edges*: edges that when switched from closed to open, change the occurrence of the event. In particular, it is possible to prove that for crossing events of a rectangle of size R and edges that are far from the boundary of the rectangle, the probability of being pivotal is of the order of the probability $\pi_4(p;R)$ that in the ball of radius R around a given edge e, the two extremities of e belong to different clusters that both reach the boundary of said ball (when $p = p_c$, we simply write $\pi_4(R)$). This property was used crucially in [Kes87] and ultimately leads to Kesten's scaling relation $L(p)^2\pi_4(L(p)) \asymp (p - p_c)^{-1}$. The description in terms of pivotal edges is wrong for random-cluster models with $q > 1$ as the covariance between an edge and crossing events at scale R is no longer of the order of $\pi_4(p;R)$; see [DGP14].

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Driven by this different phenomenology, in this paper we introduce a *new interpretation* of the covariance valid for every $q > 1$ encoding how much an edge is influenced by boundary conditions at a distance R or, equivalently, how fast the model mixes.

Definition 1.5 (Mixing rate). For $1 < q \le 4$, $1 \le r < R$, $p \in (0,1)$ and e an edge incident to the origin, write

$$\Delta_p(R) := \phi^1_{\Lambda_R,p}[\omega_e] - \phi^0_{\Lambda_R,p}[\omega_e], \tag{1.10}$$

$$\Delta_p(r,R) := \phi^1_{\Lambda_R,p}[\mathcal{C}(\Lambda_r)] - \phi^0_{\Lambda_R,p}[\mathcal{C}(\Lambda_r)]. \tag{1.11}$$

The quantity $\Delta_p(R)$, to which we now refer as the *mixing rate*, will be crucial in our study, as it will replace the amplitude $\pi_4(p;R)$ of standard pivotal events in the study of Bernoulli percolation. As such, it is very important to derive some of its properties.

Theorem 1.6 (Properties of the mixing rate). *Fix* $1 < q \le 4$.

(i) (Mixing rate/covariance connection) *For every* $\eta > 0$*, every* $p \in (0,1)$ *and every* η*-regular quad* $(\mathcal{D}, a, b, c, d)$ *at scale* $R \le L(p)$ *(below, the constants in* $\asymp$ *depend on* η*),*

$$\mathrm{Cov}_p[\omega_e; \mathcal{C}(\mathcal{D})] \asymp \Delta_p(R), \qquad \forall e \in \Lambda_{2R} \text{ at a distance } \eta R \text{ of } \partial\mathcal{D}, \tag{1.12}$$

$$\mathrm{Cov}_p[\omega_e; \mathcal{C}(\mathcal{D})] \asymp \Delta_p(|e|)^2/\Delta_p(R) \qquad \forall e \in \Lambda_{L(p)} \setminus \Lambda_{2R}, \tag{1.13}$$

$$\mathrm{Cov}_p[\omega_e; \mathcal{C}(\mathcal{D})] \lesssim \exp[-c(\eta)|e|/L(p)]\Delta_p(L(p))^2/\Delta_p(R) \qquad \forall e \notin \Lambda_{L(p)}. \tag{1.14}$$

(ii) (Quasi-multiplicativity) *For every* $p \in (0,1)$ *and* $1 \le r \le \rho \le R \le L(p)$*,*

$$\Delta_p(r,\rho)\Delta_p(\rho,R) \asymp \Delta_p(r,R). \tag{1.15}$$

(iii) (Stability below the characteristic length) *For every* $p \in (0,1)$ *and* $1 \le R \le L(p)$*,*

$$\Delta_p(R) \asymp \Delta_{p_c}(R). \tag{1.16}$$

(iv) (Comparison to pivotality) *There exists* $\varepsilon > 0$ *such that for every* $1 \le r \le R \le L(p)$*,*

$$\Delta_p(r,R) \gtrsim (R/r)^\varepsilon \pi_4(R)/\pi_4(r). \tag{1.17}$$

(v) (Mixing interpretation) *For every* $1 \le 2r \le R \le L(p)$*,*

$$\Delta_p(r,R) \asymp \max\Big\{\Big|\frac{\phi_p[A \cap B]}{\phi_p[A]\phi_p[B]} - 1\Big| \ :\ A \in \mathcal{F}(\Lambda_r) \text{ and } B \in \mathcal{F}(\mathbb{Z}^2 \setminus \Lambda_R)\Big\}, \tag{1.18}$$

where $\mathcal{F}(S)$ *is the* σ*-algebra generated by the edges with both endpoints in* S*.*

The proof of this theorem is the main innovation of the paper. It is based on new increasing couplings between random-cluster models. While coupling Bernoulli percolation at different parameters is fairly straightforward, coupling different random-cluster models can be quite elaborate. In this paper, we develop several increasing couplings between random-cluster models (typically one at p_c and one at p, or one at p_c and another at p_c, but conditioned on an event) that satisfy various properties.

In the previous theorem, Properties (i)–(v) have crucial interpretations. Property (i) will have the following important application: it states that the covariance between an edge that is deep inside an η-regular quad and the crossing event of said quad is of the order of $\Delta_p(R)$. Some more refined results (in particular near the boundary) can be obtained; see Lemma 5.3 and the remarks below. Property (ii) is an analogue of the quasi-multiplicativity of probabilities of arm events and will be popping up everywhere in the applications of $\Delta_p(R)$, in particular when trying to estimate the covariance between

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a crossing event and an edge close to the boundary of the quad. Property (iii) states the stability below the characteristic length for the mixing rate, analogously to that proved by Kesten for the four-arm event probability. Property (iv) shows that replacing $\pi_4(p;R)$ by $\Delta_p(R)$ is really necessary, as the trivial bound stating that the covariance is larger than or equal to pivotality is polynomially far from being sharp for any $q > 1$. Finally, (v) justifies the reference to mixing in the name of Δ_p, as it links this quantity to the error term in the ratio-weak mixing.

We finish our comments on this theorem with a crucial observation. When trying to compute asymptotics for the covariance between an edge and a crossing event, (i) and (ii) imply that it suffices to understand for every $\varepsilon > 0$ the limit of $\Delta_{p_c}(\varepsilon R, R)$ as R tends to infinity. Indeed, these limits allow us to estimate the covariance up to arbitrarily small polynomial terms and therefore estimate the critical exponent. This is very useful as the covariance itself is not easily expressed in terms of properties of large interfaces of the critical system, while $\Delta_{p_c}(\varepsilon R, R)$ (which is equal to $1-2\phi^0_{\Lambda_R,p_c}[\mathcal{C}(\Lambda_{\varepsilon R})]+o(1)$ by duality) is a quantity that can be derived from the scaling limit of the critical model, for instance using the conjectural convergence to the Conformal Loop Ensemble – see also Question 3 below.

It is tempting to deduce from the previous theorem that when $q > 1$, the derivative of crossing probabilities of η-regular quads at scale $R \le L(p)$ is of order $R^2\Delta_p(R)$ (exactly like it is of order $R^2\pi_4(R)$ for Bernoulli percolation). *This statement is not always correct and illustrates the subtle but deep difference with Bernoulli percolation*. Indeed, there is a competition on the right-hand side of equation (1.9) between two possible scenarios:

- The collective contribution of edges in $\mathcal{D}$ is the main part of the right-hand side. In this case, we expect the derivative at p_c to exist and to be of order $R^2\Delta_{p_c}(R)$. Moreover, it may be proved in this case that the derivative is stable within the critical window.
- The collective contribution of edges far from $\mathcal{D}$ is the main part of the right-hand side. In this case, the derivative at p_c is infinite. For $p \neq p_c$, the contribution comes mostly from edges at distance $L(p)$, and the derivative is of order $L(p)^2\Delta_p(L(p))^2/\Delta_p(R)$.

Taking into account estimates from Lemma 5.3 to handle covariances with edges near the boundary, an accurate estimate of the derivative, valid in all scenarios, is therefore given by the following statement.

Corollary 1.7. *Fix $1 < q \le 4$ and $\eta > 0$. For every $p \in (0,1)$ and every η-regular quad $(\mathcal{D}, a, b, c, d)$ at scale $R \le L(p)$,*

$$\tfrac{\mathrm{d}}{\mathrm{d}p}\phi_p[\mathcal{C}(\mathcal{D})] \asymp R^2\Delta_p(R) + \sum_{\ell=R}^{L(p)} \ell\Delta_p(\ell)\Delta_p(R,\ell), \tag{1.19}$$

where the constants in $\asymp$ depend on η.

Looking at this sum formula for the derivative, one sees that whether the derivative is governed by the collective contribution of edges in or close to $\mathcal{D}$ or by that of edges far from $\mathcal{D}$ is related to the way $\ell\Delta_p(\ell)$ decays or not as ℓ tends to infinity. This can also be related to whether the specific heat blows up or not at p_c, as will be seen in the next section. Note that this up-to-constant formula unravelled a third possible scenario where each scale contributes the same amount. This scenario happens for the random-cluster model with $q = 2$, in which the derivative blows up logarithmically in $L(p)$ – the logarithmic blowup of the specific heat may be deduced from the explicit form of the free energy [Ons44].

To derive this corollary, one will need an important result that is reminiscent of the classical claim that the four-arm exponent is strictly smaller than 2 for Bernoulli percolation.

Proposition 1.8 (Lower bound on $\Delta_p(r,R)$). *For every $1 < q \le 4$, there exists $\delta > 0$ such that for every $1 \le r \le R \le L(p)$,*

$$\Delta_p(r,R) \gtrsim (r/R)^{2-\delta}. \tag{1.20}$$

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While the rest of the paper relies on fairly generic assumptions on the percolation model at hand, the previous proposition harvests a much more specific property of the random-cluster model on $\mathbb{Z}^2$, namely the parafermionic observable. For $1 \le q < 4$, the result will follow from crossing estimates that were recently obtained in [DMT20] using this observable. These crossing estimates are uniform in boundary conditions and (possibly fractal) domains. The byproduct of the analysis in [DMT20] is that $\pi_4(R)/\pi_4(r)$ is bounded from below by $(r/R)^{2-\delta}$ and therefore, by (iv), so is $\Delta_p(r, R)$. For $q = 4$, the crossing estimates are not uniform in boundary conditions, and a more specific analysis, also based on the parafermionic observable, must be performed. It is the subject of Section 6.2 in this paper.

1.4. Scaling relations

In continuous phase transitions, natural observables of the model decay algebraically. The behaviour at and near criticality is thus expected to be encoded by various *critical exponents* $\alpha, \beta, \gamma, \delta, \eta, \nu, \zeta, \iota, \xi_1$ and ξ_4 defined as follows (below, $o(1)$ denotes a quantity tending to 0):

$$
\begin{aligned}
f''(p) &= |p - p_c|^{-\alpha+o(1)} && \text{as } p \to p_c,\\
\theta(p) &= (p - p_c)^{\beta+o(1)} && \text{as } p \searrow p_c,\\
\chi(p) &= |p - p_c|^{-\gamma+o(1)} && \text{as } p \to p_c,\\
\phi^1_{p_c,h}[0 \longleftrightarrow \mathfrak{g}] &= h^{1/\delta+o(1)} && \text{as } h \to 0,\\
\phi_{p_c}[0 \longleftrightarrow x] &= |x|^{-\eta+o(1)} && \text{as } |x| \to \infty,\\
\pi_1(R) &= R^{-\xi_1+o(1)} && \text{as } R \to \infty,\\
\xi(p) &= |p - p_c|^{-\nu+o(1)} && \text{as } p \to p_c,\\
\phi_{p_c}[|\mathsf{C}| \ge n] &= n^{-\zeta+o(1)} && \text{as } n \to \infty,\\
\Delta_{p_c}(R) &= R^{-\iota+o(1)} && \text{as } R \to \infty,\\
\pi_4(R) &= R^{-\xi_4+o(1)} && \text{as } R \to \infty,
\end{aligned}
$$

where all the quantities above were already defined in previous sections, except that $0 \longleftrightarrow \mathfrak{g}$ is the event that 0 is connected to the ghost by an open path, $|\mathsf{C}|$ is the number of vertices in the cluster C of the origin, and $f(p)$ and $\chi(p)$ are the thermodynamical quantities, respectively called the *free-energy* and the *susceptibility* defined[1] by

$$
\begin{aligned}
f(p) &:= \lim_{n\to\infty} -\tfrac{1}{|\Lambda_n|} \log Z^0(\Lambda_n, p),\\
\chi(p) &:= \phi_p[|\mathsf{C}|\mathbb{1}_{|\mathsf{C}|<\infty}].
\end{aligned}
$$

Let us mention that the first equation only applies when $f''(p)$ diverges as p approaches p_c, which is to say that the phase transition is of second order.

These exponents are quantities of central interest in physics and have been the object of many studies. A beautiful prediction (again, see e.g., [EF63, Fis64, Wid65]) is that these exponents should depend on each other via *scaling relations*:

$$\eta = 2\xi_1, \tag{R1}$$

$$\zeta = \xi_1/(2 - \xi_1), \tag{R2}$$

$$\delta = (2 - \xi_1)/\xi_1, \tag{R3}$$

$$\beta = \nu\xi_1, \tag{R4}$$

[1]The definition of $f(p)$ for $q = 1$ is slightly different and is given by $f(p) := \phi_p[1/|\mathsf{C}|]$.

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$$\gamma = (2 - 2\xi_1)\nu, \tag{R5}$$

$$\alpha = 2 - 2\nu. \tag{R6}$$

An important feature of the relations above is that they are independent of q: the exponents vary from model to model, but not the formulae. Relations (**R1–5**) were proved for Bernoulli percolation (i.e., cluster-weight $q = 1$) in a milestone paper by Harry Kesten [Kes87] without any of them being computed or, indeed, even shown to exist (see also [Nol08, GPS18, DMT20b]). For the random-cluster with $q = 2$, critical exponents were calculated independently [MW83, DGP14, Dum13] and observed to satisfy (**R1–6**). Let us mention that similar relations should hold in all dimensions that are below the so-called *upper-critical dimension* (with certain values of 2 replaced by the dimension d). We refer to a paper by Borgs, Chayes, Kesten and Spencer [BCKS99] (see also [BCKS01]) for a discussion of this phenomenon for Bernoulli percolation.

Kesten's analysis in the case of Bernoulli percolation relied on another scaling relation, sometimes referred to as *Kesten's scaling relation*, stating that $\nu(2-\xi_4) = 1$ for $q = 1$. It was observed in [DGP14] that this equality fails for $q = 2$ (one can also check this using the table gathering the predicted exponents below). We will show that it should be replaced by the following *generalised Kesten's scaling relation*:

$$\nu(2 - \iota) = 1. \tag{R7}$$

Note that the second property of Theorem 1.6 shows that $\xi_4 > \iota$ so that $\nu(2 - \xi_4) = 1$ fails not only at $q = 2$ but *for every* $q > 1$.

Before discussing the main results, let us mention the predicted values for the different exponents. The three first scaling relations enable us to express δ, η and ζ in terms of ξ_1 only. This is particularly interesting since ξ_1 is measurable in terms of the scaling limit of interfaces at criticality. The relations given by equations (**R6**) and (**R7**) link α, ν and ι. This is again very useful since it was noted in the previous section how ι can be obtained from the understanding of the scaling limit of interfaces. An alternative approach to computing these three exponents would be to first obtain α, which may perhaps be derived using exact integrability of the random-cluster model; see [Bax89, Section 12.8] and Section 1.6 for more details. Finally, equations (**R4**) and (**R5**) express β and γ in terms of ξ_1 and ν so that one can obtain all the exponents from ξ_1 and ι (or α).

Conformal invariance enables us to predict that the scaling limit of the random-cluster model with cluster-weight $q \in [0, 4]$ is related to CLE(κ) (see [SSW09] and the discussion in [GW19]), from which ξ_1 and ι can be deduced. This leads to the following table, where all the exponents are expressed in terms of κ.

Exponent	**Definition**	$q \in [0,4]$	$q = 1$	$q = 2$	$q = 3$	$q = 4$
κ	$\kappa(q) := 4\pi/\arccos(-\frac{\sqrt{q}}{2})$	κ	6	$\frac{16}{3}$	$\frac{24}{5}$	4
α	$f''(p) = \|p - p_c\|^{-\alpha+o(1)}$	$\frac{2}{3}\frac{16-3\kappa}{8-\kappa}$	$-\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$
β	$\theta(p) = (p - p_c)^{\beta+o(1)}$	$\frac{3\kappa-8}{12\kappa}$	$\frac{5}{36}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{12}$
γ	$\chi(p) = \|p_c - p\|^{-\gamma+o(1)}$	$\frac{4}{3\kappa} + \frac{16}{3(8-\kappa)} - \frac{1}{2}$	$\frac{43}{18}$	$\frac{7}{4}$	$\frac{13}{9}$	$\frac{7}{6}$
δ	$\theta(p_c, h) = h^{1/\delta+o(1)}$	$\frac{(8+\kappa)(3\kappa+8)}{(8-\kappa)(3\kappa-8)}$	$\frac{91}{5}$	15	14	15
η	$\phi^0_{p_c}[0 \longleftrightarrow x] = \|x\|^{-\eta+o(1)}$	$\frac{(8-\kappa)(3\kappa-8)}{16\kappa}$	$\frac{5}{24}$	$\frac{1}{4}$	$\frac{4}{15}$	$\frac{1}{4}$
ν	$\xi(p) = \|p_c - p\|^{-\nu+o(1)}$	$\frac{8}{3(8-\kappa)}$	$\frac{4}{3}$	1	$\frac{5}{6}$	$\frac{2}{3}$
ζ	$\phi^0_{p_c}[\|\mathsf{C}\| \geq n] = n^{\zeta+o(1)}$	$\frac{(8-\kappa)(3\kappa-8)}{(8+\kappa)(3\kappa+8)}$	$\frac{5}{91}$	$\frac{1}{15}$	$\frac{1}{14}$	$\frac{1}{15}$
ξ_1	$\pi_1(R) = R^{-\xi_1+o(1)}$	$\frac{(8-\kappa)(3\kappa-8)}{32\kappa}$	$\frac{5}{48}$	$\frac{1}{8}$	$\frac{2}{15}$	$\frac{1}{8}$
ξ_4	$\pi_4(R) = R^{-\xi_4+o(1)}$	$-\frac{\kappa}{8} + 4 + \frac{6}{\kappa}$	$\frac{5}{4}$	$\frac{39}{24}$	$\frac{33}{20}$	2
ι	$\Delta(R) = R^{-\iota+o(1)}$	$\frac{3\kappa-8}{8}$		1	$\frac{4}{5}$	$\frac{1}{2}$

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In this paper, we prove (**R1–7**) for the random-cluster model with general cluster-weights $q \in [1,4]$, except for equation (**R6**) when α is negative. We insist on the fact that the random-cluster models belong to different universality classes when q varies from $q = 1$ to $q = 4$, so that this paper provides the first generic derivation of these relations for different universality classes. As in [Kes87], we do not claim to show that any of these exponents exist, nor do we compute their values; the actual statements of the scaling relations with no reference to the exponents are given in the following three theorems. Nonetheless, if one makes the assumption of algebraic decay with the proper exponent, the next statements imply the scaling relations mentioned above.

Below, we assume that $1 < q \le 4$ as the case $q = 1$ is already known. We start with the two simplest scaling relations given by equations (**R1**) and (**R2**) involving only quantities at $p = p_c$ and $h = 0$. The theorem is an easy consequence of uniform crossing estimates obtained for the random-cluster at criticality; see, for example, [DST17]. While the result is not especially complicated, we chose to include it here for completeness. Introduce the following quantity for every $n > 0$,

$$\varphi(n) := \min\{r \in \mathbb{N} : r^2\pi_1(r) \ge n\}. \tag{1.21}$$

Theorem 1.9 (Scaling relations at criticality). *Fix* $1 < q \le 4$. *For* $x \in \mathbb{Z}^2$ *and* $n \ge 1$*,*

$$\phi_{p_c}[0 \longleftrightarrow x] \asymp \pi_1(|x|)^2, \tag{1.22}$$

$$\phi_{p_c}[|\mathsf{C}| \ge n] \asymp \pi_1(\varphi(n)). \tag{1.23}$$

We now turn to the scaling relation given by equation (**R3**) involving the magnetic field. For $q = 2$, equation (**R3**) was proved in [CGN14] using the GKS inequality, but this inequality is not available for general random-cluster models. A fact that came as a surprise to us is that equation (**R3**) can be derived for every $1 \le q \le 4$ without referring to any other result of the paper (see Section 8.3).

Theorem 1.10 (Scaling relation with magnetic field). *Fix* $1 < q \le 4$. *For* $h > 0$*,*

$$\phi_{p_c,h}[0 \longleftrightarrow \mathfrak{g}] \asymp \pi_1(\varphi(\tfrac{1}{h})). \tag{1.24}$$

The scaling relations in equations **R4**–**R7** are the most difficult as they involve the random-cluster model at p near p_c and rely heavily on the stability in the near-critical regime.

Theorem 1.11 (Scaling relations near-critical regime). *Fix* $1 < q \le 4$. *For* $p > p_c$ *(and* $p \ne p_c$ *for the second),*

$$\theta(p) \asymp \pi_1(\xi(p)), \tag{1.25}$$

$$\chi(p) \asymp \xi(p)^2\pi_1(\xi(p))^2, \tag{1.26}$$

$$\Delta_{p_c}(\xi(p)) \asymp \xi(p)^{-2}|p - p_c|^{-1}, \tag{1.27}$$

$$f''(p) \asymp \sum_{\ell \le \xi(p)} \ell\Delta_{p_c}(\ell)^2. \tag{1.28}$$

Note that assuming that ι exists, we get very different behaviour depending on whether it is smaller or larger than 1 or, correspondingly, whether α is positive or negative: that is, whether the random-cluster model undergoes a second-order or higher-order phase transition. The former occurs when $\nu < 1$: that is, conjecturally when $q \in (2,4]$. When $\nu = 1$, which is conjectured to correspond only at $q = 2$, all the exponents are known and $f''(p)$ blows up logarithmically (in particular, it satisfies the scaling relation as well). When $\nu > 1$, $f''(p)$ should remain bounded in the vicinity of p_c, and the phase transition becomes of third order (or higher). The exponent α may still be defined using the third derivative of f,

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which is supposed to diverge at p_c. We are currently only able to derive an upper bound on f'''; the lower bound is unavailable even for Bernoulli percolation. We refer to Remark 8.4 for details.

1.5. Three complementary results on the Potts models

This section gathers three satellite results that are of interest on their own and do not necessarily fit into the story line of the previous sections. For $q = 2$, it is already known that the value of ξ_1 is equal to $1/8$, so our paper provides a new proof of the following immediate corollary using the Edwards-Sokal coupling [Gri06, Section 1.4]. We include it since the Ising model with a magnetic field, contrary to the case $h = 0$, is not integrable and hence is notoriously difficult to study. As mentioned previously, it was obtained in [CGN14] using alternative arguments.

Theorem 1.12. *Let $m(\beta, h)$ be the spontaneous magnetisation of the Ising model on $\mathbb{Z}^2$ at inverse-temperature β and magnetic field h. For every $h \in (0, 1)$,*

$$m(\beta_c, h) \asymp h^{1/15}. \tag{1.29}$$

Let us mention here that for $q = 2$, we can derive from the estimate $\xi(p) \asymp |p - p_c|^{-1}$ (this result was obtained numerous times; see [BD12b] for a short proof) the following result.

Theorem 1.13. *For $q = 2$, we have that for every p,*

$$L(p) \asymp |p - p_c|^{-1} \tag{1.30}$$

and for every $R \le L(p)$ and e and f two edges at a distance R from each other,

$$\Delta_p(R) \asymp R^{-1}, \tag{1.31}$$

$$\mathrm{Cov}_p(\omega_e, \omega_f) \asymp \Delta_{p_c}(R)^2 \asymp R^{-2}. \tag{1.32}$$

In particular, we deduce that

$$f''(p) \asymp \log\left(\tfrac{1}{|p-p_c|}\right). \tag{1.33}$$

Of course, the last estimate can be obtained by differentiating the exact formula for the free energy due to [Ons44]. To the best of our knowledge, the other estimates are new.

When $1 \le q \le 3$, it was proved in [DMT20] that $\pi_1(R)\pi_4(R) \ge cR^{c-2}$ for every $R \ge 1$ and some constant $c > 0$. In Remark 6.8, we show that $\pi_1(R)\Delta_{p_c}(R) \ge cR^{c-2}$ also for $q = 4$. From these inequalities, using equations (1.25), (1.27) and (1.20), one may deduce the result below, which should be understood as $\beta < 1$ for $1 \le q \le 3$ and $q = 4$. The result for $q = 1$ (that is, for Bernoulli percolation) was already obtained by Kesten and Zhang [KZ87]; we expect $\beta < 1$ to be valid for all $1 \le q \le 4$.

Theorem 1.14 (Nondifferentiability of the order parameter). *For every $1 \le q \le 3$ or $q = 4$, there exists $c > 0$ such that for every $p \ge p_c$,*

$$\theta(p) \ge c(p - p_c)^{1-c}. \tag{1.34}$$

In particular, the spontaneous magnetisation $m(\beta)$ of the 2-, 3- and 4-state Potts models satisfies $m(\beta) \ge c(\beta - \beta_c)^{1-c}$ for $\beta \ge \beta_c$.

1.6. Open questions

The present paper opens many doors in the study of the critical regime of random-cluster models (and, more generally, planar-dependent percolation models). We now mention a few open questions that in our

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opinion deserve attention. We refrain from asking the obvious question of proving conformal invariance of the model and focus on questions directly related to the current work.

Let us start with a question concerning scaling relations: whether one can prove equation (**R6**) when $q \le 2$. As mentioned above, in this case $f''(p)$ is expected to remain bounded when p tends to p_c, but one may consider the behaviour of $f'''(p)$ to make sense of α. Remark 8.4 of the present paper shows that the critical exponent α defined like this satisfies $2\nu \le 2 - \alpha$, leaving the following question open (note that this is also open for $q = 1$).

Question 1. Prove that for every $1 \le q < 2$, one has $2\nu \ge 2 - \alpha$.

Another natural question is to derive critical exponents for random-cluster models. The scaling relations enable us to deduce certain exponents from others, and we may therefore choose which exponents to try to derive. From this point of view, the exponent α is particularly tempting since it directly implies ν and also since exact integrability often provides physicists and mathematicians with closed formulae that may lead to α. We refer to [Bax89] for more details on this and summarise the discussion in the following question.

Question 2. Obtain α using exact integrability to understand the near-critical behaviour of the free energy.

Another approach consists in deriving the exponents having as a basis the assumption of conformal invariance. As already discussed, the scaling limit of the family of boundaries of clusters should then be described by a Conformal Loop Ensemble. The parameter of the CLE may be identified via crossing probabilities and self-duality as the unique $\kappa \ge 4$ that satisfies $\sqrt{q} = -2\cos(4\pi/\kappa)$ [MilWer18].

Due to the present paper, ξ_1 and ι are sufficient to derive the other exponents. Moreover, they both should be computable using the scaling limit of the critical model, while the exponents β, γ, δ and ν involve values of $p \ne p_c$ and should not therefore be accessible directly from the critical scaling limit. The question of deriving ξ_1 assuming the convergence to CLE of the interfaces has already been discussed in the literature [LSW02]. In light of the quasi-multiplicativity property of $\Delta_{p_c}(r, R)$, the following question seems tractable.

Question 3. Compute ι assuming that the scaling limit of the critical model is described by CLE(κ).

Let us finish this section by mentioning that [DGP14] emphasises a self-organised mechanism in the way new edges occur as p increases in Grimmett's monotone coupling (see [Gri06, DGP14] for details). The authors argued that edges appear in clouds and that the understanding of these clouds would be crucial towards the construction of the near-critical scaling limit; would anybody manage to construct the conformally invariant scaling limit at p_c? The current work answers a number of questions and conjectures asked in this paper (including Conjectures 4.1 and 4.2 since $\xi(p)$ is explicitly known; see, for example, [BD12b] and references therein) but does not provide direct insight on the structure of these clouds. We therefore include the following question.

Question 4. What does the present work tell us about clouds (in the sense of [DGP14]) in Grimmett's monotone coupling?

Almost everything is known about the random-cluster model with $q = 2$ on the square lattice, since the conformal invariance of the model and its interfaces was proved [Smi10, CDHKS14]. It is therefore only natural to discuss the question of the construction of the near-critical scaling limit in this context, especially since one expects subtle differences with the corresponding result for Bernoulli percolation (see [GPS18]).

Question 5. Construct the near-critical scaling limit of the model: that is, the limits of random-cluster models on $\frac{1}{R}\mathbb{Z}^2$ at p such that R is on the order of $\lambda L(p)$, where λ is a fixed strictly positive parameter. One may start by studying the case of $q = 2$.

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Recently, the rotational invariance of the critical random-cluster model was obtained in [DKKMO20]. This rotational invariance is expected to carry over to the near-critical regime. The arguments developed here, combined with those of [DGP14], should be relevant for the next question.

Question 6. Prove that the near-critical scaling limit of the model is invariant under rotations.

Organisation of the paper

Section 2 provides the necessary background to our paper. Section 3 studies the dependency of crossing probabilities on boundary conditions (see Theorem 3.6) and introduces the notion of boosting pairs of boundary conditions. Section 4 contains the proof of points (ii), (iv) and (v) of Theorem 1.6. This is the core of our paper and indeed its biggest innovation. Section 5 initiates the connection between the quantity Δ_p and covariances, in particular proving Theorem 1.6(i). Section 6 shows the lower bound on Δ_p given by Proposition 1.8. Section 7 contains the proof of the stability below the correlation length: Theorem 1.4 and Theorem 1.6(iii). Finally, Section 8 contains the derivation of the scaling relations.

A word about constants

We will often work with $p \in (0,1)$ and a spatial scale $R \le L(p)$. Unless stated otherwise, constants $c, (c_i)_{i\ge0}, C$ and $(C_i)_{i\ge0}$ are assumed uniform in $(p;R)$ as above, with the assumption that p is not close to 0 or 1. They are, however, allowed to depend on the threshold δ_L used in the definition of $L(p)$; recall that this threshold is assumed small but fixed. We do not discuss the dependence in q of constants, but the careful reader will notice that they may be rendered uniform in q, potentially outside the vicinity of 1 and 4.

We reiterate that the constants in the notation $\asymp$, $\lesssim$ and $\gtrsim$ also follow the same principle.

2. Preliminaries

This section briefly recalls some tools for the study of the planar random-cluster model. Some sections are new, for instance Section 2.4. We recommend that the reader quickly browses this section, even if they are already comfortable with the basics of the random-cluster model.

2.1. Elementary properties of the random-cluster model

We will use standard properties of the random-cluster model. They can be found in [Gri06], and we only recall them briefly below. Fix a subgraph $G = (V,E)$ of $\mathbb{Z}^2$.

Monotonic properties. An event A is called *increasing* if for any $\omega \le \omega'$ (for the partial ordering on $\{0,1\}^E$), $\omega \in A$ implies that $\omega' \in A$. Fix $q \ge 1$, $1 \ge p' \ge p \ge 0$, $h' \ge h \ge 0$ and some boundary conditions $\xi' \ge \xi$, where $\xi' \ge \xi$ means that any wired vertices in ξ are also wired in ξ'. Then for every increasing events A and B,

$$\phi^{\xi}_{G,p,h}[A\cap B] \ge \phi^{\xi}_{G,p,h}[A]\phi^{\xi}_{G,p,h}[B], \tag{FKG}$$

$$\phi^{\xi}_{G,p,h'}[A] \ge \phi^{\xi}_{G,p,h}[A], \tag{h-MON}$$

$$\phi^{\xi}_{G,p'}[A] \ge \phi^{\xi}_{G,p}[A], \tag{p-MON}$$

$$\phi^{\xi'}_{G,p,h}[A] \ge \phi^{\xi}_{G,p,h}[A]. \tag{CBC}$$

The inequalities above will, respectively, be referred to as the *FKG inequality*, the *monotonicity in h* and *p* and the *comparison between boundary conditions*.

Spatial Markov property. For any configuration $\omega' \in \{0,1\}^E$ and any $F \subset E$,

$$\phi^{\xi}_{G,p,h}[\cdot_{|F} \mid \omega_e = \omega'_e, \forall e \notin F] = \phi^{\xi'}_{H,p,h}[\cdot], \tag{SMP}$$

where H denotes the graph spanned by the edge-set F and ξ' the boundary conditions on H defined as follows: x and y on ∂H are wired if they are connected in $(\omega')^{\xi}_{|E\setminus F}$.

Dual model. Define the dual graph $G^* = (V^*, E^*)$ of G in the usual way: place dual sites at the centres of the faces of G, and for every bond $e \in E$, place a dual bond e^* between the two dual sites corresponding to faces bordering e. Given a subgraph configuration ω, construct a configuration ω^* on G^* by declaring any bond of the dual graph to be open (respectively, closed) if the corresponding bond of the primal lattice is closed (respectively, open) for the initial configuration. The new configuration is called the *dual configuration* of ω. The dual model on the dual graph given by the dual configurations then corresponds to a random-cluster measure with the same parameter q, a dual parameter p^* satisfying

$$\frac{p^* p}{(1-p^*)(1-p)} = q$$

and dual boundary conditions. We do not want to discuss too much the details of how dual boundary conditions are defined (we refer to [Gri06]) and simply mention that the dual of free boundary conditions are the wired ones and vice versa. Note that the critical point is self-dual in the sense that $p_c^* = p_c$.

Loop model. The loop representation of a configuration on G is supported on the *medial graph* of G defined as follows. Let $(\mathbb{Z}^2)^\diamond$ be the *medial* lattice with vertex-set given by the midpoints of edges of $\mathbb{Z}^2$ and edges between pairs of nearest vertices (i.e., vertices at a distance $\sqrt{2}/2$ from each other). It is a rotated and rescaled version of $\mathbb{Z}^2$. For future reference, note that the faces of $(\mathbb{Z}^2)^\diamond$ contain either a vertex of $\mathbb{Z}^2$ or one of $(\mathbb{Z}^2)^*$. The edges of the medial lattice $(\mathbb{Z}^2)^\diamond$ are considered oriented in a counterclockwise direction around each face containing a vertex of $\mathbb{Z}^2$. Let $G^\diamond$ be the subgraph of $(\mathbb{Z}^2)^\diamond$ spanned by the edges of $(\mathbb{Z}^2)^\diamond$ adjacent to a face corresponding to a vertex of G.

Let ω be a configuration on G. Draw self-avoiding paths on $G^\diamond$ as follows: a path arriving at a vertex of the medial lattice always takes a $\pm\pi/2$ turn at vertices so as not to cross the edges of ω or ω^* (see Figure 14). The loop configuration thus defined is formed of possibly several paths going from boundary to boundary, as well as disjoint loops; together these form a partition of the edges of $G^\diamond$.

2.2. Crossing and arm event probabilities below the characteristic length

As is often the case when investigating the critical behaviour of lattice models, we will need to use crossing estimates in rectangles and more generally in quads, as well as estimates on certain universal and non-universal critical exponents. Such crossing estimates initially emerged in the study of Bernoulli percolation in the late 1970s under the coined name of Russo-Seymour-Welsh theory [Rus78, SW78].

The main technical tool we will use is the following result on crossing estimates and arm events probabilities.

Theorem 2.1 (Crossing estimates below the characteristic length). *For $\rho, \varepsilon > 0$, there exist $c' > 0$ and $c = c(\rho,\varepsilon) > 0$ such that for every p, every $1 \le n \le L(p)$, every graph G containing $[-\varepsilon n, (\rho+\varepsilon)n] \times [-\varepsilon n, (1+\varepsilon)n]$ and every boundary conditions ξ,*

$$c \le \phi^{\xi}_{G,p}[\mathcal{C}([0,\rho n]\times[0,n])] \le 1-c. \tag{RSW}$$

Moreover, if A_n denotes the event that there exists an open circuit surrounding Λ_n in Ann$(n, 2n)$*,*

$$\phi^0_{\mathrm{Ann}(n,2n),p}[A_n] \ge c' > 0. \tag{RSW$'$}$$

Since the result is not formally proved anywhere, we include it here. It basically consists in gathering different known results.

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Proof. We start with equation (RSW). By duality and comparison between boundary conditions given by equation (CBC), it suffices to show that for $p < p_c$ and $\xi = 0$, we have that

$$\phi^0_{\overline{R},p}[\mathcal{C}(R)] \geq c,$$

where $R := [0, \rho n] \times [0, n]$ and $\overline{R} = [-\varepsilon n, (\rho+\varepsilon)n] \times [-\varepsilon n, (1+\varepsilon)n]$.

The RSW theorem extracted from [DT19] gives the existence of $C = C(\rho) > 0$ such that for every n and p,

$$\phi_p[\mathcal{C}(R)] \geq \tfrac{1}{C}\phi_p[\mathcal{C}(\Lambda_n)]^C \geq \tfrac{1}{C}\delta_L^C,$$

where in the second inequality, we used the definition of $L(p)$ and the fact that $n \leq L(p)$.

Consider the event $\mathcal{E}^*$ that there exists a dual-open circuit in the annulus $\overline{R} \setminus R$ surrounding R. Then equations (CBC) and (p-MON) and the fact that $\mathcal{E}^*$ is decreasing imply that $\phi_p[\mathcal{E}^*|\mathcal{C}(R)] \geq \phi^1_{\overline{R}\setminus R, p_c}[\mathcal{E}^*]$. The result of [DST17] states that the latter probability is bounded from below by $c_0 = c_0(\rho, \varepsilon) > 0$ independently of n. The spatial Markov property and the comparison between boundary conditions allow us to conclude that

$$\phi^0_{\overline{R},p}[\mathcal{C}(R)] \geq \phi_p[\mathcal{C}(R)|\mathcal{E}^*] \geq \phi_p[\mathcal{E}^*|\mathcal{C}(R)]\phi_p[\mathcal{C}(R)] \geq \frac{c_0}{C}\delta_L^C.$$

This concludes the proof of equation (RSW). For equation (RSW′), use the FKG inequality and the fact that there exists a circuit in $\mathrm{Ann}(n, 2n)$ surrounding the origin if the rectangle $[-\frac{5}{3}n, \frac{5}{3}n] \times [\frac{4}{3}n, \frac{5}{3}n]$ as well as its rotations by angles $\frac{\pi}{2}$, π and $\frac{3\pi}{2}$ are crossed in the long direction. □

The previous theorem has classical applications for the probability of so-called arm events. A self-avoiding path of type 0 or 1 connecting the inner to the outer boundary of an annulus is called an *arm*. We say that an arm is *of type* 1 if it is composed of primal edges that are all open and *of type* 0 if it is composed of dual edges that are all dual-open. For $k \geq 1$ and $\sigma \in \{0,1\}^k$, define $A_\sigma(r, R)$ to be the event that there exist k *disjoint* arms from the inner to the outer boundary of $\mathrm{Ann}(r, R)$, which are of type $\sigma_1, \dots, \sigma_k$ when indexed in counterclockwise order.

To simplify the notation, we introduce $\pi_\sigma(p; r, R)$ for the ϕ_p-probability of $A_\sigma(r, R)$. We drop p or r from the notation when $p = p_c$ or r is the smallest integer such that $\pi_\sigma(p; r, R) > 0$ for all $R > r$. Finally, when $\sigma = 1010\dots$ is alternating 0 and 1 and has length k, we write the subscript k instead of σ. For every σ, $\pi_\sigma(R)$ decays algebraically with R [DST17], and the scale invariance prediction suggests the existence of a critical exponent ξ_σ such that $\pi_\sigma(R) = R^{-\xi_\sigma + o(1)}$ as R tends to infinity.

We also introduce $A^+_\sigma(r, R)$ to be the same event as $A_\sigma(r, R)$, except that the paths must lie in the upper half-plane $\mathbb{H} := \mathbb{Z} \times \mathbb{Z}_+$ and are indexed starting from the right-most. Introduce its probability $\pi^+_\sigma(p; r, R)$ and the associated exponent ξ^+_σ.

We will need the following near-critical estimates on certain arm event probabilities.

Proposition 2.2 (Estimates on certain arm events). *Fix* $1 \leq q \leq 4$. *There exists* $c > 0$ *such that for* $p \in (0, 1)$ *and* $1 \leq r \leq R \leq L(p)$,

$$\pi_2(p; r, R) \gtrsim (r/R)^{1-c}, \tag{2.1}$$

$$\pi_1(p_c, r, R) \gtrsim (r/R)^{1/2-c} \quad \textit{and} \quad \pi_1(p; r, R) \lesssim (r/R)^c. \tag{2.2}$$

Proof. The bound given by equation (2.1) follows from the fractal structure of interfaces, which in turn follows from equation (RSW) and [AB99, Theorem 1.3]. The argument is classical, and we omit it.

The inequality on the right of equation (2.2) is a standard consequence of equation (RSW). The one on the left follows readily from equation (2.1), the FKG inequality and self-duality. □

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Proposition 2.3 (Quasi-multiplicativity of arm event probabilities). *Fix $1 \le q \le 4$. For every $1 \le r \le \rho \le R \le L(p)$ and $k = 1$ or $k \ge 2$ even,*

$$\pi_k(p; r, \rho)\pi_k(p; \rho, R) \asymp \pi_k(p; r, R). \tag{2.3}$$

Proof. This is a standard consequence of equation (RSW). For $k = 1$, the proof is simple; for $k \ge 2$ even, the proof is more tedious but is identical to that for $q = 1$ [Kes87]. □

The proof of equation (2.3) goes through the so-called separation of arms (for alternating arm events), which is also a consequence of equation (RSW). We direct the reader to [Kes87, Nol08, CDH16] for details.

2.3. Couplings via exploration

In this section, we present a technique for coupling different random-cluster measures in an increasing fashion by exploring the graph edge by edge, which we formalise using decision trees as follows. Consider a graph $G = (V, E)$ with n edges and $U = (U_e)_{e\in E}$ a family of independent uniform random variables in $[0, 1]$. For a n-tuple $e = (e_1, \dots, e_n)$ of edges and for $t \le n$, write $e_{[t]} = (e_1, \dots, e_t)$ (with the convention $e_{[0]} = \emptyset$) and $U_{[t]} = (U_{e_1}, \dots, U_{e_t})$.

Definition 2.4 (Decision tree, stopping time). A *decision tree* is a pair $\mathbf{T} = (e_1, (\psi_t)_{2\le t\le n})$, where $e_1 \in E$, and for each $2 \le t \le n$ the function ψ_t takes a pair $(e_{[t-1]}, U_{[t-1]})$ as an input and returns an element $e_t \in E \setminus \{e_1, \dots, e_{t-1}\}$. A *stopping time* for $\mathbf{T}$ is a random variable τ taking values in $\{1, \dots, n, \infty\}$, which is such that $\{\tau \le t\}$ is measurable in terms of $(e_{[t]}, U_{[t]})$.

We will say that the decision tree *reveals* one by one the edges of E; the edges $e_{[t]}$ are the edges *revealed* at time t. Less formally, a decision tree takes U as an input and reveals edges one after the other. It always starts from the same fixed $e_1 \in E$ (which corresponds to the root of the decision tree) and then queries the value of U_{e_1}. After that, it continues inductively as follows: at step $t > 1$, the function ψ_t, which should be interpreted as the decision rule at time t, takes the locations and the values of the revealed edges at time $t - 1$ and decides which edge to reveal next.

Remark 2.5. The theory of (random) decision trees played a key role in computer science (we refer the reader to the survey [BW02]) but also found many applications in other fields of mathematics. In particular, random decision trees (sometimes called randomised algorithms) were used in [SS10] to study the noise sensitivity of Boolean functions, for instance in the context of percolation theory. It was also used in [DRT19] in combination with the OSSS inequality (which was originally introduced in [OSSS05]) to prove sharpness of random-cluster models.

Decision trees may be used to construct random-cluster measures in a step-by-step fashion. This technique is generic and may be applied to so-called monotonic measures (see, e.g., [Gri06]). A key feature of this construction is that it enables one to do it with two (or more) random-cluster measures simultaneously. In this case, the decision tree produces couplings of these measures. Since we are mostly interested in couplings, we directly explain the construction for a pair of configurations. For $1 \le t \le n$, we extend the notation $e_{[t]}$ and $U_{[t]}$ with the notation $\omega_{[t]} = (\omega_{e_1}, \dots, \omega_{e_t})$. Below, we use the notation G_t for the graph G minus the edges $e_1, \dots, e_t$.

Proposition 2.6. *Fix a finite subgraph $G = (V, E)$ of $\mathbb{Z}^2$. Consider $0 \le p \le p' \le 1$, $q \ge 1$ and $\xi \le \xi'$ boundary conditions. Let $\mathbf{T} = (e_1, (\psi_t)_{2\le t\le n})$ be a decision tree and $(U_e)_{e\in E}$ be a set of independent and identically distributed uniform random variables under some measure $\mathbb{P}_{\mathbf{T}}$. Define $\omega, \omega' \in \{0, 1\}^E$ by the following inductive procedure: for every $0 \le t < n$,*

$$\omega_{e_{t+1}} := \mathbb{1}(U_{e_{t+1}} \ge \phi^{\xi_t}_{G_t,p}[e_{t+1} \text{ closed}]);$$
$$\omega'_{e_{t+1}} := \mathbb{1}(U_{e_{t+1}} \ge \phi^{\xi'_t}_{G_t,p'}[e_{t+1} \text{ closed}]),$$

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where ξ_t and ξ'_t are the boundary conditions induced by $\omega^{\xi}_{[t]}$ and $(\omega'_{[t]})^{\xi'}$, respectively (when $t = 0$, these are ξ and ξ'). Then $\mathbb{P}_{\mathbf{T}}$-almost surely, for every stopping time τ for **T***, we have that*

- $\omega_{[\tau]} \le \omega'_{[\tau]}$,
- *conditionally on $(\tau, \omega_{[\tau]}, \omega'_{[\tau]})$, ω and ω' on G_τ have law $\phi^{\xi_\tau}_{G_\tau,p}$ and $\phi^{\xi'_\tau}_{G_\tau,p'}$.*

Note that for $\tau = 0$, we obtain that ω and ω' have laws $\phi^{\xi}_{G,p}$ and $\phi^{\xi'}_{G,p'}$, respectively, and that $\omega \le \omega'$ a.s. The procedure above may be applied to infinite-volume measures as long as **T** is such that all edges are eventually queried.

Proof. That $\omega_{[\tau]} \le \omega'_{[\tau]}$ is proved by induction and uses the monotonic property of random-cluster measures mentioned in Section 2.1. That $\omega_{[\tau]}$ and $\omega'_{[\tau]}$ have the right laws is also proved by induction, using the spatial Markov property given by equation (SMP). □

Remark 2.7. While the definition indicates that **T** looks at $U_{[t]}$ in order to decide the next queried edge e_{t+1} (and hence $U_{e_{t+1}}$), we will often describe **T** as choosing e_{t+1} as a function of $\omega_{[t]}$ and $\omega'_{[t]}$, which are in turn functions of $(e_{[t]}, U_{[t]})$.

Remark 2.8. Due to Proposition 2.6, we may construct an increasing coupling between $\phi^{\xi}_{G,p}$ and $\phi^{\xi'}_{G,p'}$ by switching between decision trees at stopping times. Indeed, if we start the coupling by following a decision tree **T** but stop the procedure at some stopping time τ, then we may complete it with any increasing coupling of $\phi^{\xi_\tau}_{G_\tau,p}$ and $\phi^{\xi'_\tau}_{G_\tau,p'}$. We will often use this property, sometimes continuing with a specific coupling, other times with an arbitrary one.

We now discuss a few examples of decision trees and the couplings they produce.

Example 1

The deterministic decision tree **T** for which the order $e_1, \ldots, e_n$ is fixed.

Example 2

The decision tree **T** explores the clusters of ∂G in ω'. Formally, this decision tree is defined using a growing sequence $\partial G = V_0 \subset V_1 \subset \cdots \subset V$ that represents the sets of vertices that the decision tree found to be connected to ∂G at time t.

Fix an arbitrary ordering of the edges in E, and set $V_0 = \partial G$. Now, for $t \ge 0$, assume that $e_{[t]}$ and $V_t \subset V$ have been constructed, and distinguish between two cases:

- If there exists an unrevealed edge connecting a vertex $x \in V_t$ to a vertex $y \notin V_t$, then reveal $e_{t+1} = xy$ (if several choices for xy exists, choose the smallest one in the chosen ordering) and set

$$V_{t+1} := \begin{cases} V_t \cup \{y\} & \text{if } \omega'_{e_{t+1}} = 1, \\ V_t & \text{otherwise.} \end{cases}$$

- If no edge as above exists, then set e_{t+1} to be the smallest $e \in E \setminus e_{[t]}$ for the chosen ordering, and set $V_{t+1} := V_t$.

The coupling $\mathbb{P}_{\mathbf{T}}$ has the following useful property when $p = p'$. If τ denotes the first time the decision tree finds no unrevealed edge between V_t and V_t^c (note that τ is a stopping time), then all edges bounding the unrevealed region $E \setminus e_{[\tau]}$ are closed in $\omega'_{[\tau]}$ and hence also in $\omega_{[\tau]}$. As a consequence, at every subsequent step in the coupling process, edges will be sampled with the same rule in the two configurations; hence the configurations will be equal on $E \setminus e_{[\tau]}$. Equivalently, they will only (possibly) differ for edges that are connected to ∂G in ω', thus leading to the following conclusion when combined with equation (RSW).

Proposition 2.9 (Mixing). *There exists $c_{\text{mix}} > 0$ such that for every $p \in (0,1)$ and every $r \le R$ with R/r large enough, every $G \supset \Lambda_R$ and every event A depending on edges in Λ_r, we have that for every two boundary conditions ξ and ψ,*

$$|\phi^{\psi}_{G,p}[A] - \phi^{\xi}_{G,p}[A]| \le (r/R)^{c_{\text{mix}}}\, \phi^{\psi}_{G,p}[A]. \tag{2.4}$$

One may be surprised at first sight not to see any reference to the characteristic length in this statement, yet one should remember that the rate of decay is in fact *faster* when p is away from p_c. The previous proposition simply states a universal bound on the rate of mixing valid for every $p \in (0,1)$.

Notice also that in equation (2.4), the event A is not assumed increasing, nor is there any assumption of ordering between the boundary conditions ξ and ψ. This absence of assumptions would greatly simplify the proof.

The above has the following immediate corollary, which we will use throughout the paper.

Corollary 2.10. *There exists a constant $C_{\text{mix}} > 1$ such that for every $p \in (0,1)$, every $R \ge 1$, every two boundary conditions ξ and ψ on $\Lambda_{C_{\text{mix}}R}$ and every event A depending on the edges in Λ_R,*

$$\tfrac{1}{2}\phi^{\xi}_{\Lambda_{C_{\text{mix}}R},p}[A] \le \phi^{\psi}_{\Lambda_{C_{\text{mix}}R},p}[A] \le 2\phi^{\xi}_{\Lambda_{C_{\text{mix}}R},p}[A]. \tag{2.5}$$

In particular, for any graph G containing $\Lambda_{C_{\text{mix}}R}$, any boundary condition ξ and any two events A and B depending on the edges inside Λ_R and outside $\Lambda_{C_{\text{mix}}R}$, respectively,

$$\phi^{\xi}_{G,p}[A \cap B] \asymp \phi^{\xi}_{G,p}[A]\phi^{\xi}_{G,p}[B]. \tag{Mix}$$

Proof of Proposition 2.9. By duality, it suffices to prove the statement for $p \le p_c$. Set $\rho = \sqrt{rR}$, and let Ω be a subgraph of G containing Λ_ρ. We first compare the probability of A under free boundary conditions to that under arbitrary boundary conditions ψ. For reasons that will be apparent later, we do this on Ω.

For some boundary conditions ψ on $\partial\Omega$, using the increasing coupling $\mathbb{P}_{\mathbf{T}}$ between $\phi^0_{\Omega,p}$ and $\phi^{\psi}_{\Omega,p}$ described above, we find

$$\begin{aligned}
\phi^{\psi}_{\Omega,p}[A] - \phi^{0}_{\Omega,p}[A] &\le \mathbb{P}_{\mathbf{T}}[\omega \notin A \text{ but } \omega' \in A]\\
&\le \phi^{\psi}_{\Omega,p}[\omega' \in A \text{ and } \partial\Lambda_r \overset{\omega'}{\longleftrightarrow} \partial\Omega]\\
&\le \phi^{1}_{\Omega\setminus\Lambda_r,p_c}[\partial\Lambda_r \longleftrightarrow \partial\Omega]\phi^{\psi}_{\Omega,p}[A]\\
&\le (r/R)^c \phi^{\psi}_{\Omega,p}[A],
\end{aligned}$$

for some constant $c > 0$. The first inequality is due to the property of the coupling, the second to equations (SMP), (CBC) and (p-MON) and the third to equations (2.2) and (2.5). In conclusion,

$$\phi^{\psi}_{\Omega,p}[A] \le \phi^{0}_{\Omega,p}[A]/(1-(r/R)^c). \tag{2.6}$$

The above applies in particular to $\Omega = G$; let us now obtain a converse bound in this case.

Start by observing that for any fixed Ω as above,

$$\phi^{0}_{G,p}[A] = \sum_{\xi \text{ b.c. on } \partial\Omega} \phi^{\xi}_{\Omega,p}[A]\phi^{0}_{G,p}[\omega_{|G\setminus\Omega} \text{ induces } \xi \text{ on } \partial\Omega] \le \phi^{0}_{\Omega,p}[A]/(1-(r/R)^c). \tag{2.7}$$

Now fix some boundary conditions ψ on G. For a configuration ω on G, let $\Omega(\omega)$ be the set of vertices that are *not* connected to $\mathbb{Z}^2 \setminus \Lambda_{R-1}$. Then

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$$
\begin{aligned}
\phi^{\psi}_{G,p}[A] &\geq \phi^{\psi}_{G,p}[A, \partial\Lambda_\rho \not\longleftrightarrow \partial\Lambda_R] \\
&= \sum_{\Omega \supset \Lambda_\rho} \phi^0_{\Omega,p}[A]\, \phi^{\psi}_{G,p}[\Omega(\omega) = \Omega] \\
&\geq (1 - (r/R)^c)\phi^0_{G,p}[A]\phi^{\psi}_{G,p}[\partial\Lambda_\rho \not\longleftrightarrow \partial\Lambda_R] \\
&\geq (1 - (r/R)^c)^2 \phi^0_{G,p}[A],
\end{aligned}
$$

where in the second inequality, we used equation (2.7) and the fact that

$$
\sum_{\Omega \supset \Lambda_\rho} \phi^{\psi}_{G,p}[\Omega(\omega) = \Omega] = \phi^{\psi}_{G,p}[\partial\Lambda_\rho \not\longleftrightarrow \partial\Lambda_R].
$$

In the last inequality, we used that

$$
\phi^{\psi}_{G,p}[\partial\Lambda_\rho \longleftrightarrow \partial\Lambda_R] \leq \phi^{\psi}_{G,p_c}[\partial\Lambda_\rho \longleftrightarrow \partial\Lambda_R] \leq (\rho/R)^{2c} = (r/R)^c
$$

for some constant $c > 0$, by equation (2.2).

Using the inequality above and equation (2.6) applied to G, we conclude that

$$
\left|\phi^{\psi}_{G,p}[A] - \phi^0_{G,p}[A]\right| \leq 2(r/R)^c \phi^0_{G,p}[A].
$$

Applying the above to two arbitrary boundary conditions ψ and ξ on ∂G and using the triangular inequality, we conclude that for all r/R large enough,

$$
\left|\phi^{\psi}_{G,p}[A] - \phi^{\xi}_{G,p}[A]\right| \leq 4(r/R)^c \phi^0_{G,p}[A] \leq 4(r/R)^c(1 + 2(r/R)^c)\phi^{\psi}_{G,p}[A].
$$

By assuming again that r/R is small enough and modifying the constant in the exponent, we may eliminate the prefactor $4(1 + 2(r/R)^c)$ and obtain equation (2.4). □

Proof of Corollary 2.10. By taking C_{mix} large enough and applying equation (2.4), one directly obtains equation (2.5). Then equation (Mix) may be deduced by equation (SMP) applied to $\Lambda_{C_{\text{mix}}R}$. □

Example 3
Alternatively, one may consider the decision tree **T** that explores the dual clusters of ∂G^* in ω^*. We do not define this decision tree formally as it is almost identical to that of the previous example. We simply mention that when coupling two measures with $p = p'$ using **T**, differences only occur for edges that are connected in ω^* to ∂G.

Remark 2.11. In spite of the constructions above, we are unaware of the existence of a coupling of random-cluster models with boundary conditions $\xi \leq \xi'$ and the same edge-parameter $p = p'$ that combines the properties of Examples 2 and 3: namely, a coupling for which only edges connected in both ω' and ω^* to ∂G may have different states in the two configurations.

Remark 2.12. Even though the uniform variables $(U_e)_{e\in E}$ are attached to the edges, the order in which these are revealed by **T** influences the final couple of configurations (ω, ω'). Indeed, consider $G = \Lambda_R$, parameters $p = p'$ and boundary conditions $\xi = 0$ and $\xi' = 1$; let e be one of the edges containing the origin. In the coupling produced with the decision tree of Example 1, ω_e may differ from ω'_e when e is not connected to ∂G in ω', while this is impossible with the one produced by Example 2.

2.4. Equivalence $L(p) - \xi(p)$: proof of Theorem 1.3

We will show the following proposition.

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Proposition 2.13. *There exist $c, C > 0$ such that for every $p < p_c$ and $x \in \mathbb{Z}^2$,*

$$\exp[-C|x|/L(p)] \lesssim \phi_p[\Lambda_{L(p)} \longleftrightarrow \Lambda_{L(p)}(x)] \lesssim \exp[-c|x|/L(p)]. \tag{2.8}$$

Before proving this proposition, we explain how it implies the theorem.

Proof of Theorem 1.3. For $p < p_c$, the proof is immediate thanks to the definition of $\xi(p)$ and the fact that

$$p^{2|\Lambda_{L(p)}|}\phi_p[\Lambda_{L(p)} \longleftrightarrow \Lambda_{L(p)}(x)] \le \phi_p[0 \longleftrightarrow x] \le \phi_p[\Lambda_{L(p)} \longleftrightarrow \Lambda_{L(p)}(x)] \tag{2.9}$$

(we use the FKG inequality on the left).

For $p > p_c$, we proceed by duality. Notice that

$$\begin{aligned}\pi_1(p;n) - \theta(p) &= \phi_p[0 \longleftrightarrow \partial\Lambda_n \text{ and } 0 \not\longleftrightarrow \infty] \\ &\le \sum_{k\ge 0} \phi_p[(k+\tfrac{1}{2},0) \overset{\omega^*}{\longleftrightarrow} \partial\Lambda_{k\vee n}(k,0)] \lesssim \exp[-cn/L(p^*)].\end{aligned}$$

Indeed, any configuration contributing to the second probability contains a dual circuit of length at least n, surrounding 0 and passing through some point $(k+\frac{1}{2},0)$ on the horizontal axis. The last inequality is due to the subcritical case already established.

Conversely, due to equation (CBC),

$$\phi_p[0 \longleftrightarrow \partial\Lambda_n \text{ and } 0 \not\longleftrightarrow \infty] \ge \phi^0_{\Lambda_{C_{\mathrm{mix}}n},p_c}[0 \longleftrightarrow \partial\Lambda_n]\phi_p[\Lambda_{C_{\mathrm{mix}}n} \not\longleftrightarrow \infty].$$

As n tends to infinity, the first term on the right-hand side above decays at most polynomially due to equations (2.2) and (2.5), while the second is lower-bounded by $\exp[-C|x|/L(p^*)]$ due to the subcritical case and the FKG property.

The two inequalities above show that $\xi(p) \asymp L(p^*)$. That $L(p^*) \asymp L(p)$ follows directly by duality from the definition of the characteristic length. □

We now turn to the proof of Proposition 2.13.

Proof of Proposition 2.13. Set $L = L(p)$. We assume that $x \in L\mathbb{Z}^2$; the general case can be solved similarly. We start with the lower bound. Consider the shortest family of vertices of $y_i \in L\mathbb{Z}^2$ with $0 = y_0, \dots, y_k = x$. Let $A_L(y)$ be the event that there exists a circuit in $\Lambda_{2L}(y)$ surrounding $\Lambda_L(y)$. If $A_L(y_j)$ occurs for every $0 \le j \le k$, then Λ_L is connected to $\Lambda_L(x)$. We deduce from the FKG inequality and equation (RSW) that

$$\phi_p[\Lambda_L \longleftrightarrow \Lambda_L(x)] \ge \phi_p[A_L]^{k+1} \ge \exp[-C|x|/L], \tag{2.10}$$

where the last inequality follows from equation (RSW) and $C > 0$ is some universal constant.

For the upper bound, we start by observing that by equation (2.5) and the RSW theorem from [DT19], we have that for some constant C,

$$\begin{aligned}\phi^1_{\Lambda_{2C_{\mathrm{mix}}L},p}[\Lambda_L \longleftrightarrow \partial\Lambda_{2L}] &\le 2\,\phi_p[\Lambda_L \longleftrightarrow \partial\Lambda_{2L}] \\ &\le 2\,C\phi_p[\mathcal{C}(\Lambda_L)]^{1/C} \le 2\,C\delta_L^{1/C} < 8^{-(4C_{\mathrm{mix}}+1)^2},\end{aligned} \tag{2.11}$$

provided that δ_L in the definition of $L(p)$ is chosen sufficiently small. Now, if Λ_L and $\Lambda_L(x)$ are connected, then there must exist a sequence of $N \ge |x|/L$ distinct vertices $0 = y_1, \dots, y_N = x$ contained in $L\mathbb{Z}^2$ such that

- $\|y_i - y_{i+1}\|_\infty = L$ for every i;
- $\Lambda_L(y_i)$ is connected to $\partial\Lambda_{2L}(y_i)$ for every i.

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Of these, choose the first subsequence of vertices $(y_i)_{i\in I}$ for the lexicographical order that contains $k = N(4C_{\text{mix}} + 1)^{-2}$ vertices that are all at a distance at least $4C_{\text{mix}}$ from each other (the existence of such a subsequence is due to the pigeonhole principle). The union bound over the possible choices of $y_1, \dots, y_N$ (of which there are at most 8^N), the spatial Markov property given by equation (SMP) and the comparison between boundary conditions in equation (CBC) imply that

$$\phi_p[\Lambda_L \longleftrightarrow \Lambda_L(x)] \leq \sum_{N\geq|x|/L} 8^N \phi^1_{\Lambda_{4L},p}[\Lambda_L \longleftrightarrow \partial\Lambda_{2L}]^k.$$

The desired upper bound follows from the above using equation (2.11). □

Remark 2.14. The previous proof is where the strongest condition on δ_L is imposed; recall that we also require that $\delta_L < 1/2$ to guarantee infinite characteristic length at p_c.

The next corollary is a useful estimate that we will invoke later in the article.

Corollary 2.15. *There exists $c = c(\delta_L) > 0$ such that for every $p > p_c$ and $k \geq 1$,*

$$\phi_p[\Lambda_{kL(p)} \leftrightarrow \infty] \geq 1 - \exp[-ck]. \tag{2.12}$$

Proof. Note that the previous proof implies that for some constant $c > 0$, we have that for every $p > p_c$,

$$\phi_{p^*}[\Lambda_{L(p)} \longleftrightarrow \partial\Lambda_{kL(p)}] \leq \exp[-ck].$$

By the same counting argument as in the proof of the supercritical case of Theorem 1.3, we deduce from the above that the probability that there exists a circuit in ω^* surrounding $\Lambda_{kL(p)}$ is bounded from above by $\sum_{j\geq k} \phi_{p^*}[\Lambda_{L(p)} \longleftrightarrow \partial\Lambda_{jL(p)}] \lesssim \exp[-ck]$. □

3. Boosting pairs of boundary conditions for flower domains

Fix $q \in (1, 4]$ for the whole section; we will omit it from the notation. The results of this section do not apply to $q = 1$. As p will be fixed in all statements, we also remove it from the notation of the measure and write ϕ instead of ϕ_p.

3.1. Flower domains

We start by introducing the crucial notion of the flower domain. Figure 1 may be useful in understanding the definition below.

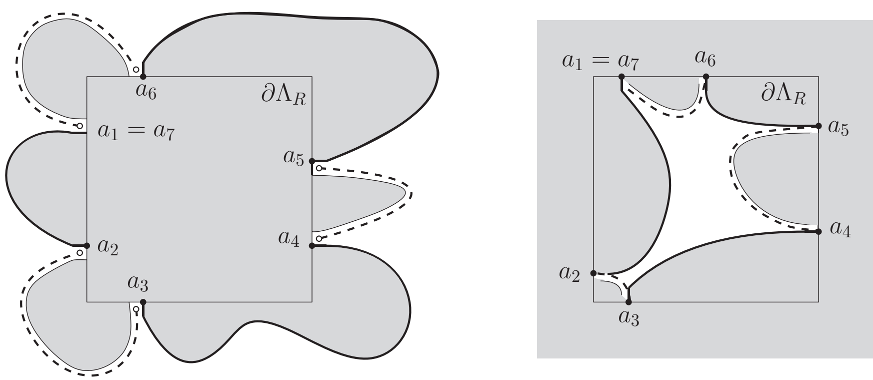

Figure 1. *An inner flower domain on the left and an outer one on the right.*

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Definition 3.1 (Flower domain). An *inner flower domain* on Λ_R is a simply connected finite domain $\mathcal{F}$ containing Λ_R, whose boundary is formed of an even sequence of arcs $(a_j a_{j+1})_{j=1,\dots,2k}$ (with the convention $a_{2k+1} = a_1$), where each point a_j is on $\partial\Lambda_R$.

An *outer flower domain* on Λ_R is the complement $\mathcal{F}$ of a simply connected finite domain, with $\Lambda_R^c \subset \mathcal{F}$ and whose boundary is formed of an even sequence of arcs $(a_j a_{j+1})_{j=1,\dots,2k}$ (with the convention $a_{2k+1} = a_1$), where each point a_j is on $\partial\Lambda_R$.

The arcs of the boundary are called *primal* and *dual petals* depending on whether j is odd or even, respectively. In both cases, we identify $\mathcal{F}$ as the graph formed of the edges strictly inside $\mathcal{F}$ plus the edges on the dual petals.

For $\eta > 0$, the flower domain $\mathcal{F}$ is said to be η-well-separated if the distance between any two distinct points a_i and a_j is greater than ηR.

A boundary condition ξ is said to be *coherent* with $\mathcal{F}$ if all vertices of any primal petal are wired together and all vertices of dual petals (except the endpoints) are wired to no other vertex of $\partial\mathcal{F}$.

Formally, flower domains should be defined as the couple formed of $\mathcal{F}$ and of the points $a_1, \dots, a_{2k}$; we will, however, allow ourselves this small abuse of notation. When considering a flower domain with a coherent boundary condition, it will be useful to view the flower domain as containing the edges of the primal petals that are conditioned to be opened. We will also often identify dual arcs $(a_j a_{j+1})$ with the dual path made of the dual edges e^* with e incident to $x \in (a_j a_{j+1})$ and $y \notin \mathcal{F}$ and assume it is made of open dual edges.

Notice that when $\mathcal{F}$ has at least four petals, several boundary conditions are coherent with $\mathcal{F}$ as different primal petals may or may not be wired together.

Example

The example that we will most commonly use is that of a flower domain revealed from the inside or outside. Consider $r < R$, and let ω be a configuration on the annulus $\mathrm{Ann}(r, R)$.

The *inner flower domain $\mathcal{F}$ from Λ_R to Λ_r* is obtained as follows. Consider all interfaces of ω contained in $\mathrm{Ann}(r, R)$ starting on $\partial\Lambda_R$; these are paths in the loop representation of the random cluster model with endpoints on $\partial\Lambda_R$ or $\partial\Lambda_r$. Write Exp for the set of edges adjacent or intersecting any such interface. Loosely speaking, these are the edges revealed during the exploration of the interfaces starting on $\partial\Lambda_R$.

If at least one such interface has an endpoint on $\partial\Lambda_r$, define $\mathcal{F}$ as being the connected component of Λ_r in $\Lambda_R \setminus \mathsf{Exp}$. Otherwise, $\mathcal{F}$ is not defined (formally, set $\mathcal{F} = \emptyset$ in this case). Observe that the number of interfaces between $\partial\Lambda_R$ and $\partial\Lambda_r$ is necessarily even. Their endpoints $\partial\Lambda_r$ naturally partition $\partial\mathcal{F}$ into primal and dual petals. See Figure 1 for an illustration.

To define the *outer flower domain from Λ_r to Λ_R*, similarly explore the interfaces starting on $\partial\Lambda_r^c$.

Let us mention an important lemma for what comes next.

Lemma 3.2. *For every $\varepsilon > 0$, there exists $\eta > 0$ such that for any $p \in (0,1)$, any $R < L(p)$ and any boundary conditions ξ,*

$$\phi^{\xi}_{\Lambda_{2R}}[\mathcal{F} \textit{ exists but is not } \eta\textit{-well-separated}] < \varepsilon,$$

when $\mathcal{F}$ denotes the inner flower domain revealed from Λ_{2R} to Λ_R, and

$$\phi^{\xi}_{\Lambda_R^c}[\mathcal{F} \textit{ exists but is not } \eta\textit{-well-separated}] < \varepsilon,$$

when $\mathcal{F}$ denotes the outer flower domain revealed from Λ_R to Λ_{2R}.

Proof. We treat the case of outer flower domains; that of inner domains can be solved similarly. For $\mathcal{F}$ to exist but not be η-well-separated, it needs to contain a primal or dual petal with $|a_i - a_{i+1}| < \eta R$. We will exclude below the possibility of a small dual petal; the case of a primal one is identical.

Divide $\partial\Lambda_{2R}$ into arcs $\ell_1, \dots, \ell_N$ of lengths between ηR and $2\eta R$ successively overlapping on a segment of length ηR, each arc being included in a single side of $\partial\Lambda_{2R}$. Let $A^{\square}_{101}(\ell_i)$ be the event that

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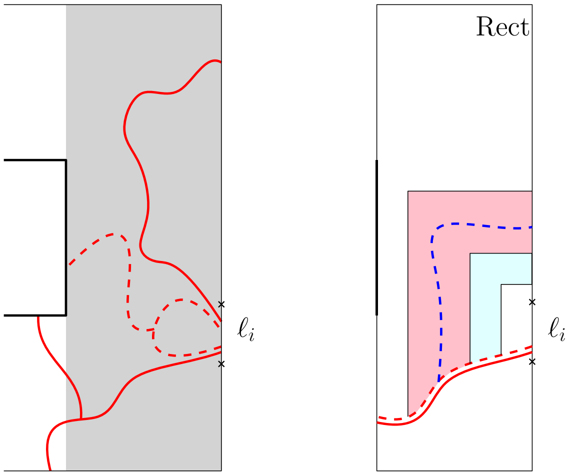

Figure 2. *Left: If there exists a dual petal with both endpoints in an interval ℓ_i, then the three-arm event $A^{\square}_{101}(\ell_i)$ occurs. Right: When $\widetilde{A}^{\square}_{10}(\ell_i)$ occurs, condition on the lowest horizontal interface crossing* Rect *from ℓ_i to its left side. Above it, we find order $\log\eta$ disjoint tubes that when dually crossed, prevent the existence of a second primal arm from ℓ_i to the left side of* Rect*. Due to equation (RSW), each such tube is dually crossed with positive probability, independently of the others. Indeed, the boundary conditions on $\partial\Lambda_R^c$, as well as those induced by the conditioning on the lowest interface, may have both free and wired parts, but only the free boundary conditions border the tubes. This suffices to obtain the polynomial term in equation (3.1).*

there exist two primal arms with a dual one in between contained in $\mathrm{Ann}(R, 2R)$ from ℓ_i to $\partial\Lambda_R$. Notice that if $\mathcal{F}$ contains a dual petal with endpoints at a distance smaller than ηR of each other, then there exists at least one arc ℓ_i for which $A^{\square}_{101}(\ell_i)$ occurs. Our goal is therefore to bound the probability of the events $A^{\square}_{101}(\ell_i)$.

Fix ℓ_i, and let $\mathrm{Rect} \subset \Lambda_{2R}$ be the rectangle of size $R \times 4R$ with one of the long sides being the side of $\partial\Lambda_{2R}$ containing ℓ_i (we use here that ℓ_i is included in one of the sides of Λ_{2R}). Define $\widetilde{A}^{\square}_{10}(\ell_i)$ as the event that there exist two paths, one primal and one dual, starting on ℓ_i and crossing Rect to its opposite side. See Figure 2 for an illustration. From equation (RSW), it is easily deduced by an exploration argument that

$$\phi^{\xi}_{\Lambda_R^c}[A^{\square}_{101}(\ell_i)] \le C_0\eta^{c_0}\,\phi^{\xi}_{\Lambda_R^c}[\widetilde{A}^{\square}_{10}(\ell_i)], \tag{3.1}$$

for universal constants $c_0, C_0 > 0$. Indeed, explore first the interface from ℓ_i to the opposite side of Rect closest to a chosen endpoint of ℓ_i. The existence of such an interface is synonymous to $\widetilde{A}^{\square}_{10}(\ell_i)$. Condition next on this interface, and bound the probability of the existence of the second primal arm as follows. Consider disjoint annuli around ℓ_i, at dyadic scales between $2\eta R$ and R, and apply equation (RSW) in each annulus to conclude that under this conditioning, the second primal arm occurs with a polynomially small probability. Observe that the conditioning on the interface induces both positive and negative information on the remaining edges, but the information favourable to the existence of a primal path is, for every dyadic annulus, at a macroscopic distance. This allows one to apply equation (RSW) in the annuli; see also the caption of Figure 2 for explanations.

To bound the probabilities of the events $\widetilde{A}^{\square}_{10}(\ell_i)$, define $\mathbf{N}$ to be the number of disjoint clusters crossing Rect. Then

$$\sum_{i=1}^{k} \phi^{\xi}_{\Lambda_R^c}[\widetilde{A}^{\square}_{10}(\ell_i)] \le 4\phi^{\xi}_{\Lambda_R^c}[\mathbf{N}] \le C_1, \tag{3.2}$$

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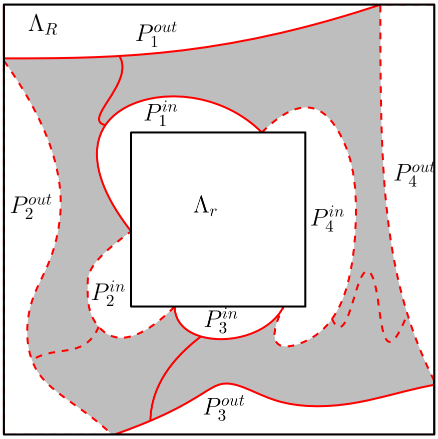

Figure 3. *A double four-petal flower domain. The two connected components of the white area are $\mathcal{F}_{in}$ and $\mathcal{F}_{out}$.*

where the first inequality is a deterministic bound (obtained by bounding, for each side of $\partial\Lambda_{2R}$, the sum of the ℓ_i included in it by the random variable $\mathbf{N}$ or a rotation of it) and the second uniform bound on the expectation of $\mathbf{N}$, which is a standard consequence of equation (RSW) sketched below: observe that there exists $c_2 > 0$ such that for every $k \geq 0$,

$$\phi^{\xi}_{\Lambda_R^c}[\mathbf{N} \geq k+1 | \mathbf{N} \geq k] \leq 1 - c_2. \tag{3.3}$$

Indeed, conditionally on the k top-most clusters crossing Rect, observe that the complement Ω in Rect of these clusters is a subset of Rect with free boundary conditions on the part of the boundary that lies strictly inside Rect. Then a dual path disconnecting the two sides of Rect in Ω occurs with probability at least $c_2 > 0$ by equation (RSW). This proves equation (3.3), which in turn implies equation (3.2).

Combining equations (3.1) and (3.2) and adding a factor 2 to account for the existence of small primal petals, we find

$$\phi^{\xi}_{\Lambda_R^c}[\mathcal{F} \text{ exists but is not } \eta\text{-well-separated}] \leq 2C_0\eta^{c_0} \sum_{i=1}^{k} \phi^{\xi}_{\Lambda_R^c}[\widetilde{A}^{\square}_{10}(\ell_i)] \leq 2C_0C_1\eta^{c_0}.$$

Fixing η small enough concludes the proof. □

Definition 3.3 (Double four-petal flower domain). Fix $1 \leq r < R$. We say that there exists a *double four-petal flower domain* between Λ_r and Λ_R if

- the outer flower domain $\mathcal{F}_{out}$ revealed from $\Lambda_{(rR)^{1/2}}$ to Λ_R exists, is $1/2$-well-separated and has exactly four petals $P_1^{out}, \dots, P_4^{out}$;
- the inner flower domain $\mathcal{F}_{in}$ revealed from $\Lambda_{(rR)^{1/2}}$ to Λ_r exists, is $1/2$-well-separated and has exactly four petals $P_1^{in}, \dots, P_4^{in}$;
- P_1^{in} is connected to P_1^{out} and P_3^{in} to P_3^{out} in $\omega \cap \mathcal{F}_{in}^c \cap \mathcal{F}_{out}^c$;
- P_2^{in} is connected to P_2^{out} and P_4^{in} to P_4^{out} in $\omega^* \cap \mathcal{F}_{in}^c \cap \mathcal{F}_{out}^c$.

See Figure 3 for an example.

The advantage of the double four-petal flower domain is that it can be revealed from $\partial\Lambda_{(rR)^{1/2}}$ towards the inside and outside and limits the interaction between the configurations in $\mathcal{F}_{in}$ and $\mathcal{F}_{out}$.

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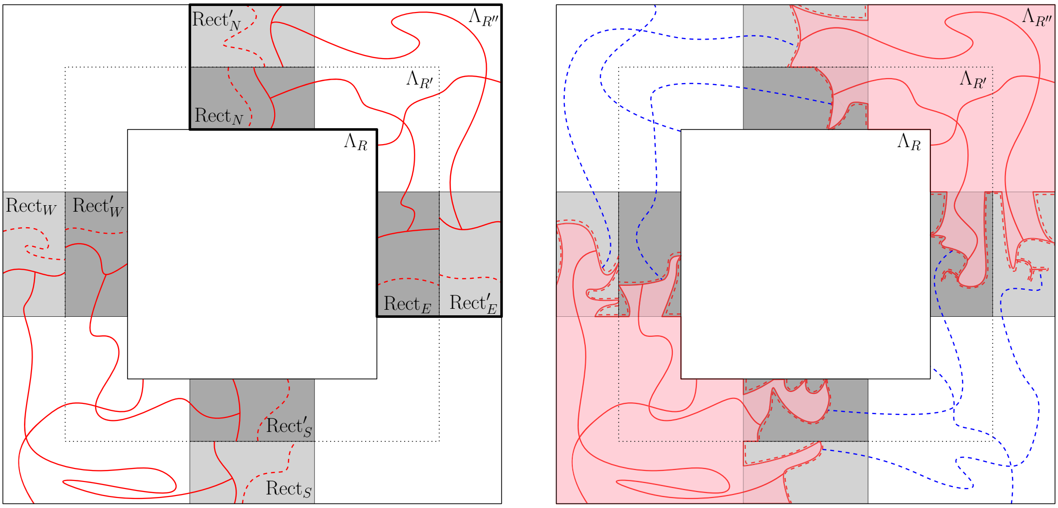

Figure 4. *Left: The event* $E_1 \cap E_3$*. The bold contour delimits* $\mathrm{Lshape}_{NW} \cup \mathrm{Lshape}'_{NW}$*. Right: The configuration in the pink area and its boundary are determined by the conditioning on* C_1, C'_1, C_3, C'_3 *and* ω *inside* $(\mathrm{Lshape}_{NE} \cup \mathrm{Lshape}'_{NE}) \setminus (\mathrm{Rect}_E \cup \mathrm{Rect}_N \cup \mathrm{Rect}'_E \cup \mathrm{Rect}'_N)$*. This part of the configuration is sufficient to ensure that* $E_1 \cap E_3$ *occurs, and the measure in the rest of the space is an FK-percolation measure with prescribed boundary conditions. The dual blue paths occur with positive probability under this conditioning; they produce* E_2 *and* E_4*. When all of* $E_1, \dots, E_4$ *occur, there exists a double four-petal flower domain between* Λ_R *and* $\Lambda_{R''}$*.*

Lemma 3.4. *For any* $\eta > 0$*, there exists* $c = c(\eta) > 0$ *such that for any* $p \in (0,1)$*, any* $R < L(p)$ *large enough and any boundary conditions* ξ *on* $\mathrm{Ann} := \mathrm{Ann}((1-\eta)R, (1+2\eta)R)$*,*

$$\phi^{\xi}_{\mathrm{Ann}}[\textit{there exists a double four-petal flower domain between } \Lambda_R \textit{ and } \Lambda_{(1+\eta)R}] > c.$$

Proof. We recommend inspecting Figure 4 while reading this proof. Set $R'' := (1+\eta)R$ and $R' := (RR'')^{1/2}$. Consider the rectangles

$$\mathrm{Rect}_E := [R, R'] \times [-R/2, R/2] \quad \text{and} \quad \mathrm{Rect}'_E := [R', R''] \times [-R/2, R/2]$$

and their rotations Rect_N, Rect_W, Rect_S and Rect'_N, Rect'_W, Rect'_S by angles $\pi/2$, π and $3\pi/2$, respectively. Also define the L-shaped regions

$$\mathrm{Lshape}_{NE} := [R, R'] \times [-R/2, R'] \cup [-R/2, R'] \times [R, R'] \quad \text{and}$$
$$\mathrm{Lshape}'_{NE} := [R', R''] \times [-R/2, R''] \cup [-R/2, R''] \times [R', R'']$$

and their rotations Lshape_{NW}, Lshape_{SW}, Lshape_{SE} and Lshape'_{NW}, Lshape'_{SW}, Lshape'_{SE} by angles $\pi/2$, π and $3\pi/2$, respectively. Let E_1 be the event that there exist

- dual crossings in Rect_E and Rect_N between Λ_R and $\partial\Lambda_{R'}$,
- dual crossings in Rect'_E and Rect'_N between $\Lambda_{R'}$ and $\partial\Lambda_{R''}$,
- primal crossings in Rect_E and Rect_N between Λ_R and $\partial\Lambda_{R'}$, connected by a primal path in Lshape_{NE},
- primal crossings in Rect'_E and Rect'_N between $\Lambda_{R'}$ and $\partial\Lambda_{R''}$, connected by a primal path in Lshape'_{NE},
- a primal path between Λ_R and $\partial\Lambda_{R''}$ contained in $(\mathrm{Lshape}_{NE} \cup \mathrm{Lshape}'_{NE}) \setminus (\mathrm{Rect}_E \cup \mathrm{Rect}_N \cup \mathrm{Rect}'_E \cup \mathrm{Rect}'_N)$.

Also, let E_3 be the rotation by π of E_1.

Due to equation (RSW), there exists some $c_0 > 0$ depending only on η such that

$$\phi^{\xi}_{\mathrm{Ann}}[E_1 \cap E_3] \geq c_0. \tag{3.4}$$

Now, when $E_1 \cap E_3$ occurs, let C_1 be the unique connected component of Lshape_{NE} that contains both crossings in Rect_E and Rect_N from Λ_R to $\partial\Lambda_{R'}$. Define C'_1 in the same way for the outer annulus and C_3 and C'_3 by the same definition applied to the rotation by π of the configuration. When $E_1 \cap E_3$ fails to occur, set $\mathsf{C}_1 = \mathsf{C}'_1 = \mathsf{C}_3 = \mathsf{C}'_3 = \emptyset$. Furthermore, when $E_1 \cap E_3$ does occur, let E_2 be the event that there exist

- a dual path between $\partial\mathsf{C}_1$ and $\partial\mathsf{C}_3$ contained in Lshape_{NW},
- a dual path between $\partial\mathsf{C}'_1$ and $\partial\mathsf{C}'_3$ contained in Lshape'_{NW},
- a dual path between Λ_R and $\partial\Lambda_{R''}$ contained in $(\mathrm{Lshape}_{NW} \cup \mathrm{Lshape}'_{NW}) \setminus (\mathrm{Rect}_N \cup \mathrm{Rect}_W \cup \mathrm{Rect}'_N \cup \mathrm{Rect}'_W)$.

Define E_4 in the same way for the configuration rotated by π.

Observe now that conditionally on any realisation of C_1, C'_1, C_3, C'_3 different from $\emptyset$ and on the configuration inside $(\mathrm{Lshape}_{NE} \cup \mathrm{Lshape}'_{NE}) \setminus (\mathrm{Rect}_E \cup \mathrm{Rect}_N \cup \mathrm{Rect}'_E \cup \mathrm{Rect}'_N)$, the configuration in the rest of the space is that of an FK-percolation with boundary conditions given by the conditioning. Indeed, the conditioning suffices to guarantee the occurrence of $E_1 \cap E_3$.

Thus, due to equation (RSW), by conditioning on C_1, C'_1, C_3, C'_3 and the configuration inside $(\mathrm{Lshape}_{NE} \cup \mathrm{Lshape}'_{NE}) \setminus (\mathrm{Rect}_E \cup \mathrm{Rect}_N \cup \mathrm{Rect}'_E \cup \mathrm{Rect}'_N)$ and then averaging the result, we find that

$$\phi^{\xi}_{\mathrm{Ann}}[E_2 \cap E_4 \,|\, E_1 \cap E_3] \geq c_1 \tag{3.5}$$

for some $c_1 > 0$ depending only on η. Thus, we conclude that

$$\phi^{\xi}_{\mathrm{Ann}}[E_1 \cap E_2 \cap E_3 \cap E_4] \geq c_0 c_1. \tag{3.6}$$

Finally, notice that when $E_1 \cap \cdots \cap E_4$ occurs, there exists a double four-petal flower domain between Λ_R and $\Lambda_{R''}$. Indeed, the events E_1 and E_3 guarantee the existence of two primal petals for $\mathcal{F}^{\mathrm{in}}$ and two primal petals for $\mathcal{F}^{\mathrm{out}}$, with endpoints in the rectangles $\mathrm{Rect}_{.}$ and $\mathrm{Rect}'_{.}$, respectively. Moreover, E_2 and E_4 guarantee that these primal petals are separated by exactly one dual petal on each side. □

3.2. *Boosting pair of boundary conditions*

The goal of this section is to study how changes in boundary conditions impact crossing probabilities.

Definition 3.5. A *boosting pair* of boundary conditions for a flower domain $\mathcal{F}$ is a pair of boundary conditions (ξ, ξ') such that

- ξ and ξ' are compatible with $\mathcal{F}$;
- $\xi \leq \xi'$;
- there exist two primal petals of $\mathcal{F}$ that are wired together in ξ' but not in ξ.

In a slight abuse of notation, we will henceforth also call a pair (ζ, ζ') of boundary conditions on $\mathcal{F}$*boosting* if there exists a boosting pair (ξ, ξ') of boundary conditions on $\mathcal{F}$ (in the sense of the definition above) such that $\zeta \leq \xi < \xi' \leq \zeta'$. In particular, ζ and ζ' need not be compatible with $\mathcal{F}$.

Recall from Section 2.2 that A_R is the event that there exists a circuit in $\mathrm{Ann}(R, 2R)$ surrounding 0. The object of this section is to prove the following theorem.

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Theorem 3.6. *Fix $q \in (1,4]$. For every $\eta > 0$, there exists $\delta = \delta(\eta, q) > 0$ such that the following holds.*

(i) *For every $p \in (0,1)$, every $R \le L(p)$, every η-well-separated inner flower domain $\mathcal{F}$ on Λ_{2R}, every boosting pair (ξ, ξ') of boundary conditions on $\mathcal{F}$ and every η-regular quad $(\mathcal{D}, a, b, c, d)$ of size R,*

$$\phi_{\mathcal{F}}^{\xi'}[\mathcal{C}(\mathcal{D})] \ge \phi_{\mathcal{F}}^{\xi}[\mathcal{C}(\mathcal{D})] + \delta, \tag{3.7}$$

$$\phi_{\mathcal{F}}^{\xi'}[A_R] \ge \phi_{\mathcal{F}}^{\xi}[A_R] + \delta. \tag{3.8}$$

(ii) *For every $p \in (0,1)$, every $R \le L(p)$, every η-well-separated outer flower domain $\mathcal{F}$ on Λ_R, every boosting pair (ξ, ξ') of boundary conditions on $\mathcal{F}$ and every η-regular quad $(\mathcal{D}, a, b, c, d)$ of size R translated in such a way that it is contained in* $\mathrm{Ann}(2R, 4R)$*,*

$$\phi_{\mathcal{F}}^{\xi'}[\mathcal{C}(\mathcal{D})] \ge \phi_{\mathcal{F}}^{\xi}[\mathcal{C}(\mathcal{D})] + \delta, \tag{3.9}$$

$$\phi_{\mathcal{F}}^{\xi'}[A_R] \ge \phi_{\mathcal{F}}^{\xi}[A_R] + \delta. \tag{3.10}$$

In light of the RSW theory, the crossing probabilities $\phi_{\mathcal{F}}^{\xi}[\mathcal{C}(\mathcal{D})]$ and $\phi_{\mathcal{F}}^{\xi'}[\mathcal{C}(\mathcal{D})]$ are bounded away from 0 and 1 by constants depending only on η. Above, we are concerned with the amount by which such crossing probabilities increase when the boundary conditions change from ξ to ξ'. Indeed, the theorem states that the increase is positive uniformly in the scale, the quad to be crossed and the boosting pair of boundary conditions. The rest of the section is dedicated to proving Theorem 3.6.

The following elementary lemma is the cornerstone for the proof. For a quad $(\mathcal{D}, a, b, c, d)$, let mix be the boundary conditions on $\mathcal{D}$ corresponding to the partitions containing (ab), (cd) and singletons, and mix′ be those containing $(ab) \cup (cd)$ and singletons.

Lemma 3.7. *For every $q > 1$, $p \in (0,1)$ and every quad $(\mathcal{D}, a, b, c, d)$, we have*

$$\phi_{\mathcal{D}}^{\mathrm{mix}'}[\mathcal{C}(\mathcal{D})] = \frac{q}{1 + (q-1)\phi_{\mathcal{D}}^{\mathrm{mix}}[\mathcal{C}(\mathcal{D})]}\phi_{\mathcal{D}}^{\mathrm{mix}}[\mathcal{C}(\mathcal{D})]. \tag{3.11}$$

Notice that the ratio on the right-hand side above is always larger than 1, and considerably so when $\phi_{\mathcal{D}}^{\mathrm{mix}}[\mathcal{C}(\mathcal{D})]$ is far from 1.

Proof. Let $w(\omega) := (\frac{p}{1-p})^{|\omega|} q^{k(\omega^{\mathrm{mix}})}$ and $w'(\omega) := (\frac{p}{1-p})^{|\omega|} q^{k(\omega^{\mathrm{mix}'})}$, and observe that

$$w(\omega) = \begin{cases} w'(\omega) & \text{if } \omega \in \mathcal{C}(\mathcal{D}), \\ q w'(\omega) & \text{if } \omega \notin \mathcal{C}(\mathcal{D}). \end{cases}$$

Now, set

$$Z[\mathcal{C}(\mathcal{D})] := \sum_{\omega \in \mathcal{C}(\mathcal{D})} w(\omega), \qquad Z[\mathcal{C}(\mathcal{D})^c] := \sum_{\omega \notin \mathcal{C}(\mathcal{D})} w(\omega),$$
$$Z'[\mathcal{C}(\mathcal{D})] := \sum_{\omega \in \mathcal{C}(\mathcal{D})} w'(\omega), \qquad Z'[\mathcal{C}(\mathcal{D})^c] := \sum_{\omega \notin \mathcal{C}(\mathcal{D})} w'(\omega).$$

Then

$$\begin{aligned}\phi_{\mathcal{D}}^{\mathrm{mix}'}[\mathcal{C}(\mathcal{D})] = \frac{Z'[\mathcal{C}(\mathcal{D})]}{Z'[\mathcal{C}(\mathcal{D})] + Z'[\mathcal{C}(\mathcal{D})^c]} &= \frac{Z[\mathcal{C}(\mathcal{D})]}{Z[\mathcal{C}(\mathcal{D})] + \frac{1}{q} Z[\mathcal{C}(\mathcal{D})^c]} \\ &= \frac{\phi_{\mathcal{D}}^{\mathrm{mix}}[\mathcal{C}(\mathcal{D})]}{\phi_{\mathcal{D}}^{\mathrm{mix}}[\mathcal{C}(\mathcal{D})] + \frac{1}{q}(1 - \phi_{\mathcal{D}}^{\mathrm{mix}}[\mathcal{C}(\mathcal{D})])},\end{aligned}$$

which is the desired equality. □

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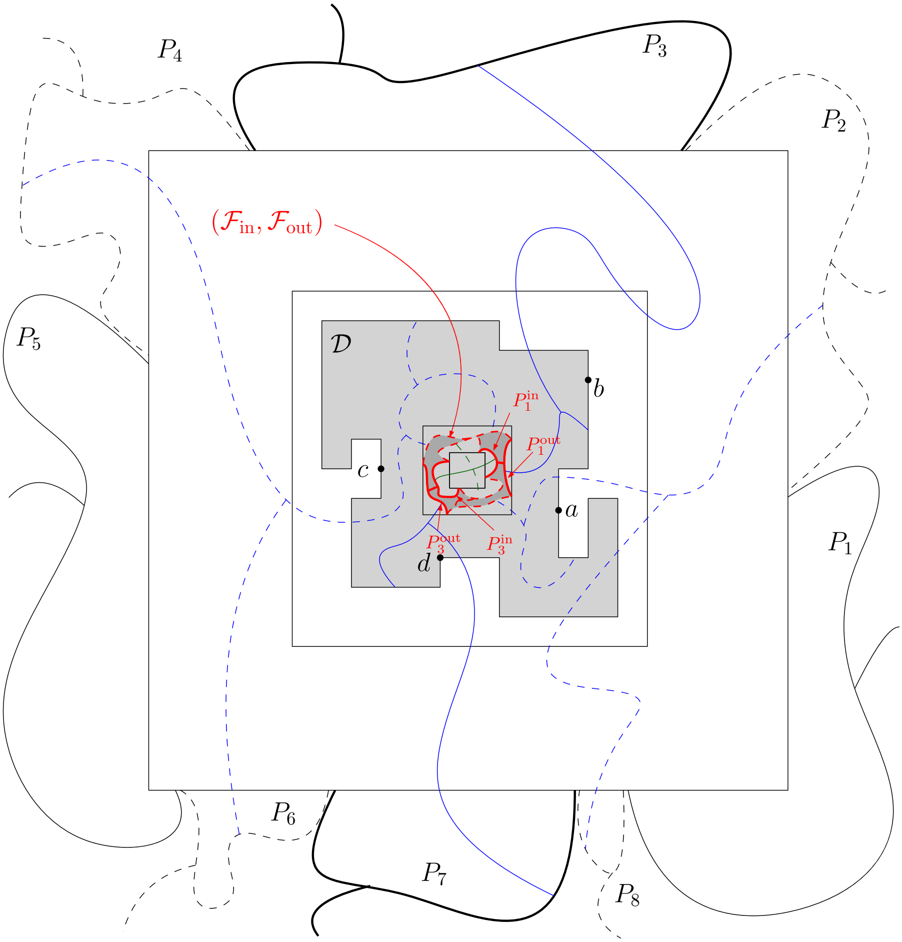

Figure 5. *The domain $\mathcal{D}$ with the four marked points a, b, c, d in grey, contained in the flower domain $\mathcal{F}$. In this case, P_3 is assumed to be connected to P_7 in ξ' but not in ξ. Red depicts the double flower domain $(\mathcal{F}_{\text{in}}, \mathcal{F}_{\text{out}})$, which is revealed first. Then conditionally on the realisation of $(\mathcal{F}_{\text{in}}, \mathcal{F}_{\text{out}})$, the conditions for the event E to occur are depicted in blue (note that the blue connections from $\mathcal{F}_{\text{out}}$ to $\mathcal{F}$ do not necessarily need to cross the arcs (ab) and (cd) of $\mathcal{D}$). At the time τ of the procedure, the red and blue parts have been revealed and only the inside of $\mathcal{F}_{\text{in}}$ is unrevealed. Then the event $\mathcal{C}(\mathcal{D})$ depends on the connection inside $\mathcal{F}_{\text{in}}$ between its primal petals, which, with positive probability, are connected in ω' but not in ω (see the green paths).*

Proof of Theorem 3.6. We will focus on inner flower domains; the case of outer flower domains is very similar. Let $\mathcal{F}$ be an η-well-separated inner flower domain on Λ_{2R}, and let (ξ, ξ') be a boosting pair of boundary conditions. Write $P_1, \ldots, P_{2k}$ for the petals of $\mathcal{F}$ in counterclockwise order, indexed in such a way that P_1 is primal. Fix i and j odd such that P_i is wired to P_j in ξ' but not in ξ. Below, $c_0, \ldots, c_4$ will denote strictly positive constants depending only on η.

We start by proving equation (3.7). We recommend taking a look at Figure 5. Fix an η-regular quad $(\mathcal{D}, a, b, c, d)$ of size R, and translate everything in such a way that the box $\Lambda_{\eta R}$ is included in $\mathcal{D}$. Consider the event E that there exists a double four-petal flower domain $(\mathcal{F}_{\text{in}}, \mathcal{F}_{\text{out}})$ between $\Lambda_{\eta R/4}$ and $\Lambda_{\eta R/2}$ and that

- P_1^{out} and P_3^{out} are connected to P_i and P_j in $\mathcal{F} \cap \mathcal{F}_{\text{out}}$, respectively;
- P_1^{out} and P_3^{out} are not connected to each other or any other primal petal of $\mathcal{F}$ in $\mathcal{F} \cap \mathcal{F}_{\text{out}}$;

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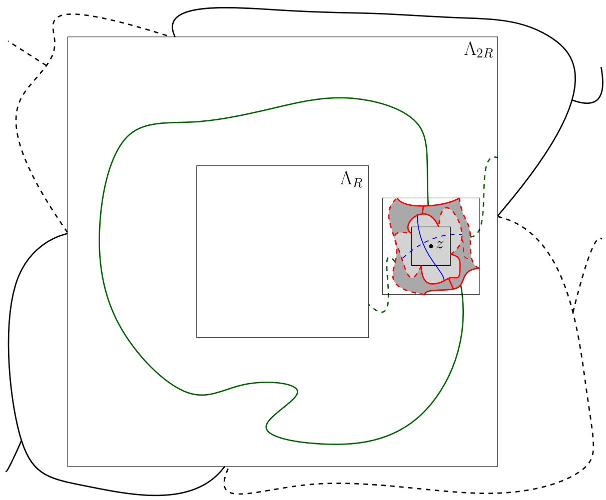

Figure 6. *A depiction of the annulus configuration in the proof of equation (3.8). In black, the flower domain $\mathcal{F}$. The red part corresponds to the double flower domain $(\mathcal{F}_{\text{in}}, \mathcal{F}_{\text{out}})$ revealed at time τ_1. The blue part corresponds to what is revealed at time τ_2: that is, a connection in ω' between two wired petals of $\mathcal{F}_{\text{in}}$ together with a dual connection in ω^* between two free petals. Then after time τ_2, we construct the event H (in green).*

- P_1^{out} and P_3^{out} are connected to the arcs (ab) and (cd) in $\mathcal{D} \cap \mathcal{F}_{\text{out}}$, respectively;
- P_2^{out} and P_4^{out} are dually connected to the arcs (bc) and (da) in $\mathcal{D} \cap \mathcal{F}_{\text{out}}$.

The separation of petals together with equation (RSW) and Lemma 3.4 give

$$\phi_{\mathcal{F}}^{\xi}[E] \geq c_0 > 0. \tag{3.12}$$

Consider the coupling $\mathbb{P}_{\mathbf{T}}$ between $\phi_{\mathcal{F}}^{\xi}$ and $\phi_{\mathcal{F}}^{\xi'}$ obtained by first exploring $(\mathcal{F}_{\text{in}}, \mathcal{F}_{\text{out}})$ in ω, then revealing all the edges in $\mathcal{F}_{\text{out}} \cap \mathcal{F}$, and then revealing all those inside $\mathcal{F}_{\text{in}}$. Let τ be the stopping time corresponding to the end of the second step of this exploration.

Suppose that $\omega_{[\tau]} \in E$. Then $\mathcal{D}$ is crossed in ω if and only if the petals P_1^{in} and P_3^{in} are connected inside $\mathcal{F}_{\text{in}}$. Write $\zeta < \zeta'$ for the two boundary conditions on $\mathcal{F}_{\text{in}}$, which are coherent with the flower domain structure (with P_1^{in} wired to P_3^{in} in ζ' but not in ζ). Then the boundary conditions induced by $\omega_{[\tau]}$ and ξ on $\mathcal{F}_{\text{in}}$ are ζ. Moreover, since $\omega' \geq \omega$, and due to the wiring of P_i and P_j in ξ', $\omega'_{[\tau]}$ induces the boundary conditions on $\mathcal{F}_{\text{in}}$ that dominate ζ'. Thus,

$$\begin{aligned}
\phi_{\mathcal{F}}^{\xi'}[\mathcal{C}(\mathcal{D})] - \phi_{\mathcal{F}}^{\xi}[\mathcal{C}(\mathcal{D})] &= \mathbb{P}_{\mathbf{T}}[\omega' \in \mathcal{C}(\mathcal{D}),\ \omega \notin \mathcal{C}(\mathcal{D})] \\
&\geq \mathbb{E}_{\mathbf{T}}\Big[\big(\phi_{\mathcal{F}_{\text{in}}}^{\zeta'}[P_1^{\text{in}} \longleftrightarrow P_3^{\text{in}}] - \phi_{\mathcal{F}_{\text{in}}}^{\zeta}[P_1^{\text{in}} \longleftrightarrow P_3^{\text{in}}]\big)\mathbf{1}_{\omega_{[\tau]} \in E}\Big] \\
&\geq c_1 \mathbb{P}_{\mathbf{T}}[\omega_{[\tau]} \in E] \geq c_1 c_0 > 0,
\end{aligned}$$

where $c_1 > 0$ is given by Lemma 3.7 and equation (RSW). This concludes the proof of equation (3.7).

We turn to equation (3.8) and refer the reader to Figure 6. Write z for the point $(3R/2, 0)$. Consider the coupling $\mathbb{P}_{\mathbf{T}}$ between $\phi_{\mathcal{F}}^{\xi}$ and $\phi_{\mathcal{F}}^{\xi'}$ obtained by first exploring the double flower domain $(\mathcal{F}_{\text{in}}, \mathcal{F}_{\text{out}})$ in ω between $\Lambda_{R/16}(z)$ and $\Lambda_{R/8}(z)$, then the configurations inside $\mathcal{F}_{\text{in}}$ and finally those in $\mathcal{F}_{\text{out}}$. If no double flower domain exists, reveal all remaining edges in arbitrary order. Write τ_1 for the stopping time marking the end of the exploration of $(\mathcal{F}_{\text{in}}, \mathcal{F}_{\text{out}})$ and τ_2 for the stopping time after further exploring $\mathcal{F}_{\text{in}}$.

Condition on a pair of configurations $(\omega_{[\tau_1]}, \omega'_{[\tau_1]})$ such that the double flower domain exists, and write $\zeta < \zeta'$ for the two boundary conditions on $\mathcal{F}_{\text{in}}$ that are coherent with the flower domain structure.

The configuration ω inside $\mathcal{F}_{\text{in}}$ is sampled according to a convex combination of $\phi^{\zeta}_{\mathcal{F}_{\text{in}}}$ and $\phi^{\zeta'}_{\mathcal{F}_{\text{in}}}$. Indeed, the coefficient λ for the latter measure is given by the probability that P_1^{out} is connected to P_3^{out} in $\mathcal{F}_{\text{out}}$, including via the boundary conditions ξ. Similarly, the law of ω' inside $\mathcal{F}_{\text{in}}$ dominates a convex combination of $\phi^{\zeta}_{\mathcal{F}_{\text{in}}}$ and $\phi^{\zeta'}_{\mathcal{F}_{\text{in}}}$, with the coefficient λ' for the latter given by the probability that P_1^{out} is connected to P_3^{out} in $\mathcal{F}_{\text{out}}$, including via the boundary conditions ξ'.

By an argument similar to the first point of the proof involving the event E, it may be shown that there exists a uniformly positive probability $c_2 > 0$ for P_1^{out} and P_3^{out} to be connected in $\mathcal{F}_{\text{out}}$ to P_i and P_j, respectively, but not to each other. Thus, $\lambda' \geq \lambda + c_2$. Applying Lemma 3.7 and equation (RSW), we find

$$\begin{aligned}\mathbb{P}_{\mathbf{T}}[P_1^{\text{in}} \xleftrightarrow{\omega'\cap\mathcal{F}_{\text{in}}} P_3^{\text{in}} \text{ but } P_1^{\text{in}} \overset{\omega\cap\mathcal{F}_{\text{in}}}{\not\longleftrightarrow} P_3^{\text{in}} \,|\, \omega_{[\tau_1]}, \omega'_{[\tau_1]} \text{ s.t. } (\mathcal{F}_{\text{in}}, \mathcal{F}_{\text{out}}) \text{ exists}]\\ \geq (\phi^{\zeta'}_{\mathcal{F}_{\text{in}}}[P_1^{\text{in}} \xleftrightarrow{\mathcal{F}_{\text{in}}} P_3^{\text{in}}] - \phi^{\zeta}_{\mathcal{F}_{\text{in}}}[P_1^{\text{in}} \xleftrightarrow{\mathcal{F}_{\text{in}}} P_3^{\text{in}}])(\lambda' - \lambda) \geq c_3,\end{aligned}$$

for some $c_3 > 0$.

Write F for the event that $(\mathcal{F}_{\text{in}}, \mathcal{F}_{\text{out}})$ exists and that P_1^{in} is connected to P_3^{in} inside $\mathcal{F}_{\text{in}}$ in ω' but not in ω. This event is measurable in terms of the configurations at the stopping time τ_2. Finally, write H for the event that in ω,

- in $\mathcal{F}_{\text{out}} \cap \text{Ann}(R, 2R)$, P_1^{out} is connected to P_3^{out} by a primal path,
- P_2^{out} is connected to Λ_R by a dual path and
- P_4^{out} is connected to Λ_{2R}^c by a dual path.

Notice that if H occurs, the primal path connecting P_1^{out} to P_3^{out} needs to 'wind around' Λ_R. Thus, H may be understood as the connection between P_1^{in} and P_3^{in} in $\mathcal{F}_{\text{in}}$ being 'pivotal' for A_R (for ω). By equation (RSW),

$$\mathbb{P}_{\mathbf{T}}[\omega \in H \,|\, (\omega_{[\tau_2]}, \omega'_{[\tau_2]}) \text{ such that } F \text{ occurs}] > c_4,$$

for some $c_4 > 0$. Moreover, when F and H occur, then A_R occurs for ω' but not for ω. Thus,

$$\phi^{\xi'}_{\mathcal{F}}[A_R] - \phi^{\xi}_{\mathcal{F}}[A_R] \geq \mathbb{P}_{\mathbf{T}}[\omega \in H \text{ and } F] \geq c_3\, c_4 > 0,$$

which is the desired conclusion. □

Remark 3.8. The proof of equation (3.8) may appear contradictory, as we are first arguing that P_1^{out} and P_3^{out} may appear wired in ω' but not in ω and then we focus on the event $\omega \in H$, which ensures that P_1^{out} and P_3^{out} are connected in both ω and ω'. The reader should keep in mind that the configurations ω and ω' in $\mathcal{F}_{\text{in}}$ are sampled before sampling the configurations in $\mathcal{F}_{\text{out}}$, and their laws are obtained by averaging over the possible configurations in $\mathcal{F}_{\text{out}}$.

Alternatively, one may imagine a coupling where $(\mathcal{F}_{\text{in}}, \mathcal{F}_{\text{out}})$ is revealed first, then the configurations in $\mathcal{F}_{\text{out}}$ are revealed, then those in $\mathcal{F}_{\text{in}}$ are revealed, and finally the configurations in $\mathcal{F}_{\text{out}}$ are resampled. In this context, we are investigating the situation where in the first sampling of the configurations in $\mathcal{F}_{\text{out}}$, P_1^{out} and P_3^{out} are connected to P_i and P_j, respectively, but not to each other, then, in the sampling inside $\mathcal{F}_{\text{in}}$, P_1^{in} and P_3^{in} are connected in ω' but not in ω, and finally, in the second sampling in $\mathcal{F}_{\text{out}}$, H occurs for ω.

3.3. Crossing quads produce boosting pairs

This section is concerned with the following result, which roughly states that conditioning on the existence of the crossing of a quad has the same effect as a boosting pair of boundary conditions.

Proposition 3.9. *For any $\eta > 0$, there exists a constant $c = c(\eta, q) > 0$ such that the following holds. Fix p, and let $(\mathcal{D}, a, b, c, d)$ be an η-regular discrete quad at scale $R \leq L(p)$. Then there exists a coupling via decision trees $\mathbb{P}$ of ϕ and $\phi[\cdot\,|\mathcal{C}(\mathcal{D})]$ and a stopping time τ such that when $\tau < \infty$, $\mathcal{F}_\tau = \mathbb{Z}^2 \setminus e_{[\tau]}$*

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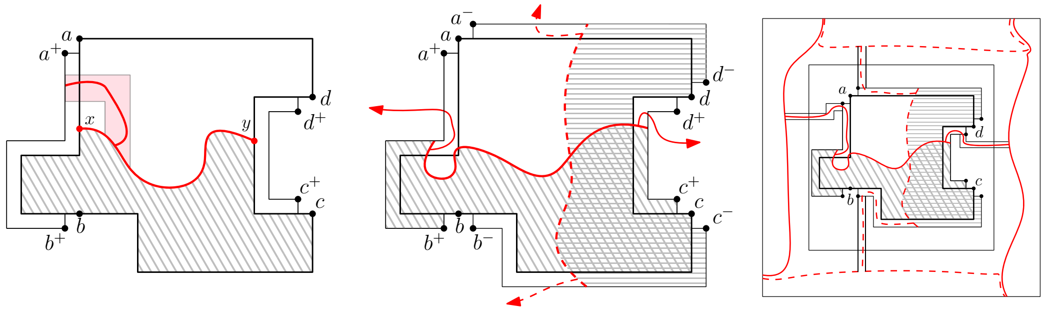

Figure 7. *Left: For a fixed lowest crossing Γ of $\mathcal{D}$ with the endpoint x at a distance at least $\sqrt{mR}$ from a, one may identify order $\log R/m$ disjoint quads – such as the pink one – which, if crossed, create a connection between Γ and (a^+b^+). Each such quad has a uniformly positive probability of being crossed, independently of the other quads. Middle: At the time $\tau_2 < \infty$, when Γ^+ and Γ^- have been revealed, the region of $\mathcal{D}^+ \setminus \mathcal{D}$ above Γ^+ and that of $\mathcal{D}^- \setminus \mathcal{D}$ left of Γ^- are unrevealed. Right: The unrevealed regions may be used to connect Γ^+ to the primal petals of $\mathcal{F}$ in ω and Γ^- to the dual petals of $\mathcal{F}$ in ω^*. This ensures that the boundary conditions induced by $\omega'_{[\tau]}$ on $\mathcal{F}$ are a boost of those induced by $\omega_{[\tau]}$.*

is a $1/2$-well-separated outer flower domain on Λ_{2R} (with $\Lambda_{3R/2} \subset \mathcal{F}_\tau^c$), and the boundary conditions induced by $\omega'_{[\tau]}$ on $\mathcal{F}_\tau$ are a boost of those induced by $\omega_{[\tau]}$. Finally,

$$\mathbb{P}[\tau < \infty] \geq c.$$

Furthermore, the same is true when $\mathcal{C}(\mathcal{D})$ is replaced by $A_{R/2}$.

The proof of Proposition 3.9 is based on the following lemma, which allows us to 'lengthen' crossings at a small cost. For an η-regular discrete quad $(\mathcal{D}, a, b, c, d)$ at some scale R and for some $m \leq \eta R/4$, define two modified quads $(\mathcal{D}^+, a^+, b^+, c^+, d^+)$ and $(\mathcal{D}^-, a^-, b^-, c^-, d^-)$ as follows (see Figure 7 for an illustration). The domains $\mathcal{D}^+$ and $\mathcal{D}^-$ are formed by the union of $\mathcal{D}$ with the set of edges of $\mathbb{Z}^2 \setminus \mathcal{D}$ that are at a ℓ^∞ distance at most m from the arcs (ab) and (cd) (respectively (bc) and (da)), but at least distance m from a, b, c and d. The point a^+ is the first point of $\partial\mathcal{D}^+$ in counterclockwise order after a that is at a distance m from $\partial\mathcal{D}$; the point b^+ is the last such point before b. The points c^+ and d^+ are defined similarly in terms of c and d, respectively. A similar definition applies to a^-, b^-, c^- and d^-.

Remark 3.10. The choice of $m \leq \eta R/4$ and the fact that $\mathcal{D}$ is η-regular guarantee that $\mathcal{D}^+ \setminus \mathcal{D}$ and $\mathcal{D}^- \setminus \mathcal{D}$ are at a distance at least $\eta R/4$ from each other and are each made of two separate connected components. In particular, the complement of $\mathcal{D}^+ \cup \mathcal{D}^-$ is connected.

Notice that $(\mathcal{D}^+, a^+, b^+, c^+, d^+)$ and $(\mathcal{D}^-, a^-, b^-, c^-, d^-)$ are both discrete quads and that

$$\mathcal{C}(\mathcal{D}^+) \subset \mathcal{C}(\mathcal{D}) \subset \mathcal{C}(\mathcal{D}^-).$$

While the lemma below is valid for all $m < \eta R/4$, we will eventually use it for $m = c_0 R$ for some small constant c_0. As such, $\mathcal{D}^+$ and $\mathcal{D}^-$ are roughly[2] η'-regular at scale R for $\eta' = c_0\eta \ll \eta$.

Lemma 3.11. *For any $\eta > 0$, there exist $C = C(\eta) > 0$, $\varepsilon = \varepsilon(\eta) > 0$ such that the following holds. For any $p \in (0, 1)$, $R \leq L(p)$, $m < \eta R/4$ and any η-regular discrete quad $(\mathcal{D}, a, b, c, d)$ at scale $R \leq L(p)$,*

$$\phi[\mathcal{C}(\mathcal{D})] - \phi[\mathcal{C}(\mathcal{D}^+)] \leq C(m/R)^\varepsilon, \tag{3.13}$$

$$\phi[\mathcal{C}(\mathcal{D}^-)] - \phi[\mathcal{C}(\mathcal{D})] \leq C(m/R)^\varepsilon. \tag{3.14}$$

[2]Formally, $\mathcal{D}^+$ and $\mathcal{D}^-$ may spill over into Λ_R^c, so formally these domains are η'-regular at scale $2R$.

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Proof. We will focus on the first inequality; the second one is the same inequality applied to the dual model. Since $\mathcal{C}(\mathcal{D}^+) \subset \mathcal{C}(\mathcal{D})$, we are searching for an upper bound on $\phi[\mathcal{C}(\mathcal{D}) \setminus \mathcal{C}(\mathcal{D}^+)]$. When $\mathcal{C}(\mathcal{D})$ occurs, let Γ be the 'lowest' crossing of $\mathcal{D}$: that is, the open path closest to the arc (bc) with endpoints $x \in (ab)$ and $y \in (cd)$. Write $\mathsf{Under}(\Gamma)$ for the set of edges of $\mathcal{D}$ between Γ and (bc).

If $\mathcal{C}(\mathcal{D}) \setminus \mathcal{C}(\mathcal{D}^+)$ occurs, then at least one of the following four events needs to occur:

(i) x is at a distance at most $\sqrt{mR}$ from a;
(ii) x is at a distance at least $\sqrt{mR}$ from a, but Γ is not connected to (a^+b^+) in $\mathcal{D}^+$;
(iii) y is at a distance at most $\sqrt{mR}$ from d;
(iv) y is at a distance at least $\sqrt{mR}$ from d, but Γ is not connected to (c^+d^+) in $\mathcal{D}^+$.

Next, we bound the probability of each of the four events described above.

If the event in (i) occurs, there exists a primal arm, namely Γ, contained in $\mathcal{D}$ from $\Lambda_{\sqrt{mR}}(a)$ to distance ηR; and therefore, using equation (2.2), we get

$$\phi[(i)\text{ occurs}] \le \pi_1(p; \sqrt{mR}, \eta R) \le C(\eta)(m/R)^{\varepsilon}.$$

To control the probability of the event (ii), we claim that there exist constants $C, \varepsilon > 0$ such that

$$\phi[\Gamma \overset{\omega\cap(\mathcal{D}^+\setminus\mathsf{Under}(\Gamma))}{\not\longleftrightarrow} (a^+b^+) \mid \Gamma \text{ and } \omega \text{ on } \mathsf{Under}(\Gamma)] \le C(m/R)^{\varepsilon}.$$

Indeed, by considering the intersections of $\mathcal{D}^+$ with the dyadic annuli contained between $\Lambda_m(x)$ and $\Lambda_{\sqrt{mR}}(x)$, one may create order $\log R/m$ disjoint quads such that when any one of them is crossed by a primal path, that induces a connection between Γ and (d^+a^+) in $\mathcal{D}^+$. Then equation (RSW) applies in these quads as the conditioning above only induces negative information at a macroscopic distance from the quads; see Figure 7 (left diagram). Therefore, that none of the quads is primally crossed occurs with a probability that is exponential in the number of quads or equivalently polynomial in m/R.

The bounds above also apply to the events in (iii) and (iv). When combined using a union bound, we find

$$\phi[\mathcal{C}(\mathcal{D}) \setminus \mathcal{C}(\mathcal{D}^+)] \le 4C(m/R)^{\varepsilon}.$$

Changing the value of C gives the proof. □

Remark 3.12. Alternatively, one may use bounds on the probability of the three-arm event in the half-plane to prove the above. We prefer the strategy above as it adapts easily to fractal quads of bounded extremal distance; the cases (i) and (ii) (and (iii) and (iv), respectively) should then be distinguished using extremal distance rather than the geometric average of m and R. Above, the quad $\mathcal{D}$ is assumed to be η-regular simply for convenience.

Proof of Proposition 3.9. Fix $\eta > 0$, and let $C = C(\eta)$ and $\varepsilon = \varepsilon(\eta)$ be the constants given by Lemma 3.11 for this value of η. Below, $c_0, \dots, c_3$ denote strictly positive constants depending only on η.

For p and R with $R \le L(p)$, let $m = c_0 R$ be such that

$$C(m/R)^{\varepsilon} < \tfrac{1}{4} \inf_{\mathcal{D}} \phi[\mathcal{C}(\mathcal{D})]\phi[\mathcal{C}(\mathcal{D})^c],$$

where the infimum is taken over all η-regular quads $\mathcal{D}$ at scale R. By equation (RSW), the infimum is uniformly positive, and therefore $c_0 > 0$ above does indeed depend only on η.

Fix $\mathcal{D}$ as in the statement of the proposition. Then in any increasing coupling $\mathbb{P}$ of $\phi[\cdot]$ and $\phi[\cdot\,|\mathcal{C}(\mathcal{D})]$ (note that $\mathcal{C}(\mathcal{D})$ occurs automatically for ω'),

$$\begin{aligned}&\mathbb{P}[\omega' \in \mathcal{C}(\mathcal{D}^+),\ \omega \notin \mathcal{C}(\mathcal{D}^-)]\\ &\qquad \ge \mathbb{P}[\omega' \in \mathcal{C}(\mathcal{D}), \omega \notin \mathcal{C}(\mathcal{D})] - \mathbb{P}[\omega \in \mathcal{C}(\mathcal{D}^-), \omega \notin \mathcal{C}(\mathcal{D})] - \mathbb{P}[\omega' \in \mathcal{C}(\mathcal{D}), \omega' \notin \mathcal{C}(\mathcal{D}^+)]\end{aligned}$$

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$$\geq 1 - \phi[\mathcal{C}(\mathcal{D})] - \phi[\mathcal{C}(\mathcal{D}^-) \setminus \mathcal{C}(\mathcal{D})] - \frac{\phi[\mathcal{C}(\mathcal{D}) \setminus \mathcal{C}(\mathcal{D}^+)]}{\phi[|\mathcal{C}(\mathcal{D})]}$$
$$\geq \phi[\mathcal{C}(\mathcal{D})^c] - C(m/R)^\varepsilon - \frac{C(m/R)^\varepsilon}{\phi[\mathcal{C}(\mathcal{D})]}$$
$$\geq \tfrac{1}{2}\phi[\mathcal{C}(\mathcal{D})^c] \geq c_1 > 0.$$

Next, we create an increasing coupling $\mathbb{P}$ between ϕ and $\phi[\cdot\,|\mathcal{C}(\mathcal{D})]$ using the specific decision tree described below. Start by exploring the lowest crossing Γ^+ in $\mathcal{D}^+$ from (a^+b^+) to (c^+d^+) (that is, the crossing closest to (b^+c^+)) in the larger configuration ω'. Write τ_1 for the stopping time when this crossing is found; set $\tau_1 = \infty$ if no such crossing exists. If $\tau_1 < \infty$, explore the 'right-most' dual crossing Γ^- in $\mathcal{D}^-$ from (b^-c^-) to (d^-a^-) (that is, the one closest to (c^-d^-)) in the configuration ω^*; define τ_2 for the stopping time when this crossing is found (if $\tau_1 = \infty$ or no such crossing is found, set $\tau_2 = \infty$). Notice that

$$\mathbb{P}[\tau_2 < \infty] = \mathbb{P}[\omega' \in \mathcal{C}(\mathcal{D}^+), \omega \notin \mathcal{C}(\mathcal{D}^-)] \geq c_1.$$

Assuming that $\tau_2 < \infty$, the revealed edges $e_{[\tau_2]}$ are those of $\mathcal{D}^+$ below Γ^+ and those of $\mathcal{D}^-$ right or Γ^-. In particular, the edges of $\mathcal{D}^+ \setminus \mathcal{D}$ that are above Γ^+, as well as those of $\mathcal{D}^- \setminus \mathcal{D}$ that are left of Γ^-, are unrevealed. See Figure 7 (middle diagram).

Next, explore the double four-petal flower domain $(\mathcal{F}_{\rm in}, \mathcal{F}_{\rm out})$ between $\Lambda_{3R/2}$ and Λ_{2R} in ω; let τ_3 be the stopping time marking the end of this exploration (with $\tau_3 = \infty$ if no double four-petal flower domain exists or if $\tau_2 = \infty$). Due to Lemma 3.4,

$$\mathbb{P}[\tau_3 < \infty \,|\, (\omega_{[\tau_2]}, \omega'_{[\tau_2]}) \text{ s.t. } \tau_2 < \infty] \geq c_2.$$

Finally, reveal the configurations in the unrevealed regions of $\mathcal{F}_{\rm in}$, and write τ_4 for the stopping time marking the end of this stage. Let H be the event that $P_1^{\rm in}$ and $P_3^{\rm in}$ are connected by paths of $\omega' \cap (\mathcal{F}_{\rm in}^c \setminus \mathcal{D})$ to Γ^+ and $P_2^{\rm in}$ and $P_4^{\rm in}$ are connected by paths of $\omega^* \cap (\mathcal{F}_{\rm in}^c \setminus \mathcal{D})$ to Γ^-. Due to equation (RSW) (see Figure 7, right diagram),

$$\mathbb{P}[H \,|\, (\omega_{[\tau_3]}, \omega'_{[\tau_3]}) \text{ s.t. } \tau_3 < \infty] \geq c_3.$$

Special care should be taken as the primal connections occur in ω' while the dual ones occur in ω. This may be easily overcome by considering connections in predetermined disjoint regions.[3]

Now, if $\tau_3 < \infty$ and H occurs, then $P_1^{\rm out}$ and $P_3^{\rm out}$ are disconnected in $\omega \cap \mathcal{F}_{\rm out}^c$ but are connected in $\omega' \cap \mathcal{F}_{\rm out}^c$. Set $\tau = \tau_3$ if the above occurs and $\tau = \infty$ otherwise. Then τ satisfies the requirements of the proposition and

$$\mathbb{P}[\tau < \infty] \geq c_1 c_2 c_3 > 0.$$

Finally, let us discuss the case where $\mathcal{C}(\mathcal{D})$ is replaced by $A_{R/2}$, for which the proof is much simpler. Consider the following increasing coupling of $\omega \sim \phi$ and $\omega' \sim \phi[\cdot\,|A_{R/2}]$. Start by exploring the double four-petal flower domain $(\mathcal{F}_{\rm in}, \mathcal{F}_{\rm out})$ between $\Lambda_{3R/2}$ and Λ_{2R} in ω; write τ_1 for the stopping time marking the end of this exploration. When $\tau_1 < \infty$, reveal the configurations inside $\mathcal{F}_{\rm in}$ in any order. At this stage, due to equation (RSW), there is a uniformly positive probability that $P_1^{\rm in}$ and $P_3^{\rm in}$ are both connected to $\Lambda_{R/2}$ but not to each other in $\omega \cap \mathcal{F}_{\rm in}$. It is then deterministic that $P_1^{\rm in}$ is connected to $P_3^{\rm in}$ by an open path of ω' inside $\mathcal{F}_{\rm in}$. Thus, when this event occurs, the boundary conditions induced by ω and ω' on $\mathcal{F}_{\rm out}$ form a boosting pair, and we declare τ to be the end of this stage of the exploration; otherwise, set

[3]One may be tempted to ask for both the primal and dual connections to occur in the same configuration: for instance, in ω. This would be conceptually simpler but would require a stronger RSW result, as the sections of Γ^+ outside of $\mathcal{D}$ are wired in ω' but not in ω. This stronger RSW result is true for $q < 4$ (see [DMT20]), but it is expected to be wrong for $q = 4$.

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$\tau = \infty$. Finally, Lemma 3.4 and the use of (RSW) mentioned above show that $\tau < \infty$ with uniformly positive probability. □

We conclude this section with the following lemma, which is a straightforward application of equation (RSW) and will be useful in future sections. It is independent of the rest of the section, but we include it here as it also deals with the probability that $\tau < \infty$.

Lemma 3.13. *For any $\eta > 0$, there exists a constant $c = c(\eta) > 0$ such that the following holds. Fix p and $\mathcal{F}$ an η-well-separated outer flower domain on Λ_R for some $R \le L(p)$. Let ξ, ξ' be a boosting pair of boundary conditions on $\mathcal{F}$ and x be a point of* $\mathrm{Ann}(2R, 4R)$. *Then there exists a coupling via decision trees $\mathbb{P}$ of $\phi^{\xi}_{\mathcal{F}}$ and $\phi^{\xi'}_{\mathcal{F}}$ and a stopping time τ such that when $\tau < \infty$, $\mathcal{F}_\tau = \mathcal{F} \setminus e_{[\tau]}$ is a $1/2$-well-separated inner flower domain on $\Lambda_{R/4}(x)$, and the boundary conditions induced by $\omega'_{[\tau]}$ on $\mathcal{F}_\tau$ are a boost of those induced by $\omega_{[\tau]}$. Finally,*

$$\mathbb{P}[\tau < \infty] \ge c.$$

The proof is similar to (yet much easier than) the one of Theorem 3.6.

Proof. Write P_i, P_j for two petals of $\mathcal{F}$ that are wired in ξ' but not in ξ.

Start by exploring the double four-petal flower domain $(\mathcal{F}_{\text{in}}, \mathcal{F}_{\text{out}})$ between $\Lambda_{R/4}(x)$ and $\Lambda_{R/2}(x)$. If no such double flower domain exists, set $\tau = \infty$ and proceed arbitrarily. If $(\mathcal{F}_{\text{in}}, \mathcal{F}_{\text{out}})$ exists, continue by revealing the configurations in $\mathcal{F} \cap \mathcal{F}_{\text{out}}$. Write H for the event that in $\omega \cap \mathcal{F}_{\text{out}}$, P_1^{out} is connected to P_i, P_3^{out} is connected to P_j, but P_1^{out} and P_3^{out} are not connected to each other or any other petal of $\mathcal{F}$. Equation (RSW) implies that

$$\mathbb{P}[H \mid \mathcal{F}_{\text{in}}, \mathcal{F}_{\text{out}}] \ge c, \tag{3.15}$$

where $c > 0$ depends only on η. If H occurs, set τ to be the stopping time at which the configurations in $\mathcal{F}_{\text{out}}$ have been revealed; otherwise, set $\tau = \infty$.

Then due to Lemma 3.4 and equation (3.15), $\mathbb{P}[H] \gtrsim c$. Finally, when $\tau < \infty$, the boundary conditions induced by $\omega'_{[\tau]}$ on $\mathcal{F}_\tau = \mathcal{F}_{\text{in}}$ are indeed a boost of those induced by $\omega_{[\tau]}$, since P^1_{in} is connected to P^3_{in} in $\omega'_{[\tau]}$ but not in $\omega_{[\tau]}$. □

4. Properties of the mixing rate

Fix $q \in (1, 4]$ and $\eta > 0$ for the whole of this section; all constants, including those in $\lesssim$ and $\asymp$, may depend on η. In this section, we always work with a single edge-parameter $p \in (0, 1)$, and therefore we often omit it from the measure $\phi_{G,p}$ for notational convenience.

4.1. Noncoupling induces boosting boundary conditions

The main result of this subsection is the following; it is the cornerstone of the other results proved later in this and other sections.

Theorem 4.1. *For any $\eta > 0$, $p \in (0, 1)$ and $4r < R \le L(p)$.*

(i) *Let $\mathcal{G}$ be an η-well-separated inner flower domain on Λ_R and (ξ, ξ') be a boosting pair of boundary conditions on $\mathcal{G}$. There exists an increasing coupling $\mathbb{P}$ of $\phi^{\xi}_{\mathcal{G},p}$ and $\phi^{\xi'}_{\mathcal{G},p}$ on $\mathcal{G} \setminus \Lambda_r$ via decision trees, and a stopping time τ with the following property. When $\tau < \infty$, $\mathcal{F}_\tau = \mathcal{G} \setminus e_{[\tau]}$ is a $1/2$-well-separated inner flower domain on Λ_r, and the boundary conditions induced by $(\omega'_{[\tau]})^{\xi'}$ on $\mathcal{F}_\tau$ are a boost of those induced by $\omega^{\xi}_{[\tau]}$. Moreover, if we write ζ and ζ' for the boundary conditions* *induced by* $(\omega')^{\xi'}$ *and* ω^{ξ} *on* $\partial\Lambda_r$*, then*

$$\mathbb{P}[\tau < \infty] \asymp \mathbb{P}[\zeta \neq \zeta'].$$

(ii) *Let* $\mathcal{G}$ *be an* η*-well-separated outer flower domain on* Λ_r *and* (ξ,ξ') *be a boosting pair of boundary conditions on* $\mathcal{G}$*. There exists an increasing coupling* $\mathbb{P}$ *of* $\phi^{\xi}_{\mathcal{G},p}$ *and* $\phi^{\xi'}_{\mathcal{G},p}$ *on* $\mathcal{G}\cap\Lambda_R$ *via decision trees, and a stopping time* τ *with the following property. When* $\tau<\infty$*,* $\mathcal{F}_\tau = \mathcal{G}\setminus e_{[\tau]}$ *is a* $1/2$*-well-separated outer flower domain on* Λ_R*, and the boundary conditions induced by* $(\omega'_{[\tau]})^{\xi'}$ *on* $\mathcal{F}_\tau$ *are a boost of those induced by* $\omega^{\xi}_{[\tau]}$*. Moreover, if we write* ζ *and* ζ' *for the boundary conditions induced by* $(\omega')^{\xi'}$ *and* ω^{ξ} *on* $\partial\Lambda_R$*, then*

$$\mathbb{P}[\tau < \infty] \asymp \mathbb{P}[\zeta \neq \zeta'].$$

Remark 4.2. In the previous theorem, we sample edges only outside Λ_r in Case (i) and only inside Λ_R in Case (ii), but of course once this is done, one may use an arbitrary coupling inside Λ_r or outside Λ_R to obtain couplings in the whole flower domain $\mathcal{G}$. We stated the previous theorem in this setting to be able to reuse the coupling in applications we have in mind.

Remark 4.3. The proofs of Theorem 4.1 (i) and (ii) may be readily adapted to the FK-percolation measure on any graphs containing $\mathcal{G}\setminus\Lambda_{r/2}$ and $\mathcal{G}\cap\Lambda_{2R}$, respectively, as they essentially only rely on the RSW property in $\mathrm{Ann}(r,R)$.

The first point of the theorem should be understood as follows. We reveal the configurations ω and ω' in the coupling $\mathbb{P}$ starting from the outside and moving inwards; while doing this, we follow the difference between the boundary conditions that the configurations induce on the unrevealed region. If this difference survives until the whole of $\mathcal{G}\setminus\Lambda_r$ is revealed, then there is a positive probability that it survives as a significant difference, namely in the form of a boosting pair of boundary conditions on a well-separated flower domain.

The second point is analogous, with the revealment starting inside and moving outwards.

Proof of Theorem 4.1. We will only prove point (i); the proof of point (ii) is identical. Fix $p\in(0,1)$, $4r\le R\le L(p)$ and $\mathcal{G}$, ξ and ξ' as in the statement. All constants below are independent of r, R, $\mathcal{G}$, ξ and ξ' unless explicitly stated. When referring to connections in configurations ω and ω' with laws $\phi^{\xi}_{\mathcal{G},p}$ and $\phi^{\xi'}_{\mathcal{G},p}$, respectively, we will implicitly include connections that use the boundary conditions. In other words, we omit the superscript in the notation ω^{ξ} and $(\omega')^{\xi'}$.

First, for any increasing coupling $\mathbb{P}$ between $\phi^{\xi}_{\mathcal{G}}$ and $\phi^{\xi'}_{\mathcal{G}}$ obtained by a decision tree, and any stopping time τ with the properties of the theorem, $\mathbb{P}[\tau<\infty]\le\mathbb{P}[\zeta\neq\zeta']$. Indeed, the requirement that the boundary conditions induced by $\omega'_{[\tau]}$ on $\mathcal{F}_\tau$ are a boost of those induced by $\omega_{[\tau]}$ imposes that the boundary conditions induced by ω and ω' on Λ_r are distinct. The rest of the proof is dedicated to the converse bound.

Assume for simplicity that $R=4^k r$ for some integer $k\ge 2$. By monotonicity, it suffices to treat the case where ξ and ξ' are identical, with the exception of two petals that are wired together in ξ' but not in ξ. Assume this is the case, and write P_1 and P_3 for the two primal petals of $\mathcal{G}$ that are wired together in ξ' but not in ξ (contrary to what the notation suggests, P_1 and P_3 need not be separated by a single dual petal).

Below, we describe an increasing coupling $\mathbf{P}$ of $\phi^{\xi}_{\mathcal{G}}$ and $\phi^{\xi'}_{\mathcal{G}}$ on $\mathcal{G}\setminus\Lambda_r$ obtained through a decision tree. The actual coupling of the theorem is a slight variation of $\mathbf{P}$ that we will describe at the end of the proof.

At any time t, write $\mathcal{F}_t$ for the connected component of Λ_r in $\mathcal{G}\setminus e_{[t]}$. We will abusively consider that all edges of $\mathcal{G}$ that are not revealed at time t and that are not part of $\mathcal{F}_t$ are revealed instantaneously. Thus, from now on, we have $\mathcal{F}_t=\mathcal{G}\setminus e_{[t]}$. We will identify $\omega_{[t]}$ and $\omega'_{[t]}$ to the boundary conditions they induce on $\mathcal{F}_t$.

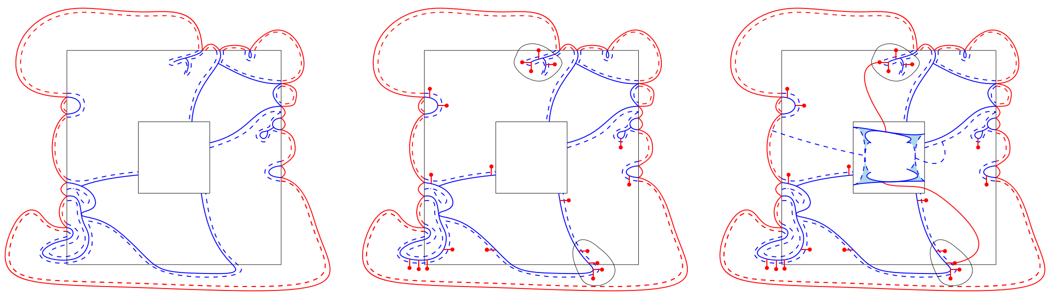

Figure 8. *Red solid lines represent open edges in ω', while blue ones represent open edges in ω. Dotted lines represent closed (or, equivalently, open dual edges) in the configurations corresponding to their colour. Left: At the end of stage j, the revealed edges are those of the cluster of P_1 and P_3 in $\omega' \cap \Lambda^c_{4\rho}$ and its boundary. Then by time $\tau_{j+1/2}$, the interfaces in ω starting at the wired vertices previously exposed are revealed until they touch $\Lambda_{2\rho}$. At this time, the unrevealed region is a flower domain that is likely well-separated. Middle: Assuming that the primal petals of $\mathcal{F}_{\tau_{j+1/2}}$ are all wired together in ω, we turn our attention to the set $\mathcal{A}$ of points that lie on dual petals of $\mathcal{F}_{\tau_{j+1/2}}$ but are connected to P_1 or P_3 in $\omega'_{[\tau_{j+1/2}]}$. If this set has a good probability of being connected to Λ_ρ, then we call $\tau_{j+1/2}$ promising. Right: For $\tau_{j+1/2}$ promising, we may connect two separate regions of $\mathcal{A}$ to the two primal external petals of a double flower-domain at a smaller scale. Then these petals will be connected in ω' but not in ω.*

Write T for the first time t when $\omega_{[t]}$ and $\omega'_{[t]}$ induce the same boundary conditions on $\mathcal{F}_t$, and $T = \infty$ if the boundary conditions are never the same. If $T < \infty$, the configurations ω, ω' in $\mathcal{F}_T$ are identical, regardless of the decision tree used after T. Thus, when this occurs, reveal the rest of the edges using lexicographical order.

The coupling proceeds in several stages numbered $j = 0, \ldots, k$. If at any point T occurs, the procedure described below stops, and the revealment by lexicographical order is used. Stage j corresponds to revealing information in $\Lambda^c_{4^{k-j}r} := \mathcal{G} \setminus \Lambda_{4^{k-j}r}$. We will write τ_j for the stopping time that marks the end of stage j. At time τ_j, the revealed edges are those of the cluster of P_1 and P_3 in $\omega' \cap \Lambda^c_{4^{k-j}r}$, any edges of $\Lambda^c_{4^{k-j}r}$ adjacent to this cluster, as well as any edges separated from Λ_r by the two categories of edges mentioned above.

Let us now describe precisely the revealment algorithm.

Revealment algorithm

Stage 0: Reveal the cluster of P_1 and P_3 inside $\omega' \cap \Lambda^c_R$ in arbitrary order. Let τ_0 be the stopping time marking the end of this stage.

Stage $j+1$: Fix $j \geq 0$, and assume the coupling defined up to time τ_j (see Figure 8 for an illustration). Stage $j+1$ is itself formed of two steps. As already mentioned, this is only valid when $T > \tau_j$. Write $\rho = 4^{k-j-1}r$. In $\omega'_{[\tau_j]}$, the boundary of $\mathcal{F}_{\tau_j}$ is formed of dual arcs outside of $\Lambda_{4\rho}$ with endpoints on $\partial\Lambda_{4\rho}$, along with points on $\partial\Lambda_{4\rho}$ that are connected in ω'_{τ_j} to P_1 or P_3. Call the latter points the 'wired' points of $\partial\mathcal{F}_{\tau_j}$. Since T has not yet occurred, there exists at least one wired point on $\partial\mathcal{F}_{\tau_j}$.

First, reveal all interfaces in $\omega \cap \mathcal{F}_{\tau_j} \cap \Lambda^c_{2\rho}$ that start at wired points of $\partial\mathcal{F}_{\tau_j}$. This may be done by tracking the left and right boundaries of the clusters of each wired point of $\partial\mathcal{F}_{\tau_j}$ until they finish on $\partial\mathcal{F}_{\tau_j}$ or reach $\partial\Lambda_{2\rho}$. Write $\tau_{j+1/2}$ for the stopping time that marks the end of this step. Formally, if T occurs before time $\tau_{j+1/2}$, set $\tau_{j+1/2} = \infty$.

Next, explore the cluster of P_1 and P_3 inside $\omega' \cap \Lambda^c_\rho$, and write τ_{j+1} for the stopping time marking the end of this stage (set $\tau_{j+1} = \infty$ if T occurs before the end of this stage). In fact, we will sometimes require that the clusters are revealed in a certain order (see the coupling P defined by Lemma 4.5 for details).

After stage k: Assuming that $T > \tau_k$, reveal all remaining edges in lexicographical order.

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Each stopping time $\tau_{j+1/2}$ will be declared promising or not (see the precise definition below) depending on the configuration at that stage and on some constant threshold $\delta > 0$ to be fixed below. Fix $0 \le j < k$, and assume $\tau_{j+1/2} < \infty$ (otherwise, $\tau_{j+1/2}$ is not promising). Then $\mathcal{F}_{\tau_{j+1/2}}$ is either a simply connected domain containing $\Lambda_{2\rho}$ and $\omega_{[\tau_{j+1/2}]}$ induces free boundary conditions on it, or it is an inner flower domain on $\Lambda_{2\rho}$ and $\omega_{[\tau_{j+1/2}]}$ induces coherent boundary conditions on it. The former occurs when none of the revealed interfaces in the first step of stage j reaches $\Lambda_{2\rho}$.

First, we analyse the case where $\omega_{[\tau_{j+1/2}]}$ induces free boundary conditions $\mathcal{F}_{\tau_{j+1/2}}$. Write $\mathcal{A}$ for the set of vertices on $\partial\mathcal{F}_{\tau_{j+1/2}}$ that are connected to $P_1 \cup P_3$ in $\omega'_{[\tau_{j+1/2}]}$. Notice that due to the exploration procedure, for any edge uv with $u \in \partial\mathcal{F}_{\tau_{j+1/2}}$ and $v \in \mathcal{F}^c_{\tau_{j+1/2}}$, v is connected in $\omega'_{[\tau_{j+1/2}]}$ to P_1 or P_3. Thus, $\mathcal{A}$ is exactly the set of vertices $u \in \partial\mathcal{F}_{\tau_{j+1/2}}$ for which there exists a revealed edge uv with $v \in \mathcal{F}^c_{\tau_{j+1/2}}$ that is open in $\omega'_{[\tau_{j+1/2}]}$. In particular, in the boundary conditions induced by $\omega'_{[\tau_{j+1/2}]}$ on $\mathcal{F}_{\tau_{j+1/2}}$, all vertices of $\mathcal{A}$ are wired together, and all other boundary vertices are free (that is, they are not wired to any other boundary vertices).

In this case, we call $\tau_{j+1/2}$ *promising* if

$$\phi_{\mathcal{F}_{\tau_{j+1/2}}}^{\omega'_{[\tau_{j+1/2}]}}[\mathcal{A} \leftrightarrow \Lambda_\rho] \ge \delta.$$

Next, we turn our attention to the case where $\mathcal{F}_{\tau_{j+1/2}}$ is a flower domain. Write $\mathcal{A}$ for the set of points on the dual petals of $\mathcal{F}_{\tau_{j+1/2}}$ that are connected to P_1 or P_3 in $\omega'_{[\tau_{j+1/2}]}$. By the same argument as in the previous case, in the boundary conditions imposed by $\omega'_{[\tau_{j+1/2}]}$ on $\mathcal{F}_{\tau_{j+1/2}}$, there exists a single wired component formed of $\mathcal{A}$ along with all primal petals of $\mathcal{F}_{\tau_{j+1/2}}$.

If $\mathcal{F}_{\tau_{j+1/2}}$ is not δ-well-separated, then we say that $\tau_{j+1/2}$ is *not promising*. If $\mathcal{F}_{\tau_{j+1/2}}$ is δ-well-separated and contains at least two primal petals that are not wired in $\omega_{[\tau_{j+1/2}]}$, then we call $\tau_{j+1/2}$ promising. Finally, if $\mathcal{F}_{\tau_{j+1/2}}$ is δ-well-separated and all its primal petals are wired together in $\omega_{[\tau_{j+1/2}]}$, we call $\tau_{j+1/2}$ *promising* if

$$\phi_{\mathcal{F}_{\tau_{j+1/2}}}^{\omega'_{[\tau_{j+1/2}]}}[\mathcal{A} \leftrightarrow \Lambda_\rho] \ge \delta.$$

For formal reasons, set $\tau_{-1/2} = 0$ and call it promising. We now state two results that are instrumental in the proof of the theorem. Roughly speaking, the first one states that if $\tau_{j+1/2}$ is not promising, then it is very likely for T to arise before τ_{j+1}. The second lemma states that as soon as $\tau_{j+1/2}$ is promising, there exists a coupling guaranteeing a fairly good probability that the flower domain on $\partial\Lambda_r$ is $1/2$-well-separated and the boundary conditions in ω and ω' induced on this domain correspond to a boosting pair.

Lemma 4.4. *For any $\varepsilon > 0$, we may choose $\delta = \delta(\varepsilon) > 0$ (independent of r, R, p, $\mathcal{G}$, ξ and ξ') so that for every $0 \le j < k$,*

$$\mathbf{P}[\tau_{j+1/2} \textit{ not promising and } T > \tau_{j+1} \,|\, \omega_{[\tau_j]}, \omega'_{[\tau_j]}] < \varepsilon.$$

Lemma 4.5. *For any $\delta > 0$, there exists $c(\delta) > 0$ with the following property. Fix some $-1 \le j \le k-1$ and a realisation of $\tau_{j+1/2}$, $\omega_{[\tau_{j+1/2}]}$, $\omega'_{[\tau_{j+1/2}]}$ and $\mathcal{F}_{\tau_{j+1/2}}$ for which $\tau_{j+1/2}$ is promising. Then there exists an increasing coupling* P *of $\phi_{\mathcal{F}_{\tau_{j+1/2}}}^{\omega_{[\tau_{j+1/2}]}}$ and $\phi_{\mathcal{F}_{\tau_{j+1/2}}}^{\omega'_{[\tau_{j+1/2}]}}$ via decision trees and a stopping time σ with the following property. When $\sigma < \infty$, $\mathcal{F}_\sigma$ is a $1/2$-well-separated inner flower domain on Λ_r, and the boundary conditions induced by $\omega'_{[\sigma]}$ on $\mathcal{F}_\sigma$ are a boost of those induced by $\omega_{[\sigma]}$. Moreover,*

$$\mathrm{P}[\sigma < \infty] \ge c(\delta)\pi_4(p; r, 4^{k-j}r). \tag{4.1}$$

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Remark 4.6. It is essential in the second lemma that $c(\delta)$ is allowed to depend on δ but that it only appears as a multiplicative constant in equation (4.1). Indeed, the upper bound in equation (4.1) depends on the ratio between the scales of $\mathcal{F}_{\tau_{j+1/2}}$ and $\mathcal{F}_\sigma$ in a way that is uniform in δ.

Before proving the two lemmas, let us conclude the proof of the theorem. Fix $\varepsilon > 0$ so that $\pi_4(p; r, 4^j r) \geq (2\varepsilon)^j$ for all $j \geq 0$ with $4^j r \leq L(p)$. Due to equation (RSW), ε may be chosen independently of r or p. Let $\delta = \delta(\varepsilon)$ be the quantity given by Lemma 4.4 for this value of ε. Below, c_0 and c_1 stand for strictly positive constants that may depend on ε but not on r, R or $\mathcal{G}$.

First, observe that due to Lemma 4.5 and Theorem 3.6, there exists $c_0 > 0$ such that for any $1 \leq j \leq k$,

$$\mathbf{P}[T = \infty] \geq \phi_{\mathcal{G}}^{\xi'}[\mathcal{C}(\Lambda_r)] - \phi_{\mathcal{G}}^{\xi}[\mathcal{C}(\Lambda_r)] \geq c_0\,(2\varepsilon)^j\,\mathbf{P}[\tau_{k-j-1/2} \text{ promising}]. \tag{4.2}$$

Indeed, the first inequality follows from the more general observation that $\mathbf{P}[T = \infty]$ bounds the distance in total variation between the restrictions of $\phi_{\mathcal{G}}^{\xi'}$ and $\phi_{\mathcal{G}}^{\xi}$ to Λ_r. For the second inequality, consider the increasing coupling of $\phi_{\mathcal{G}}^{\xi'}$ and $\phi_{\mathcal{G}}^{\xi}$ obtained by following $\mathbf{P}$ up to the stopping time $\tau_{k-j-1/2}$ and then, if $\tau_{k-j-1/2}$ is promising, using the coupling P of Lemma 4.5 and applying Theorem 3.6.

Next, we claim that there exists some $\ell \geq 1$ such that if $T = \infty$, then with positive probability, there exists one promising stopping time among the last ℓ ones. It is essential here that ℓ is independent of R/r. Indeed, fix $\ell \geq 1$, and observe that

$$\begin{aligned}
&\mathbf{P}[T = \infty \text{ and } \tau_{k-1/2}, \dots, \tau_{k-\ell+1/2} \text{ not promising}]\\
&\quad = \sum_{j=\ell}^{k} \mathbf{P}[T = \infty;\ \tau_{k-1/2}, \dots, \tau_{k-j+1/2} \text{ not promising; } \tau_{k-j-1/2} \text{ promising}]\\
&\quad \leq \sum_{j=\ell}^{k} \varepsilon^{j-1} \mathbf{P}[\tau_{k-j-1/2} \text{ promising}]\\
&\quad \leq c_0\,\mathbf{P}[T = \infty] \sum_{j=\ell}^{k} 2^{-j} \leq c_0\, 2^{-\ell+1}\,\mathbf{P}[T = \infty].
\end{aligned}$$

(Note that this is where we use the convention that $\tau_{-1/2}$ is promising.) The first inequality is obtained by repeated applications of Lemma 4.4; the second is due to equation (4.2). Thus, if we fix $\ell \geq -\log_2 c_0 + 2$, then indeed

$$\mathbf{P}[\text{at least one of } \tau_{k-1/2}, \dots, \tau_{k-\ell+1/2} \text{ is promising} \,|\, T = \infty] \geq \tfrac{1}{2}. \tag{4.3}$$

We are now ready to define the coupling $\mathbb{P}$ of the theorem. Follow $\mathbf{P}$ up to the first promising stopping time $\tau_{j+1/2}$ with $j \geq k - \ell$. Then follow the coupling P of Lemma 4.5; write τ_{final} for the stopping time described in said lemma. If $\mathbf{P}$ does not encounter a promising stopping time $\tau_{j+1/2}$ with $j \geq k - \ell$, set $\tau_{\text{final}} = \infty$. Then due to Lemma 4.5 and equation (4.3),

$$\mathbb{P}[\tau_{\text{final}} < \infty \,|\, T = \infty] \geq \tfrac{1}{2} c_1\, c_2^{\ell}.$$

Since ε and ℓ are independent of r and R, the right-hand side is bounded away from 0 uniformly in $r \leq R/4$. Moreover, τ_{final} satisfies the conditions stated in the theorem. Multiply the above by $\mathbb{P}[T = \infty] = \mathbb{P}[\zeta \neq \zeta']$ to obtain the desired inequality. □

Proof of Lemma 4.4. Fix $\varepsilon > 0$, and fix a realisation of τ_j, $\omega_{[\tau_j]}$ and $\omega'_{[\tau_j]}$. Lemma 3.2 ensures that by choosing $\delta > 0$ small enough, we have

$$\mathbf{P}[\mathcal{F}_{\tau_{j+1/2}} \text{ is a flower domain, which is not } \delta\text{-well-separated} \,|\, \omega_{[\tau_j]}, \omega'_{[\tau_j]}] < \varepsilon/2.$$

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Suppose now that either $\mathcal{F}_{\tau_{j+1/2}}$ is a δ-well-separated flower domain or $\omega_{\tau_{j+1/2}}$ induced free boundary conditions on it. In either case, if $\tau_{j+1/2}$ is not promising, it is because $\mathcal{A}$ has a conditionally small probability of being connected to Λ_ρ in ω'.

Continue the coupling $\mathbf{P}$ by revealing first the connected component C of $\mathcal{A}$ in $\omega' \cap \Lambda_\rho^c$ and then the rest of the connected components of P_1 and P_3. If C does not reach Λ_ρ, then it is entirely surrounded by closed edges of ω'. Thus, the boundary conditions in ω and ω' on the complement of C are identical, which is to say that T occurs before τ_{j+1}. Thus

$$\mathbf{P}[T > \tau_{j+1} \,|\, \omega_{[\tau_{j+1/2}]}, \omega'_{[\tau_{j+1/2}]}] \leq \phi_{\mathcal{F}_{\tau_{j+1/2}}}^{\omega'_{[\tau_{j+1/2}]}}[\mathcal{A} \leftrightarrow \Lambda_\rho] < \delta.$$

By choosing $\delta > 0$ small enough, the right-hand side may be rendered smaller than $\varepsilon/2$. Apply the union bound to obtain the desired inequality. □

Proof of Lemma 4.5. Fix $\delta > 0$ and a realisation $\mathcal{F}$ of $\mathcal{F}_{\tau_{j+1/2}}$, with boundary conditions ζ and ζ' induced by $\omega_{[\tau_{j+1/2}]}$ and $\omega'_{[\tau_{j+1/2}]}$, respectively, on $\mathcal{F}$. Assume that $\mathcal{F}$, ζ and ζ' satisfy the assumptions of the lemma. Three situations need to be analysed; we will proceed in increasing order of difficulty. Set $\rho := 4^{k-j} r$.

Case 1: $\mathcal{F}$ is a flower domain with two petals wired in ζ' but not in ζ. First, assume that $\mathcal{F}$ is a δ-well-separated flower domain containing two primal petals P_i and P_j that are not wired together in ζ. Recall that P_i and P_j are necessarily wired together in ζ'.

Then P is constructed as follows. Attempt to explore the double four-petal flower domain $(\mathcal{F}_{\rm in}, \mathcal{F}_{\rm out})$ between Λ_r and Λ_{2r} for the configuration ω. If no such double four-petal flower domain exists, set $\sigma = \infty$, and reveal the remaining edges in lexicographical order. If it exists, reveal the configurations in $\mathcal{F}_{\rm out} \cap \mathcal{F}$, and let σ be the stopping time marking the end of this stage.

Let H be the event that the double four-petal flower domain $(\mathcal{F}_{\rm in}, \mathcal{F}_{\rm out})$ exists and that the two primal petals $P_1^{\rm out}$ and $P_3^{\rm out}$ of $\mathcal{F}_{\rm out}$ are connected in $\omega'_{[\sigma]}$ to P_i and P_j, respectively, while P_i and P_j are not connected to each other or any primal petal of $\mathcal{F}$ in $\omega_{[\sigma]}$. It is a standard consequence of the separation of arms that there exists a constant $c(\delta) > 0$ such that

$$\mathrm{P}[H \,|\, (\mathcal{F}_{\rm in}, \mathcal{F}_{\rm out}) \text{ double four-petal flower domain}] \geq c(\delta)\, \pi_4(p; r, 4\rho).$$

Lemma 3.4 ensures that the event in the conditioning has uniformly positive probability, and hence so does H. Finally, notice that when H occurs, the boundary conditions induced by $\omega'_{[\sigma]}$ on $\partial\mathcal{F}_{\rm in}$ are a boost of those induced by $\omega_{[\sigma]}$. This concludes the proof in this case.

Case 2: $\mathcal{F}$ is not a flower domain. Next, assume that $\mathcal{F}$ is not a flower domain and therefore that $\zeta = 0$. Start the coupling by attempting to reveal the double four-petal flower domains $(\mathcal{F}_{\rm in}, \mathcal{F}_{\rm out})$ between Λ_r and Λ_{2r} and $(\overline{\mathcal{F}}_{\rm in}, \overline{\mathcal{F}}_{\rm out})$ between Λ_ρ and $\Lambda_{3\rho/2}$, respectively, for the configuration ω (if $\rho = r$, only perform the latter exploration). If these two double four-petal flower domains exist, proceed by revealing the configurations in $\overline{\mathcal{F}}_{\rm in} \cap \mathcal{F}_{\rm out}$; call σ_0 the stopping time marking the end of this stage.

Let H be the event that the two double four-petal flower domains above exist, that $\overline{P}_1^{\rm in}$ is connected to $P_1^{\rm out}$ and $\overline{P}_3^{\rm in}$ to $P_3^{\rm out}$ in $\omega \cap \overline{\mathcal{F}}_{\rm in} \cap \mathcal{F}_{\rm out}$ and that $\overline{P}_2^{\rm in}$ is connected to $P_2^{\rm out}$ and $\overline{P}_4^{\rm in}$ to $P_4^{\rm out}$ in $\omega^* \cap \overline{\mathcal{F}}_{\rm in} \cap \mathcal{F}_{\rm out}$. As in the first case, it is a consequence of the separation of arms and Lemma 3.4 that there exists $c_0 > 0$ such that

$$\mathrm{P}[H] \geq c_0\, \pi_4(p; r, 4\rho). \tag{4.4}$$

Next we reveal the configurations in $\mathcal{F} \cap \overline{\mathcal{F}}_{\rm out}$ in a fashion described below. Write H' for the event that $\overline{P}_1^{\rm out}$ and $\overline{P}_3^{\rm out}$ are connected to each other in $\omega' \cap \overline{\mathcal{F}}_{\rm out}$ but not in $\omega \cap \overline{\mathcal{F}}_{\rm out}$. Observe that if H and H' occur, then the boundary conditions induced on $\mathcal{F}_{\rm in}$ by $\omega \cap \mathcal{F}_{\rm in}$ and $\omega' \cap \mathcal{F}_{\rm in}$, respectively, form a boosting pair. At this stage, it will be useful to introduce the following claim.

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Claim 4.7. There exist $c_1 = c_1(\delta) > 0$ and four disjoint domains $D_1, \dots, D_4$ contained in $\mathcal{F} \cap \overline{\mathcal{F}}_{\text{out}}$ (they depend on $\mathcal{A}$) such that

$$\phi_{D_1}^{\omega'_{[\sigma_0]},0}[\mathcal{A} \longleftrightarrow \overline{P}_1^{\text{out}}] > c_1, \qquad \phi_{D_3}^{\omega'_{[\sigma_0]},0}[\mathcal{A} \longleftrightarrow \overline{P}_3^{\text{out}}] > c_1, \tag{4.5}$$

$$\phi_{D_2}^{\omega_{[\sigma_0]},1}[\partial\mathcal{F} \overset{\omega^*}{\longleftrightarrow} \overline{P}_2^{\text{out}}] > c_1, \qquad \phi_{D_4}^{\omega_{[\sigma_0]},1}[\partial\mathcal{F} \overset{\omega^*}{\longleftrightarrow} \overline{P}_4^{\text{out}}] > c_1, \tag{4.6}$$

where $\omega'_{[\sigma_0]}, 0$ (and $\omega_{[\sigma_0]}, 1$) are the boundary conditions on ∂D_i induced by the configuration equal to ω' (and ω, respectively), completed in the unrevealed region of D_i^c by closed (respectively, open) edges.

Essentially, $D_1, \dots, D_4$ should be viewed as disjoint tubes connecting the petals $\overline{P}_i^{\text{out}}$ to four disjoint regions of $\partial\mathcal{F}$. Each tube should be considered thick enough to contain either primal or dual paths with positive probability (for instance, we may think of each tube as being constructed using a finite number of rectangles of aspect ratio 2). Moreover, we should further require that the regions of $\partial\mathcal{F}$ contained in D_1 and D_3 contain sufficiently many points of $\mathcal{A}$ that a connection reaches these points with positive probability. Let us delay the proof of this technical result and finish the proof of the lemma in this case.

Fix the four domains $D_1, \dots, D_4$ given by the claim. Reveal the configuration in $\mathcal{F} \cap \overline{\mathcal{F}}_{\text{out}}$ by first revealing the configurations outside $D_1, \dots, D_4$ and then in each of the quads $D_1, \dots D_4$. Due to equation (CBC),

$$\mathrm{P}\big[\mathcal{A} \xleftrightarrow{\omega' \cap D_1} \overline{P}_1^{\text{out}}, \mathcal{A} \xleftrightarrow{\omega' \cap D_3} \overline{P}_3^{\text{out}}, \partial\mathcal{F} \xleftrightarrow{\omega^* \cap D_2} \overline{P}_2^{\text{out}}, \partial\mathcal{F} \xleftrightarrow{\omega^* \cap D_4} \overline{P}_4^{\text{out}} \,\big|\, (\omega_{[\sigma_0]}, \omega'_{[\sigma_0]})\big] \geq c_1^4,$$

whenever $(\omega_{[\sigma_0]}, \omega'_{[\sigma_0]})$ are such that H occur. Notice now that since all points of $\mathcal{A}$ are wired together in ω' outside of $\mathcal{F}$ but the entire boundary of $\mathcal{F}$ is free in ω, if the event above occurs, then so does H'. Thus,

$$\mathrm{P}[H' \cap H] \geq c_1^4 \mathrm{P}[H] \geq c_0\, \pi_4(p; r, 4\rho).$$

If we now set σ to be the stopping time at which all of $\mathcal{F}_{\text{in}}^c$ has been revealed and H' has been found to occur, and $\sigma = \infty$ otherwise, then σ satisfies the properties claimed in the lemma.

Case 3: $\mathcal{F}$ is a flower domain, but the petals wired in $\zeta'$are wired in ζ. The construction in this case is very similar to that of Case 2. Start by revealing the double four-petal flower domains $(\mathcal{F}_{\text{in}}, \mathcal{F}_{\text{out}})$ between Λ_r and Λ_{2r} and $(\overline{\mathcal{F}}_{\text{in}}, \overline{\mathcal{F}}_{\text{out}})$ between Λ_ρ and $\Lambda_{3\rho/2}$, respectively, for the configuration ω. Define H and H' in the same way as in Case 2. The only difference is in the way in which the configurations in $\mathcal{F} \cap \overline{\mathcal{F}}_{\text{out}}$ are revealed so that $\mathrm{P}[H' \mid H]$ is bounded below by a constant depending only on δ.

Write $P_1, \dots, P_{2k}$ for the petals of $\mathcal{F}$. By the union bound and the assumption on $\mathcal{A}$, there exists a free petal P_j such that

$$\phi_{\mathcal{F}}^{\zeta'}[P_j \cap \mathcal{A} \overset{\omega'}{\longleftrightarrow} \Lambda_\rho] \geq \delta/k.$$

Recall that $\mathcal{F}$ is δ-well-separated and therefore k is bounded in terms of δ only. A construction similar to that of the claim may be performed, with $\mathcal{A}$ replaced by $\mathcal{A} \cap P_j$ and $\partial\mathcal{F}$ replaced by P_j. We conclude in the same way as in Case 2. □

We now turn to the proof of Claim 4.7. This type of construction is usually tedious because of the general form of $\mathcal{F}$ but may be performed fairly explicitly. We warn the reader of the difficulties arising from the potential bottlenecks of $\mathcal{F}$ and the fact that equation (RSW) is not valid with arbitrary boundary conditions in arbitrary quads when $q = 4$. Nevertheless, with sufficient care, the domains $D_1, \dots, D_4$ may be constructed as unions of small squares $\Lambda_{c\rho n}(x)$ with $x \in c\rho\mathbb{Z}^2$ and $c > 0$ a small constant independent of ρ. We propose below an alternative, more innovative construction of $D_1, \dots, D_4$. We will use the following result from [DMT20] (see Remark 4.2 to be precise).

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The extremal distance between the arcs (ab) and (cd) of a quad $(\mathcal{D}, a, b, c, d)$ is the unique value $\ell = \ell_{\mathcal{D}}[(ab),(cd)]$ such that there exists a conformal transformation ψ from $\mathcal{D}$ (seen as a domain in the continuum) to $(0,1)\times(0,\ell)$, with a, b, c, d being mapped (by the continuous extension of ψ) to the corners of $[0,1]\times[0,\ell]$, in counterclockwise order, starting with the lower-left corner.

Proposition 4.8 (RSW in terms of extremal distance). *For all $L > 0$, there exists $\eta(L) > 0$ such that for all $1 \le q \le 4$, $p \in (0,1)$, $R < L(p)$ and $(\mathcal{D}, a, b, c, d)$ a discrete quad contained in Λ_R, if $\ell_{\mathcal{D}}[(ab),(cd)] \le L$, then*

$$\phi^{1/0}_{\mathcal{D},p}[\mathcal{C}(\mathcal{D})] \ge \eta(L),$$

where $1/0$ denotes the boundary condition on $\mathcal{D}$, where the arcs (ab) and (cd) are wired and the rest of the boundary is free.

While the result is stated only for $p = p_c$ in [DMT20], the reader will easily check in the corresponding paper that the proof extends to the near-critical regime.

Proof of Claim 4.7. Write B_s for the euclidean ball of $\mathbb{R}^2$ of radius s centred at 0 and $B_s(z)$ for its translate by $z \in \mathbb{R}^2$. Let ψ be a conformal map from $\mathcal{F} \cap \overline{\mathcal{F}}_{\text{out}}$ to some $B_1 \setminus B_s$. The existence of such a map is given by the uniformisation theorem for the topological annulus $\mathcal{F} \cap \overline{\mathcal{F}}_{\text{out}}$. Note that s is determined by $\mathcal{F} \cap \overline{\mathcal{F}}_{\text{out}}$. Moreover, since $\partial\mathcal{F}$ intersects $\Lambda_{2\rho}$ and $\partial\overline{\mathcal{F}}$ is contained in $\text{Ann}(\rho, \frac{3}{2}\rho)$, s is bounded uniformly away from 0 and 1, and the endpoints of $P_1, \dots, P_4$ are far away from each other.

Fix some small positive constant $c_0 = c_0(\delta) < (1-s)/16$, which will be chosen below and will depend only on δ. Let $a_0, \dots, a_K = a_0$ (with $K = 2\pi/c_0$) be points on the circle ∂B_1 indexed in counterclockwise order and at a distance c_0 from each other. Write $(a_i a_{i+1})$ for the arc of ∂B_1 contained between a_i and a_{i+1}; identify $\psi^{-1}(a_i a_{i+1})$ to the corresponding vertices of $\partial\mathcal{F}$. Let $\mathcal{A}_i := \mathcal{A} \cap \psi^{-1}(a_i a_{i+1})$.

Next, we argue that there exist two indices i, j with $|i-j| \ge 8$ (modulo K) such that

$$\begin{aligned}\phi^{\zeta'}_{\mathcal{F}}[\mathcal{A}_i \longleftrightarrow \psi^{-1}(B_{2c_0}(a_i))^c] &\ge c_0\delta/2,\\ \phi^{\zeta'}_{\mathcal{F}}[\mathcal{A}_j \longleftrightarrow \psi^{-1}(B_{2c_0}(a_j))^c] &\ge c_0\delta/2.\end{aligned} \tag{4.7}$$

Indeed, if the above fails, then there necessarily exists i such that for all $j \notin \{i, \dots, i+7\}$,

$$\phi^{\zeta'}_{\mathcal{F}}[\mathcal{A}_j \longleftrightarrow \psi^{-1}(B_{2c_0}(a_j))^c] < c_0\delta/2.$$

Assuming that this is the case, explore the open cluster $\mathbf{C}$ of $\mathcal{A} \setminus \psi^{-1}(a_i a_{i+8})$. Due to the small value of c_0 and to the assumption above, we conclude by the union bound that

$$\phi^{\zeta'}_{\mathcal{F}}[\mathbf{C} \text{ intersects } \Lambda_\rho] < \delta/2. \tag{4.8}$$

If $\mathbf{C}$ does not intersect Λ_ρ, the measure on $\mathcal{F} \setminus \mathbf{C}$ obtained by conditioning on $\mathbf{C}$ has free boundary conditions for all vertices adjacent to $\mathbf{C}$. Then due to Proposition 4.8 applied repeatedly to the dual model, we conclude that for c_0 small enough,

$$\phi^{\zeta'}_{\mathcal{F}}[\mathcal{A} \longleftrightarrow \Lambda_\rho \,|\, \mathbf{C} \text{ does not intersect } \Lambda_\rho] < \delta/2. \tag{4.9}$$

Combining equations (4.8) and (4.9), we conclude that $\phi^{\zeta'}_{\mathcal{F}}[\mathcal{A} \leftrightarrow \Lambda_\rho] < \delta$, which contradicts the assumption that the stopping time at which $\mathcal{F}$ was discovered was promising. Thus the existence of i, j as above, satisfying equation (4.7), is proved.

For what comes next, we refer to Figure 9. Set

$$(D_k, a_k, b_k, c_k, d_k) := \psi^{-1}[(D'_k, a'_k, b'_k, c'_k, d'_k)],$$

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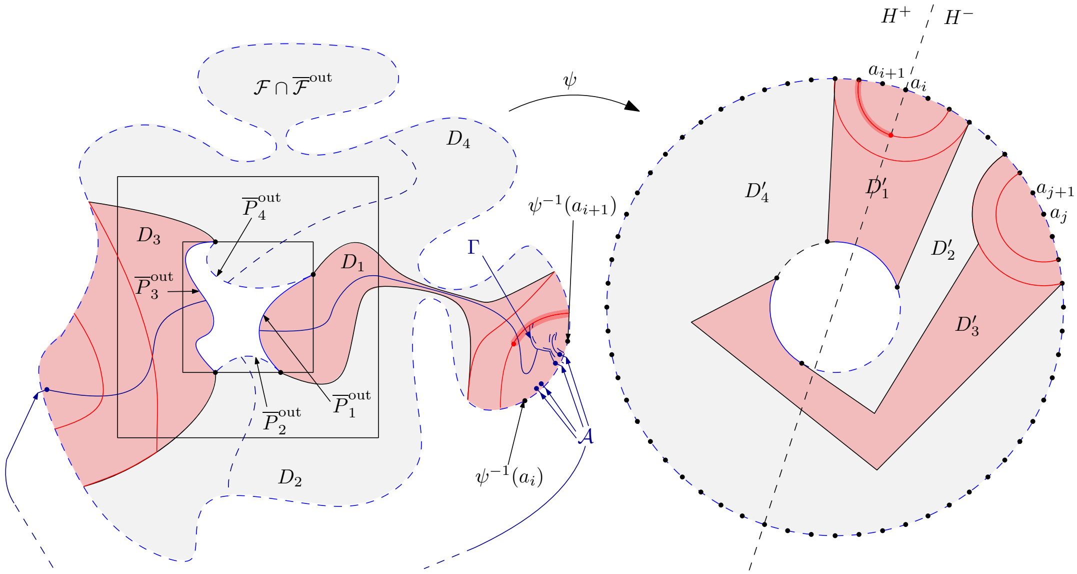

Figure 9. *The uniformisation map Ψ from $\mathcal{F}\cap\overline{\mathcal{F}}^{\mathrm{out}}$ into $B_1\setminus B_s$. On the right, the black dots denote the a_i. We depicted the balls $B_{2c_0}(a_i)$ and $B_{3c_0}(a_i)$. The four domains $D'_1,\dots,D'_4$ can be chosen in many ways. In brighter red, the path ℓ^+ and its preimage $\psi^{-1}(\ell^+)$. The path Γ from $\mathcal{A}$ into $\psi^{-1}(\ell^+)$ is drawn in dark blue.*

where the quads $(D'_k,a'_k,b'_k,c'_k,d'_k)$ satisfy:

- $D'_k\subset B_1\setminus B_s$ for every k;
- the extremal lengths $\ell_{D_k}[(a_kb_k),(c_kd_k)]$ belong to $(\kappa,1/\kappa)$ with $\kappa=\kappa(c_0)\in(0,\infty)$;
- $(a'_kb'_k)=\psi(\overline{P}_k^{\mathrm{out}})$;
- $(a'_kb'_k)$ is equal to $(a_{i-3}a_{i+3})$, $(a_{i+3}a_{j-3})$, $(a_{j-3}a_{j+3})$ and $(a_{j+3}a_{i-3})$ for $k=1,2,3,4$, respectively;
- D'_1 and D'_3 contain $B_{3c_0}(a_i)$ and $B_{3c_0}(a_j)$, respectively.

The construction of the domains $(D'_k,a'_k,b'_k,c'_k,d'_k)$ is straightforward.

We now check equations (4.5) and (4.6). We start with the latter and focus on the first inequality. The boundary conditions induced by $\{\omega_{[\sigma_0]},1\}$ on D_2 are free on $\overline{P}_2^{\mathrm{out}}$ and on $\psi^{-1}((a_{i+3}a_{j-3}))$. Therefore, the result follows directly from Proposition 4.8 and duality. We now turn to the former and focus on the first inequality. Let

$$\ell:=\partial\psi^{-1}(B_{2c_0}(a_i))^c\setminus\partial(\mathcal{F}\cap\overline{\mathcal{F}}^{\mathrm{out}}).$$

First, a mixing-type argument using Proposition 4.8 gives that

$$\phi_{D_1}^{\omega'_{[\sigma_0]},0}[\mathcal{A}_i\longleftrightarrow\ell]\geq c_2\,\phi_{\mathcal{F}}^{\zeta'}[\mathcal{A}_i\longleftrightarrow\ell]\overset{(4.7)}{\geq}c_3(\delta).$$

Second, if $\ell^\pm:=\ell\cap\psi^{-1}(H^\pm)$, where H^+ is the half-plane on the left of the line going from 0 to $\frac12(a_i+a_{i+1})$ (where left is understood when looking in the direction $\frac12(a_i+a_{i+1})$) and $H^-=\mathbb{C}\setminus H^+$, the union bound gives that for # equal to + or −,

$$\phi_{D_1}^{\omega'_{[\sigma_0]},0}[\mathcal{A}_i\longleftrightarrow\ell^\#]\geq\tfrac12\phi_{D_1}^{\omega'_{[\sigma_0]},0}[\mathcal{A}_i\longleftrightarrow\ell].$$

We assume below that # = + (the same can be done for # = −). Condition on the left-most path Γ going from $\mathcal{A}$ to ℓ^+. Then conditioned on Γ, the domain carved from D_1 by removing Γ and all the edges revealed to determine Γ has wired boundary conditions on $\overline{P}_1^{\text{out}}$, wired on Γ and boundary conditions dominating the free boundary conditions elsewhere. Since the extremal distance between $\overline{P}_1^{\text{out}}$ and Γ in this new domain is larger than $\kappa' = \kappa'(\kappa, \delta) > 0$, we deduce from Proposition 4.8 that

$$\phi_{D_1}^{\omega'_{[\sigma_0]},0}[\mathcal{A}_i \longleftrightarrow \overline{P}_1^{\text{out}} | \mathcal{A}_i \longleftrightarrow \ell^+] \geq c_4.$$

Combining the three last displayed equations gives the first inequality of equation (4.5) and therefore concludes the proof. □

4.2. *Mixing rate versus coupling*

In this section, we estimate the probability under the coupling of Theorem 4.1 that τ occurs. The relevant result of this section is the following theorem.

Theorem 4.9. *For any $p \in (0,1)$ and any $4r < R \leq L(p)$, the following holds. Fix $\mathcal{G}$ an η-well-separated inner flower domain on Λ_R or outer flower domain on Λ_r and (ξ, ξ') a boosting pair of boundary conditions on $\mathcal{G}$. Let $\mathbb{P}$ be the coupling of Theorem 4.1 (points (i) or (ii), depending on whether $\mathcal{G}$ is an inner or outer flower domain), and recall the stopping time τ associated to $\mathbb{P}$. Then*

$$\mathbb{P}[\tau < \infty] \asymp \Delta_p(r, R). \tag{4.10}$$

As a consequence, if $\mathcal{G}$ is an inner flower domain and H is either the crossing event of an η-regular quad at scale r or $H = A_{r/2}$, or if $\mathcal{G}$ is an outer flower domain and H is either the crossing event of an η-regular quad at scale R translated so that it is contained in $\text{Ann}(R, 2R)$ *or $H = A_R$, then*

$$\phi_{\mathcal{G}}^{\xi'}[H] - \phi_{\mathcal{G}}^{\xi}[H] \asymp \Delta_p(r, R). \tag{4.11}$$

When $\mathcal{G}$ is an inner flower domain, equation (4.11) should be understood as follows. Any boosting pair of boundary conditions at scale R boosts the probability of any crossing event at scale r by a quantity comparable to $\Delta_p(r, R)$. The same holds when the boundary conditions are at scale r and the crossing event is at scale R. Recall that $\Delta_p(r, R)$ was defined in terms of the boost that a specific pair of boundary conditions at scale R has on a specific crossing event at scale r. Thus, in addition to stating that the boost of any boosting pair of boundary conditions on any crossing event is comparable, the proposition also links the boost from outside in to that from inside out.

Remark 4.10. As with Theorem 4.1, Theorem 4.9 also applies to graphs containing $\mathcal{G} \setminus \Lambda_{r/2}$ and $\mathcal{G} \cap \Lambda_{2R}$, respectively. Note that the probability that $\tau < \infty$ is comparable to $\Delta_p(r, R)$, independently of the chosen graph. The proof only requires minor adaptations, which we omit here.

The rest of this section is dedicated to showing Theorem 4.9. The proof is split into several steps, each corresponding to a lemma. First, as a consequence of Theorem 4.1(i), we prove that the influence of boundary conditions at scale R on the crossing of any two regular quads at scale r is comparable.

Lemma 4.11. *Fix $p \in (0,1)$ and $2r < R \leq L(p)$. Let $\mathcal{G}$ be an η-well-separated inner flower domain on Λ_R and (ξ, ξ') be a boosting pair of boundary conditions on $\mathcal{G}$. Moreover, let H be either the crossing event of an η-regular quad at scale r or $H = A_{r/2}$. Then if $\mathbb{P}$ is the coupling of Theorem 4.1(i),*

$$\phi_{\mathcal{G}}^{\xi'}[H] - \phi_{\mathcal{G}}^{\xi}[H] \asymp \mathbb{P}[\tau < \infty] \asymp \phi_{\mathcal{G}}^{\xi'}[\mathcal{C}(\Lambda_r)] - \phi_{\mathcal{G}}^{\xi}[\mathcal{C}(\Lambda_r)]. \tag{4.12}$$

In particular,

$$\phi_{\Lambda_R}^1[H] - \phi_{\Lambda_R}^0[H] \asymp \Delta_p(r, R). \tag{4.13}$$

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A particular case of equation (4.13) shows that $\Delta_p(1,R) \asymp \Delta_p(R)$. In addition, by taking $H = \mathcal{C}(\Lambda_{r/2})$, equation (4.13) also proves that

$$\Delta_p(r,R) \asymp \Delta_p(r/2,R), \tag{4.14}$$

and more generally that replacing r by a multiple of r only affects Δ_p by a multiplicative constant.

Remark 4.12. As will be apparent from the proof, an equivalent of equation (4.12) may also be proved for outer flower domains on Λ_r and crossing events at scale R. However, an equivalent of equation (4.13) cannot be shown with the same proof, at least for now (but will be later).

Proof. Fix p, r, R, $\mathcal{G}$, ξ, ξ' and H as in the statement. Let $\mathbb{P}$ be the coupling of Theorem 4.1 (i) between $\phi^{\xi'}_{\Lambda_R}$ and $\phi^{\xi}_{\Lambda_R}$. Then using the notation of the theorem,

$$\phi^{\xi'}_{\mathcal{G}}[H] - \phi^{\xi}_{\mathcal{G}}[H] = \mathbb{P}[\omega' \in H, \omega \notin H] \le \mathbb{P}[\zeta \neq \zeta'] \lesssim \mathbb{P}[\tau < \infty].$$

Moreover, by Theorem 3.6,

$$\phi^{\xi'}_{\mathcal{G}}[H] - \phi^{\xi}_{\mathcal{G}}[H] \ge \mathbb{E}\big[\mathbb{1}_{\{\tau<\infty\}}\big(\phi^{\omega'_{[\tau]}}_{\mathcal{F}_\tau}[H] - \phi^{\omega_{[\tau]}}_{\mathcal{F}_\tau}[H]\big)\big] \gtrsim \mathbb{P}[\tau < \infty],$$

where $\mathbb{E}$ stands for the expectation associated to $\mathbb{P}$. The two displays above imply that

$$\phi^{\xi'}_{\mathcal{G}}[H] - \phi^{\xi}_{\mathcal{G}}[H] \asymp \mathbb{P}[\tau < \infty]. \tag{4.15}$$

Apply this to a generic H and to $H = \mathcal{C}(\Lambda_r)$ to obtain equation (4.12). Finally, equation (4.12) applied with $\mathcal{G} = \Lambda_R$, $\xi = 0$, $\xi' = 1$ gives equation (4.13). □

Next, we deduce an interpretation of $\Delta_p(r,R)$ as a covariance between events at scales r and R. Considering the symmetry between r and R below, this result is used to link the influence from outside in to that from inside out.

Lemma 4.13. *For every p and every $4r \le R \le L(p)$,*

$$\mathrm{Cov}[A_{r/2}, A_R] \asymp \phi^1_{\Lambda_r^c}[A_R] - \phi^0_{\Lambda_r^c}[A_R] \asymp \Delta_p(r,R). \tag{4.16}$$

Proof. We start by proving the equivalence between $\mathrm{Cov}(A_{r/2}, A_R)$ and $\Delta_p(r,R)$. By the monotonicity of boundary conditions in equation (CBC),

$$\mathrm{Cov}(A_{r/2}, A_R) = \phi[A_R](\phi[A_{r/2} \mid A_R] - \phi[A_{r/2}]) \le \phi^1_{\Lambda_R}[A_{r/2}] - \phi^0_{\Lambda_R}[A_{r/2}] \lesssim \Delta_p(r,R), \tag{4.17}$$

where the last inequality is due to Lemma 4.11.

For the converse bound, some additional work is needed. Let $\mathbb{P}$ denote the coupling between ϕ and $\phi[\cdot\,|A_{r/2}]$ inside $\Lambda_{R/2}$ produced by applying Proposition 3.9 and then Theorem 4.1(ii). Recall that ζ and ζ' denote the boundary conditions on $\Lambda^c_{R/2}$ induced by ω and ω', respectively. Complete the coupling outside of $\Lambda_{R/2}$ by an arbitrary increasing coupling. Then Theorems 3.6 and 4.1(ii) show that

$$\begin{aligned}\frac{1}{\phi[A_{r/2}]}\mathrm{Cov}(A_{r/2}, A_R) &= \mathbb{P}[\omega' \in A_R,\ \omega \notin A_R]\\ &\ge \mathbb{E}\big[\mathbb{1}_{\{\tau<\infty\}}\big(\phi^{\omega'_{[\tau]}}_{\mathcal{F}_\tau}[A_R] - \phi^{\omega_{[\tau]}}_{\mathcal{F}_\tau}[A_R]\big)\big]\\ &\gtrsim \mathbb{P}[\tau < \infty] \gtrsim \mathbb{P}[\zeta' \neq \zeta].\end{aligned} \tag{4.18}$$

Using that $\phi[A_{r/2}] \gtrsim 1$, we deducefrom the display above that $\mathrm{Cov}(A_{r/2}, A_R) \gtrsim \mathbb{P}[\zeta' \neq \zeta]$. Moreover, by equation (SMP),

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$$\begin{aligned}\mathbb{P}[\zeta' \neq \zeta] &\geq \mathbb{P}[\omega' \in A_{R/2}, \omega \notin A_{R/2}] \geq \mathrm{Cov}(A_{r/2}, A_{R/2}) \quad \text{and} \\ \mathbb{P}[\zeta' \neq \zeta] &\geq \mathbb{P}[\omega' \notin A^*_{R/2}, \omega \in A^*_{R/2}] \geq -\mathrm{Cov}(A_{r/2}, A^*_{R/2}),\end{aligned} \tag{4.19}$$

where $A^*_{R/2} = \{\Lambda_{R/2} \not\longleftrightarrow \partial\Lambda_R\}$ is the event $A_{R/2}$ applied to the dual model. Divide the equations above by $\phi[A_{R/2}]$ and $\phi[A^*_{R/2}]$, respectively, both of which are uniformly positive quantities. Using the monotonicity of boundary conditions in equations (CBC) and (4.13), we conclude that

$$\mathbb{P}[\zeta' \neq \zeta] \gtrsim \phi[A_{r/2} \,|\, A_{R/2}] - \phi[A_{r/2} \,|\, A^*_{R/2}] \geq \phi^1_{\Lambda_R}[A_{r/2}] - \phi^0_{\Lambda_R}[A_{r/2}] \gtrsim \Delta(r,R).$$

Together with equations (4.17) and (4.18), this shows that

$$\Delta(r,R) \lesssim \mathbb{P}[\zeta' \neq \zeta] \lesssim \mathrm{Cov}(A_{r/2}, A_R) \lesssim \Delta(r,R). \tag{4.20}$$

Finally, we turn to the equivalence between $\mathrm{Cov}(A_{r/2}, A_R)$ and $\phi^1_{\Lambda_r^c}[A_R] - \phi^0_{\Lambda_r^c}[A_R]$. The monotonicity of boundary conditions in equation (CBC) shows that

$$\mathrm{Cov}(A_{r/2}, A_R) \leq \phi[A_R \,|\, A_{r/2}] - \phi[A_R] \leq \phi^1_{\Lambda_r^c}[A_R] - \phi^0_{\Lambda_r^c}[A_R].$$

Conversely,

$$\begin{aligned}
\phi^1_{\Lambda_r^c}[A_R] - \phi^0_{\Lambda_r^c}[A_R] &\leq \phi[A_R \,|\, A_r] - \phi[A_R \,|\, A_r^*] && \text{by equation (CBC)} \\
&\leq (\phi[A_R \,|\, A_r] - \phi[A_R]) - (\phi[A_R \,|\, A_r^*] - \phi[A_R]) \\
&\lesssim \mathrm{Cov}(A_R, A_r) - \mathrm{Cov}(A_R, A_r^*) && \text{as } \phi[A_r] \gtrsim 1 \text{ and } \phi[A_r^*] \gtrsim 1 \\
&\lesssim (\phi[A_r \,|\, A_R] - \phi[A_r]) + (\phi[A_r^*] - \phi[A_r^* \,|\, A_R]) && \text{since } \phi[A_R] \leq 1 \\
&\lesssim \Delta_p(2r, R) && \text{by equation (CBC) and Lemma 4.11} \\
&\lesssim \Delta_p(r, R) && \text{by equation (4.14)} \\
&\lesssim \mathrm{Cov}[A_{r/2}, A_R] && \text{by equation (4.20).}
\end{aligned}$$

Note that in the second and third lines above, the second term is negative. The last two displays prove that $\mathrm{Cov}[A_{r/2}, A_R] \asymp \phi^1_{\Lambda_r^c}[A_R] - \phi^0_{\Lambda_r^c}[A_R]$. □

Notice that equations 4.17–4.20 also show that

$$\Delta_p(r,R) \gtrsim \mathrm{Cov}[A_{r/2}, A_R] \gtrsim \mathrm{Cov}[A_{r/2}, A_{R/2}] \gtrsim \Delta_p(r, R/2) \geq \Delta_p(r,R) \tag{4.21}$$

and more generally that replacing R by a constant multiple of R only affects Δ_p by a multiplicative constant.

Finally, we claim that all boosting pairs of boundary conditions at scale R influence $A_{r/2}$ by a similar amount; the same holds from the inside out.

Lemma 4.14. *Fix p and $4r \leq R \leq L(p)$. Then*

(i) *for any η-well-separated inner flower domain $\mathcal{G}$ on Λ_R and any boosting pair of boundary conditions ξ, ξ' on $\mathcal{G}$,*

$$\phi^{\xi'}_{\mathcal{G}}[A_{r/2}] - \phi^{\xi}_{\mathcal{G}}[A_{r/2}] \asymp \phi^1_{\Lambda_R}[A_{r/2}] - \phi^0_{\Lambda_R}[A_{r/2}]; \tag{4.22}$$

(ii) *for any η-well-separated outer flower domain $\mathcal{G}$ on Λ_r and any boosting pair of boundary conditions ξ, ξ' on $\mathcal{G}$,*

$$\phi^{\xi'}_{\mathcal{G}}[A_R] - \phi^{\xi}_{\mathcal{G}}[A_R] \asymp \phi^1_{\Lambda_r^c}[A_R] - \phi^0_{\Lambda_r^c}[A_R]. \tag{4.23}$$

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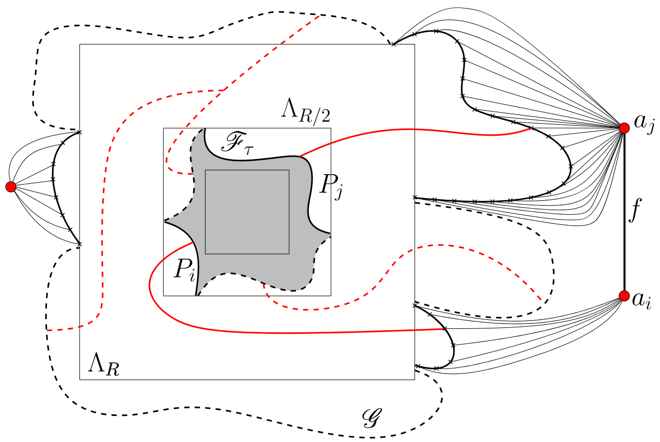

Figure 10. *The graph G. The black edges between the true vertices of the petals and the vertices in red mean they are all merged into the red vertex or, equivalently, that we added open edges between them and the red vertices. We also depicted the flower domains $\mathcal{G}$ and $\mathcal{F}_\tau$, as well as the event H that corresponds to the occurrence of the red paths.*

Proof. We will only treat point (i) as point (ii) is identical.

By the monotonicity of boundary conditions,

$$\phi_{\mathcal{G}}^{\xi'}[A_{r/2}] - \phi_{\mathcal{G}}^{\xi}[A_{r/2}] \leq \phi_{\Lambda_R}^{1}[A_{r/2}] - \phi_{\Lambda_R}^{0}[A_{r/2}].$$

We turn to the converse bound. We recommend taking a look at Figure 10. By monotonicity, we may assume that ξ and ξ' are both coherent with $\mathcal{G}$, and that there exists exactly one pair of sets in the partition ξ that are wired together in ξ' (due to the coherence condition, each such set contains at least one primal petal of $\mathcal{G}$). Consider the graph G obtained from $\mathcal{G}$ as follows. All vertices contained in a non-singleton set of the partition ξ are collapsed to a single point (in particular, all points on each primal petal of $\mathcal{G}$ are collapsed together). Write $a_1, \ldots, a_k$ for the points thus obtained. Then there exist two distinct points a_i and a_j such that the corresponding groups of petals are wired in ξ'. Finally, G is the graph obtained after the collapsing procedure described above, with an additional edge f between a_i and a_j. There is an obvious correspondence between the edges of $\mathcal{G}$ and those of $G \setminus \{f\}$, and we will identify them from now on. Let ϕ_G be the random-cluster measure on the finite graph G (note that G is not a subgraph of $\mathbb{Z}^2$).

With the construction above, $\phi_{\mathcal{G}}^{\xi}$ and $\phi_{\mathcal{G}}^{\xi'}$ are simply the restrictions of $\phi_G[\cdot|\omega_f = 0]$ and $\phi_G[\cdot|\omega_f = 1]$, respectively, to $\mathcal{G}$. As such, $\phi_G[A_{r/2}] \gtrsim 1$ and Bayes's formula imply that

$$\begin{aligned}\phi_{\mathcal{G}}^{\xi'}[A_{r/2}] - \phi_{\mathcal{G}}^{\xi}[A_{r/2}] &= \phi_G[A_{r/2}|\omega_f = 1] - \phi_G[A_{r/2}|\omega_f = 0] \\ &\geq \phi_G[A_{r/2}|\omega_f = 1] - \phi_G[A_{r/2}] \\ &\gtrsim \phi_G[\omega_f = 1|A_{r/2}] - \phi_G[\omega_f = 1]. \end{aligned} \tag{4.24}$$

Now let $\mathbb{P}$ be the coupling between ϕ_G and $\phi_G[\cdot|A_{r/2}]$ obtained as follows:

- reveal the edges of Λ_{2r} in the order dictated by Proposition 3.9. If this stage produces an outer flower domain with a boosting pair of boundary conditions, apply Theorem 4.1(ii) from scale $2r$ to scale $R/2$, up to the associated stopping time τ (see Remark 4.3 about applying Theorem 4.1(ii) in G);
- reveal all remaining edges of $\mathcal{G}$;
- reveal the state of f.

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If $\tau < \infty$, there exist two primal petals P_i and P_j of $\mathcal{F}_\tau$ that are wired in $\omega'_{[\tau]}$ but not in $\omega_{[\tau]}$. Arbitrarily choose one such pair of petals. Let $H = H(\mathcal{F}_\tau, \omega_{[\tau]}, \omega'_{[\tau]})$ be the event that a_i is connected to P_i in $\omega \cap \mathcal{F}_\tau$, that a_j is connected to P_j in $\omega \cap \mathcal{F}_\tau$ but that a_i and a_j are not connected to each other or any other primal petal of $\mathcal{F}_\tau$ in $\omega \cap \mathcal{F}_\tau$.

Since $\mathcal{F}_\tau$ is $1/2$-well-separated and $\mathcal{G}$ is η-well-separated, and since $\mathcal{F}_\tau \cap \mathcal{G}$ contains the annulus $\mathrm{Ann}(R/2, R)$, by standard applications of equation (RSW), we find that

$$\mathbb{P}(H \,|\, \tau < \infty, \mathcal{F}_\tau, \omega_{[\tau]}, \omega'_{[\tau]}) \gtrsim 1.$$

The occurrence of H may be determined before revealing the state of the edge f. Moreover, if H occurs, then the endpoints a_i and a_j of f are connected in $\omega' \setminus \{f\}$ but not in $\omega \setminus \{f\}$. Indeed, ω' dominates ω, which implies that a_i is connected to P_i (in $\omega' \cap \mathcal{F}_\tau$), which is connected to P_j (in $\omega'_{[\tau]}$), which in turn is connected to a_j (in $\omega' \cap \mathcal{F}_\tau$).

It follows that at the last step of the coupling, if $\tau < \infty$ and H occurs, there is a probability $\frac{p(1-p)(q-1)}{p+(1-p)q} > 0$ that f is closed in ω but open in ω'. To summarise, we find

$$\phi_G[\omega_f = 1|A_{r/2}] - \phi_G[\omega_f = 1] = \mathbb{P}[\omega_f = 0, \omega'_f = 1] \gtrsim \mathbb{P}[H \text{ and } \tau < \infty] \gtrsim \mathbb{P}[\tau < \infty]. \quad (4.25)$$

Finally, by Theorem 4.1(ii), the fact that $\phi_G[A_{R/2}] \gtrsim 1$ and the comparison between boundary conditions in equation (CBC), we have

$$\begin{aligned} \mathbb{P}[\tau < \infty] &\gtrsim \mathbb{P}[\omega \text{ and } \omega' \text{ induce different b.c. on } \Lambda^c_{R/2}] \quad (4.26)\\ &\geq \mathbb{P}[\omega' \in A_{R/2} \text{ and } \omega \notin A_{R/2}] \\ &\gtrsim \mathrm{Cov}_G(A_{r/2}, A_{R/2}) \gtrsim \phi^1_{\Lambda_R}[A_{r/2}] - \phi^0_{\Lambda_R}[A_{r/2}], \end{aligned}$$

where Cov_G is the covariance under ϕ_G and the last inequality is given by equations (4.13) and (4.16). Equations (4.24)-(4.26) prove

$$\phi^{\xi'}_{\mathcal{G}}[A_{r/2}] - \phi^{\xi}_{\mathcal{G}}[A_{r/2}] \gtrsim \phi^1_{\Lambda_R}[A_{r/2}] - \phi^0_{\Lambda_R}[A_{r/2}],$$

as desired. □

We are finally ready to prove Theorem 4.9.

Proof of Theorem 4.9. We start with the case where $\mathcal{G}$ is an inner flower domain on Λ_R. Recall that ξ and ξ' form a boosting pair of boundary conditions on $\mathcal{G}$ and that τ is the stopping time associated to the coupling of Theorem 4.1 between $\phi^{\xi'}_{\mathcal{G}}$ and $\phi^{\xi}_{\mathcal{G}}$. Due to Lemma 4.11, we find that

$$\mathbb{P}[\tau < \infty] \asymp \phi^{\xi'}_{\mathcal{G}}[A_{r/2}] - \phi^{\xi}_{\mathcal{G}}[A_{r/2}].$$

Now, Lemma 4.14 and equation (4.13) indicate that the right-hand side above is of order $\Delta_p(r, R)$, and equation (4.10) is proved in this case. Finally, equation (4.11) follows from another application of Lemma 4.11.

Similarly, when $\mathcal{G}$ is an outer flower domain on Λ_r, due to Lemma 4.11 applied from inside to outside (see Remark 4.12), then to Lemmata 4.14 and 4.13, respectively,

$$\mathbb{P}[\tau < \infty] \asymp \phi^{\xi'}_{\mathcal{G}}[A_R] - \phi^{\xi}_{\mathcal{G}}[A_R] \asymp \phi^1_{\Lambda^c_r}[A_R] - \phi^0_{\Lambda^c_r}[A_R] \asymp \Delta_p(r, R).$$

Another application of Lemma 4.11 also proves equation (4.11) in this case. □

4.3. Δ_p controls the mixing rate: proof of Theorem 1.6(v)

For the lower bound, equation (4.16) gives that

$$\max\left\{\left|\frac{\phi_p[A\cap B]}{\phi_p[A]\phi_p[B]}-1\right| : A\in\mathcal{F}(\Lambda_r), B\in\mathcal{F}(\mathbb{Z}^2\setminus\Lambda_R)\right\}\geq\frac{\phi_p[A_{r/2}\cap A_R]}{\phi_p[A_{r/2}]\phi_p[A_R]}-1\gtrsim\Delta_p(r,R).$$

For the upper bound, use the spatial Markov property given by equation (SMP) to get that for every $A\in\mathcal{F}(\Lambda_r)$ and $B\in\mathcal{F}(\mathbb{Z}^2\setminus\Lambda_R)$,

$$\left|\frac{\phi_p[A\cap B]}{\phi_p[A]\phi_p[B]}-1\right|\leq\max\left\{\left|\frac{\phi^{\xi'}_{\Lambda_R,p}[A]}{\phi^{\xi}_{\Lambda_R,p}[A]}-1\right| : \xi,\xi' \text{ b.c. on } \partial\Lambda_R\right\}. \tag{4.27}$$

Now, consider the coupling $\mathbb{P}$ between ω^0 and ω^1 constructed in Theorem 4.9 on $\Lambda_R\setminus\Lambda_{2r}$ and boundary conditions 0 and 1, respectively, on Λ_R (which form a boosting pair). By applying the same decision tree for the boundary conditions ξ and ξ', we obtain two additional configurations $\omega^{\xi},\omega^{\xi'}$ with laws $\phi^{\xi}_{\Lambda_R}$ and $\phi^{\xi'}_{\Lambda_R}$, respectively, such that $\omega^0\leq\omega^{\xi}\leq\omega^1$ and $\omega^0\leq\omega^{\xi'}\leq\omega^1$. If ζ and ζ' are the boundary conditions induced on $\partial\Lambda_{2r}$ by ω^0 and ω^1, we see that

$$\phi^{\xi'}_{\Lambda_R,p}[A]-\phi^{\xi}_{\Lambda_R,p}[A]\leq\mathbb{P}[\zeta\neq\zeta']\max_{\psi}\phi^{\psi}_{\Lambda_{2r},p}[A]\lesssim\Delta_p(r,R)\phi^{\xi}_{\Lambda_R,p}[A],$$

where in the second inequality we used Theorems 4.1 and 4.9 as well as the mixing property to replace $\phi^{\psi}_{\Lambda_{2r},p}[A]$ by $\phi^{\xi}_{\Lambda_R,p}[A]$. This concludes the proof. □

4.4. Quasi-multiplicativity of Δ_p: proof of Theorem 1.6(ii)

Let $\mathbb{P}$ be the coupling between $\phi^1_{\Lambda_R}$ and $\phi^0_{\Lambda_R}$ in $\mathrm{Ann}(n,R)$ given by Theorem 4.1(i), and let ζ and ζ' be the boundary conditions induced by ω and ω' on $\partial\Lambda_n$. Complete the coupling inside Λ_n by an arbitrary increasing coupling of $\phi^{\zeta}_{\Lambda_n}$ and $\phi^{\zeta'}_{\Lambda_n}$. Write $\mathbb{E}$ for the expectation associated to $\mathbb{P}$.

On the one hand, the definition of $\Delta_p(r,n)$, Theorems 4.1 and 4.9 and the comparison between boundary conditions in equation (CBC) yield

$$\begin{aligned}\Delta_p(r,R)&=\mathbb{E}[\mathbb{1}_{\{\zeta\neq\zeta'\}}(\phi^{\zeta'}_{\Lambda_n}[\mathcal{C}(\Lambda_r)]-\phi^{\zeta}_{\Lambda_n}[\mathcal{C}(\Lambda_r)])]\\&\leq\mathbb{P}[\zeta\neq\zeta'](\phi^{1}_{\Lambda_n}[\mathcal{C}(\Lambda_r)]-\phi^{0}_{\Lambda_n}[\mathcal{C}(\Lambda_r)])\\&\lesssim\Delta_p(n,R)\Delta_p(r,n).\end{aligned}$$

On the other hand, recall that when $\tau<\infty$, $\mathcal{F}_\tau$ is a 1/2-well-separated inner flower domain on Λ_n and that $\omega_{[\tau]}$ and $\omega'_{[\tau]}$ induce a boosting pair of boundary conditions on $\mathcal{F}_\tau$. Thus, Theorem 4.9 implies

$$\begin{aligned}\Delta_p(r,R)&\geq\mathbb{E}[\mathbb{1}_{\{\tau<\infty\}}(\phi^{\omega'_{[\tau]}}_{\mathcal{F}_\tau}[\mathcal{C}(\Lambda_r)]-\phi^{\omega_{[\tau]}}_{\mathcal{F}_\tau}[\mathcal{C}(\Lambda_r)])]\\&\gtrsim\mathbb{P}[\tau<\infty]\Delta_p(n,R)\\&\gtrsim\Delta_p(r,n)\Delta_p(n,R).\end{aligned}$$

□

4.5. Mixing rate versus pivotality: proof of Theorem 1.6(iv)

This section concerns the proof of equation (1.17). This property is not crucial to the rest of the paper (its use in the proof of Lemma 5.3 may be avoided), but it illustrates that Kesten's scaling relation does not extend with $\pi_4(p;r,R)$ instead of $\Delta_p(r,R)$.

Remark 4.15. Note that the weaker bound $\Delta_p(r,R)\gtrsim\pi_4(p;r,R)$ follows readily from Theorems 4.1 and 4.9. Thus, the polynomial improvement of $(R/r)^c$ is the core of equation (1.17).

We will use the following notation. Let $r < R$, $\mathcal{F}$ be an outer flower domain on Λ_r and $\mathcal{G}$ be an inner flower domain on Λ_R, each containing exactly four petals denoted by $P_1,\dots,P_4$ and $P'_1,\dots,P'_4$, respectively. Then ξ^1 (respectively, ξ^0) are the boundary conditions on $\mathcal{F}$, which are coherent with its flower domain structure and in which the primal petals P_1 and P_3 are wired (respectively, not wired) together. The similarly defined boundary conditions on $\mathcal{G}$ are written ζ^1 and ζ^0, respectively.

For $i,j\in\{0,1\}$, denote by $\phi^{\xi^i\cup\zeta^j}_{\mathcal{F}\cap\mathcal{G}}$ the measure on the subgraph $\mathcal{F}\cap\mathcal{G}$ with the boundary condition $\xi^i\cup\zeta^j$, which is the partition of $\partial(\mathcal{F}\cap\mathcal{G})=\partial\mathcal{F}\cup\partial\mathcal{G}$ given by the union of the partitions ξ^i of the inner boundary and ζ^j of the outer one.

For configurations on $\mathcal{F}\cap\mathcal{G}$, define the events

$$\begin{aligned}A_4(\mathcal{F},\mathcal{G}) &:= \{P_1 \overset{\omega}{\longleftrightarrow} P'_1,\ P_3 \overset{\omega}{\longleftrightarrow} P'_3,\ P_2 \overset{\omega^*}{\longleftrightarrow} P'_2, P_4 \overset{\omega^*}{\longleftrightarrow} P'_4\},\\ \tilde{A}_4(\mathcal{F},\mathcal{G}) &:= \{P_1 \overset{\omega'}{\longleftrightarrow} P'_1,\ P_3 \overset{\omega'}{\longleftrightarrow} P'_3,\ P_2 \overset{\omega^*}{\longleftrightarrow} P'_2, P_4 \overset{\omega^*}{\longleftrightarrow} P'_4\}.\end{aligned}$$

Also write $\phi^{\xi^i\cup 1}_{\mathcal{F}\cap\Lambda_R}$ and $\phi^{\xi^i\cup 0}_{\mathcal{F}\cap\Lambda_R}$ for the measures on $\mathcal{F}\cap\Lambda_R$ with boundary conditions ξ^i on $\partial\mathcal{F}$ and, respectively, wired and free boundary conditions on $\partial\Lambda_R$. Define $A_4(\mathcal{F},R)$ and $\tilde{A}_4(\mathcal{F},R)$ as in the last display, with $P'_1,\dots,P'_4$ all replaced by $\partial\Lambda_R$.

The following lemma states that the probability of the four-arm event $A_4(\mathcal{F},\mathcal{G})$ increases substantially if we allow the primal arms to be in ω' rather than in ω. It is particularly important in the lemma below that $\mathcal{F}$ and $\mathcal{G}$ are not assumed to be well-separated.

Lemma 4.16. *There exists $\delta>0$ such that the following holds. For any $p\in(0,1)$ and $1\le r\le L(p)$, any outer flower domain $\mathcal{F}$ on Λ_r and any inner flower domain $\mathcal{G}$ on Λ_{2r}, both containing exactly four petals, and any $i\in\{0,1\}$, there exists a coupling $\mathbb{P}$ of $\phi^{\xi^i\cup\zeta^0}_{\mathcal{F}\cap\mathcal{G}}$ and $\phi^{\xi^i\cup\zeta^1}_{\mathcal{F}\cap\mathcal{G}}$ such that*

$$\mathbb{P}[\tilde{A}_4(\mathcal{F},\mathcal{G})]\ge(1+\delta)\mathbb{P}[A_4(\mathcal{F},\mathcal{G})]. \tag{4.28}$$

Proof. Fix p, r, i, $\mathcal{F}$ and $\mathcal{G}$ as above. We will first treat the case where both flower domains are well-separated and then use it to solve the general case.

Proof when $\mathcal{F}$ and $\mathcal{G}$ are $1/2$*-well-separated.* Let $x=(3n/2,0)$, and start the coupling by exploring the double four-petal flower domain $(\mathcal{F}_{\text{in}},\mathcal{F}_{\text{out}})$ between $\Lambda_{n/8}(x)$ and $\Lambda_{n/4}(x)$ in ω. If this stage fails, reveal the rest of the configuration in an arbitrarily increasing fashion. If $(\mathcal{F}_{\text{in}},\mathcal{F}_{\text{out}})$ exists, continue by revealing the configuration inside $\mathcal{F}_{\text{in}}$. Write $\xi^0_{\text{in}}<\xi^1_{\text{in}}$ for the two boundary conditions on $\mathcal{F}_{\text{in}}$, which are coherent with the flower domain structure.

We now use the same argument as in the proof of equation (3.8) to study the connection probability between P^{in}_1 and P^{in}_3 inside $\mathcal{F}_{\text{in}}$ in ω and ω'. The conditional law of ω in $\mathcal{F}_{\text{in}}$ is

$$(1-\lambda)\phi^{\xi^0_{\text{in}}}_{\mathcal{F}_{\text{in}}}+\lambda\phi^{\xi^1_{\text{in}}}_{\mathcal{F}_{\text{in}}}\qquad\text{with}\qquad \lambda:=\phi^{\xi^i\cup\zeta^0}_{\mathcal{F}\cap\mathcal{G}}[P^{\text{out}}_1\overset{\mathcal{F}_{\text{out}}}{\longleftrightarrow}P^{\text{out}}_3\,|\,\mathcal{F}_{\text{in}},\mathcal{F}_{\text{out}}],$$

while that of ω' dominates

$$(1-\lambda')\phi^{\xi^0_{\text{in}}}_{\mathcal{F}_{\text{in}}}+\lambda'\phi^{\xi^i_{\text{in}}}_{\mathcal{F}_{\text{in}}}\qquad\text{with}\qquad \lambda':=\phi^{\xi^i\cup\zeta^1}_{\mathcal{F}\cap\mathcal{G}}[P^{\text{out}}_1\overset{\mathcal{F}_{\text{out}}}{\longleftrightarrow}P^{\text{out}}_3\,|\,\mathcal{F}_{\text{in}},\mathcal{F}_{\text{out}}].$$

Thus, $\lambda'-\lambda$ may be lower bounded by the probability that P^{out}_1 and P^{out}_3 are connected in $\mathcal{F}_{\text{out}}$ to the two primal petals of $\mathcal{G}$ but not to each other. By equation (RSW), we conclude that $\lambda'-\lambda\gtrsim1$ and by Lemma 3.7 that

$$\mathbb{P}[\omega'\in\{P^{\text{in}}_1\overset{\mathcal{F}_{\text{in}}}{\longleftrightarrow}P^{\text{in}}_3\}\text{ but }\omega\notin\{P^{\text{in}}_1\overset{\mathcal{F}_{\text{in}}}{\longleftrightarrow}P^{\text{in}}_3\}\,|\,\mathcal{F}_{\text{in}},\mathcal{F}_{\text{out}}]\gtrsim1. \tag{4.29}$$

Finally, reveal the configurations on $\mathcal{F}_{\text{out}}$.

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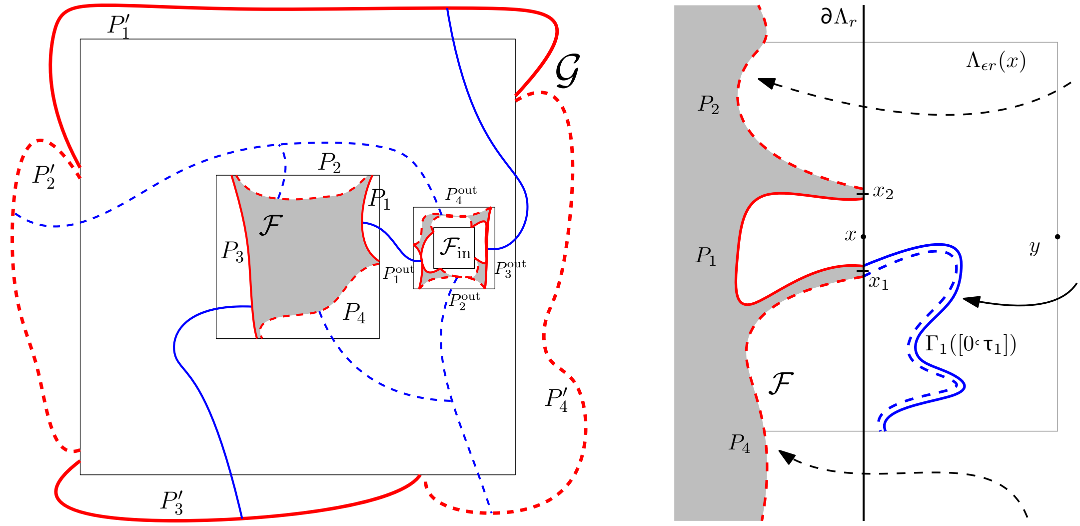

Figure 11. *Left: In the graph $\mathcal{F}\cap\mathcal{G}$, we first explore the double four-petal flower domain $(\mathcal{F}_{\text{in}},\mathcal{F}_{\text{out}})$ and then reveal the configurations in $\mathcal{F}_{\text{in}}$ and $\mathcal{F}_{\text{out}}$. If H occurs (see the blue paths), then $A_4(\mathcal{F},\mathcal{G})$ depends on the connection inside $\mathcal{F}_{\text{in}}$ between its primal petals. If this connection occurs in ω' but not in ω, then the configurations are in $\tilde{A}_4(\mathcal{F},\mathcal{G})\setminus A_4(\mathcal{F},\mathcal{G})$. Right: When P_1 is small, exploring one of the interfaces Γ_1 or Γ_2 produces a long arc that renders the probability that P_1 is connected to I_1 uniformly positive. In the picture, this is done by exploring Γ_1. However, if Γ_1 typically exits $\Lambda_{\varepsilon r}(x)$ to the right of y, we would explore Γ_2.*

Let H be the event that in $\mathcal{F}_{\text{out}}$ (see Figure 11, left diagram for an illustration):

- P_1^{out} is connected to P_1 in ω,
- P_3^{out} is connected to P_1' in ω,
- P_3 is connected to P_3' in ω,
- P_2, P_2' and P_4^{out} are connected in ω^* and
- P_4, P_4' and P_2^{out} are connected in ω^*.

In other words, H is the event that the connection between P_1^{in} and P_3^{in} inside $\mathcal{F}_{\text{in}}$ is pivotal for $A_4(\mathcal{F},\mathcal{G})$. Then equation (RSW) and the well-separation of $\mathcal{F}$, $\mathcal{F}_{\text{out}}$ and $\mathcal{G}$ imply that

$$\mathbb{P}[H\,|\,\mathcal{F}_{\text{in}},\mathcal{F}_{\text{out}}\ \text{and}(\omega,\omega')\ \text{on}\ \mathcal{F}_{\text{in}}]\gtrsim 1. \tag{4.30}$$

Now, since $A_4(\mathcal{F},\mathcal{G})$ implies the occurrence of $\tilde{A}_4(\mathcal{F},\mathcal{G})$, we conclude that

$$\mathbb{P}[\tilde{A}_4(\mathcal{F},\mathcal{G})]-\mathbb{P}[A_4(\mathcal{F},\mathcal{G})]\geq\mathbb{P}[(\mathcal{F}_{\text{in}},\mathcal{F}_{\text{out}})\ \text{exist},\ P_1^{\text{out}}\overset{\mathcal{F}_{\text{in}}}{\longleftrightarrow}P_3^{\text{out}}\ \text{in}\ \omega'\ \text{but not in}\ \omega,\ H].$$

Finally, equations (4.29) and (4.30) and Lemma 3.4 together imply that the right-hand side is larger than $c_0>0$. This concludes the proof of equation (4.28) when $\mathcal{F}$ and $\mathcal{G}$ are $1/2$-well-separated.

Proof when $\mathcal{F}$ and $\mathcal{G}$are not $1/2$*-well-separated.* We will construct the coupling $\mathbb{P}$ between $\phi^{\xi^i,\zeta^0}_{\mathcal{F}\cap\mathcal{G}}$ and $\phi^{\xi^i,\zeta^1}_{\mathcal{F}\cap\mathcal{G}}$ in two steps. We start by exploring the outer flower domain $\overline{\mathcal{F}}$ between Λ_r and $\Lambda_{5r/8}$ in ω and the inner flower domain $\overline{\mathcal{G}}$ from Λ_{2r} to $\Lambda_{7r/8}$ also in ω. Say that $(\overline{\mathcal{F}},\overline{\mathcal{G}})$ is *good* if

- $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ each contain exactly four petals and are $1/2$-well-separated;
- the four petals of $\overline{\mathcal{F}}$ are connected in $\mathcal{F}\setminus\overline{\mathcal{F}}$ to the corresponding petals of $\mathcal{F}$ by paths of alternating types in ω;

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- the four petals of $\overline{\mathcal{G}}$ are connected in $\mathcal{G} \setminus \overline{\mathcal{G}}$ to the corresponding petals of $\mathcal{G}$ by paths of alternating types in ω.

The following claim is a standard but tedious consequence of equation (RSW), similar to the separation of arms. We will sketch its proof below after completing that of Lemma 4.16.

Claim 4.17. *There exists a universal constant $c > 0$ such that*

$$\mathbb{P}[(\overline{\mathcal{F}}, \overline{\mathcal{G}}) \text{ good}] \geq c\mathbb{P}[A_4(\mathcal{F}, \mathcal{G})], \tag{4.31}$$

where the constant in $\gtrsim$ does not depend on $\mathcal{F}$ and $\mathcal{G}$.

If $(\overline{\mathcal{F}}, \overline{\mathcal{G}})$ is not good, complete (ω, ω') inside $\overline{\mathcal{F}} \cap \overline{\mathcal{G}}$ using an arbitrary increasing coupling. When $(\overline{\mathcal{F}}, \overline{\mathcal{G}})$ is good, the first case may be applied to $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ and provides a way to complete (ω, ω') inside $\overline{\mathcal{F}} \cap \overline{\mathcal{G}}$ so that

$$\mathbb{P}[\tilde{A}_4(\overline{\mathcal{F}}, \overline{\mathcal{G}}) \setminus A_4(\overline{\mathcal{F}}, \overline{\mathcal{G}}) \,|\, (\overline{\mathcal{F}}, \overline{\mathcal{G}}) \text{ good}] \gtrsim 1.$$

Moreover, notice that in this case, $A_4(\mathcal{F}, \mathcal{G})$ is equivalent to $A_4(\overline{\mathcal{F}}, \overline{\mathcal{G}})$ and $\tilde{A}_4(\mathcal{F}, \mathcal{G})$ to $\tilde{A}_4(\overline{\mathcal{F}}, \overline{\mathcal{G}})$. By summing the display above, we conclude that

$$\mathbb{P}[\tilde{A}_4(\mathcal{F}, \mathcal{G})] - \mathbb{P}[A_4(\mathcal{F}, \mathcal{G})] \gtrsim \mathbb{P}[(\overline{\mathcal{F}}, \overline{\mathcal{G}}) \text{ good}].$$

The above, together with Claim 4.17, provide the desired conclusion. □

Finally, we turn to the proof of Claim 4.17. As already mentioned, this proof is quite tedious but is only based on the RSW theory and uses somewhat standard techniques. We sketch it below.

Proof of Claim 4.17. Note that all events depend exclusively on ω, and we will ignore the configuration ω' henceforth. Write $\phi = \phi^{\xi^i \cup \zeta^0}_{\mathcal{F} \cap \mathcal{G}}$ for the law of ω.

Consider the interfaces $\Gamma_1, \dots, \Gamma_4$ separating the primal clusters of the primal petals of $\mathcal{F}$ from the dual clusters of its dual petals. View these interfaces as directed paths[4] starting from the points $x_1, \dots, x_4$ that separate the petals of $\mathcal{F}$ and indexed by $t \geq 0$. When $A_4(\mathcal{F}, \mathcal{G})$ occurs, these interfaces end on $\partial\mathcal{G}$ at the points separating its petals.

For times $t_1, \dots, t_4$, the portions of interfaces $\Gamma_1([0, t_1]), \dots, \Gamma_4([0, t_4])$ may be explored via a standard procedure, and the measure in $\mathcal{F} \cap \mathcal{G} \setminus \big(\Gamma_1([0, t_1]) \cup \dots \cup \Gamma_4([0, t_4])\big)$ is given by equation (SMP). Fix $\varepsilon = \frac{1}{128}$, and set $r' := (1 + 2\varepsilon)r$. Let $I_1 := \{r'\} \times [-r'/2, r'/2]$ be an interval along the right side of the box $\Lambda_{r'}$. Call I_2, I_3 and I_3 the rotations of I_1 by angles $\pi/2$, π and $3\pi/2$, respectively. For $c > 0$, we say that $\Gamma_1([0, t_1]), \dots, \Gamma_4([0, t_4])$ c-expose $\mathcal{F}$ if

$$\phi\Big[P_1 \xleftrightarrow{\omega \cap \Lambda_{r'}} I_1,\ P_2 \xleftrightarrow{\omega^* \cap \Lambda_{r'}} I_2,\ P_3 \xleftrightarrow{\omega \cap \Lambda_{r'}} I_3,\ P_4 \xleftrightarrow{\omega^* \cap \Lambda_{r'}} I_4 \,\Big|\, \Gamma_1([0, t_1]), \dots, \Gamma_4([0, t_4])\Big] \geq c.$$

We claim that for some universal $c > 0$, there exist stopping times $\tau_1, \dots, \tau_4$ such that $\Gamma_1([0, \tau_1]), \dots, \Gamma_4([0, \tau_4])$ are all contained in $\Lambda_{r'}$ and

$$\phi\big[\Gamma_1([0, \tau_1]), \dots, \Gamma_4([0, \tau_4]) \text{ } c\text{-expose } \mathcal{F} \,\big|\, A_4(\mathcal{F}, \mathcal{G})\big] \geq c. \tag{4.32}$$

The above should be understood as a separation statement. Indeed, it essentially states that conditionally on the interfaces $\Gamma_1, \dots, \Gamma_4$ reaching $\partial\mathcal{G}$, one may find a time when they are well-separated.

Notice that if $\mathcal{F}$ is $\varepsilon/4$-well-separated, then the above is trivial, as we may take $\tau_1 = \dots = \tau_4 = 0$ and adjust c. Thus, we only need to consider the case where at least one of the petals of $\mathcal{F}$ is small.

[4]Formally, these paths use edges of the medial lattice of $\mathbb{Z}^2$. They may be parametrised to travel along each edge in one unit of time.

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To start, assume that P_1 has endpoints x_1 and x_2 at a distance at most $\varepsilon r/4$ from each other, while all other petals have well-separated endpoints (say, at a distance $2\varepsilon r$ from each other). For simplicity of exposition, assume that both endpoints of P_1 are on the right side of Λ_r. We will then take $\tau_3 = \tau_4 = 0$ and either $\tau_1 = 0$ or $\tau_2 = 0$, as described below.

Let x denote the centre of the segment $[x_1, x_2]$. Let $y = x + (\varepsilon r, 0)$ be the midpoint of the arc $\partial\Lambda_{\varepsilon r}(x) \cap \Lambda_r^c$. Let $\tilde{\tau}_1$ be the first time when Γ_1 exits $\Lambda_{\varepsilon r}(x)$ and $\tilde{\tau}_2$ be the first time when Γ_2 exits $\Lambda_{\varepsilon r}(x)$. Observe that due to our assumption, Γ_1 and Γ_2 both start inside $\Lambda_{\varepsilon r/4}(x)$. Formally, Γ_1 may end at x_2 and never exit $\Lambda_{\varepsilon r}(x)$ – in this case, Γ_2 is the reverse of Γ_1, and we set $\tilde{\tau}_1 = \tilde{\tau}_2 = \infty$. For $A_4(\mathcal{F}, \mathcal{G})$ to occur, it is necessary that $\tilde{\tau}_1 < \infty$ and $\tilde{\tau}_2 < \infty$.

Now, when $\tilde{\tau}_1 < \infty$ and $\tilde{\tau}_2 < \infty$, $\Gamma_1(\tilde{\tau}_1)$ is a point on $\partial\Lambda_{\varepsilon r}(x)$ to the right of $\Gamma_2(\tilde{\tau}_2)$ – see Figure 11, right diagram. Thus, we have

$$\phi[\Gamma_1(\tilde{\tau}_1) \text{ right of } y \,|\, A_4(\mathcal{F}, \mathcal{G})] \geq \tfrac{1}{2} \quad \text{or} \quad \phi[\Gamma_2(\tilde{\tau}_2) \text{ left of } y \,|\, A_4(\mathcal{F}, \mathcal{G})] \geq \tfrac{1}{2}.$$

If the former is valid, take $\tau_1 = \tilde{\tau}_1$ and $\tau_2 = 0$. Otherwise, take $\tau_1 = 0$ and $\tau_2 = \tilde{\tau}_2$.

Assume that we are in the first case. When $\Gamma_1(\tau_1)$ is indeed to the right of y, equation (RSW) proves that $\Gamma_1([0, \tau_1]), \ldots, \Gamma_4([0, \tau_4])$ c-expose $\mathcal{F}$ for some universal constant $c > 0$; see also Figure 11, right diagram.

This proves equation (4.32) under the particular assumption that P_1 is small, while the other petals are big. To fully prove equation (4.32), one should also consider the cases where more than one petal is small. These may be solved by performing the same procedure two or three times, with radii $r' \gg r'' \gg r'''$. We leave the details of these to the reader.

Call $\overline{\mathcal{F}}$ good if it is 1/2-well-separated and has exactly four petals that are connected to the corresponding ones of $\mathcal{F}$ in $\mathcal{F} \setminus \overline{\mathcal{F}}$. A similar definition may be given for $\overline{\mathcal{G}}$, so that $(\overline{\mathcal{F}}, \overline{\mathcal{G}})$ is good if and only if both $\overline{\mathcal{F}}$ and $\overline{\mathcal{G}}$ are good. Now, applying Lemma 3.4 in $\mathrm{Ann}(\frac{9}{16}r, \frac{5}{8}r)$ and using equation (RSW), we deduce the existence of a universal constant $c_1 > 0$ such that

$$\phi[\overline{\mathcal{F}} \text{ is good} \,|\, \Gamma_1([0, \tau_1]), \ldots, \Gamma_4([0, \tau_4]) \ c\text{-expose } \mathcal{F}] > c_1. \tag{4.33}$$

Combining the above with equation (4.32), we conclude that

$$\phi[\overline{\mathcal{F}} \text{ is good} \,|\, A_4(\mathcal{F}, \mathcal{G})] > c_2,$$

for some universal constant $c_2 > 0$. Apply the same argument for $\mathcal{G}$ to show that

$$\phi[\overline{\mathcal{G}} \text{ is good} \,|\, A_4(\mathcal{F}, \mathcal{G}),\ \overline{\mathcal{F}} \text{ is good}] > c_2.$$

Combining the last two inequalities, we obtain the desired conclusion. □

The following is a consequence of Lemma 4.16.

Corollary 4.18. *For ρ large enough, $p \in (0,1)$ and $r < R \leq L(p)$ with $R = (\rho^2 + 2)^k r$, any outer flower domain $\mathcal{F}$ on Λ_r with exactly four petals and any $i \in \{0, 1\}$, there exists an increasing coupling $\mathbb{P}$ of $\phi^{\xi^i \cup 0}_{\mathcal{F} \cap \Lambda_R}$ and $\phi^{\xi^i \cup 1}_{\mathcal{F} \cap \Lambda_R}$ such that*

$$\mathbb{P}[\tilde{A}_4(\mathcal{F}, R)] \geq (1 + \tfrac{\delta}{2})^k\, \mathbb{P}[A_4(\mathcal{F}, R)], \tag{4.34}$$

where $\delta > 0$ is the constant given by Lemma 4.16.

Proof. The proof proceeds by induction on k. Consider the value of $\rho > 1$ fixed; it will be chosen at the end of the proof, and it will be apparent that it is independent of p, r, R or $\mathcal{F}$. The case $k = 0$ is trivially true.

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Fix $k \geq 0$, and assume that equation (4.34) holds for this value of k; we will now prove equation (4.34) for $k+1$. Let $\mathcal{F}$ be an inner flower domain on Λ_r, and fix boundary conditions $\xi \in \{\xi^0, \xi^1\}$ on $\partial\mathcal{F}$. The coupling $\mathbb{P}$ is built in three steps:

Step 1: Explore the inner flower domain from $\partial\Lambda_{2r\rho}$ to $\partial\Lambda_{2r}$ and the outer one from $\partial\Lambda_{2r\rho}$ to $\partial\Lambda_{2r\rho^2}$ in ω; call these $\mathcal{G}_{\rm in}$ and $\mathcal{G}_{\rm out}$, respectively. We will abuse notation by identifying $(\mathcal{G}_{\rm in}, \mathcal{G}_{\rm out})$ with the entire configuration (ω, ω') on $\mathcal{G}_{\rm in}^c \cap \mathcal{G}_{\rm out}^c$. Say that $(\mathcal{G}_{\rm in}, \mathcal{G}_{\rm out})$ is *good* if both $\mathcal{G}_{\rm in}$ and $\mathcal{G}_{\rm out}$ have exactly four petals $P_1^{\rm in}, \dots, P_4^{\rm in}$ and $P_1^{\rm out}, \dots, P_4^{\rm out}$, respectively, and if in $\omega \cap (\mathcal{G}_{\rm in}^c \cap \mathcal{G}_{\rm out}^c)$ there exist open paths connecting $P_1^{\rm in}$ to $P_1^{\rm out}$ and $P_3^{\rm in}$ to $P_3^{\rm out}$ and dual-open paths connecting $P_2^{\rm in}$ to $P_2^{\rm out}$ and $P_4^{\rm in}$ to $P_4^{\rm out}$. As before, we use the notation $\zeta_{\rm in}^0, \zeta_{\rm in}^1$ for the two boundary conditions on $\mathcal{G}_{\rm in}$ coherent with it being a flower domain and $\zeta_{\rm out}^0$ and $\zeta_{\rm out}^1$ for those on $\mathcal{G}_{\rm out}$.

Stage 2: If $(\mathcal{G}_{\rm in}, \mathcal{G}_{\rm out})$ is not good, we sample the rest of the configurations according to an arbitrary increasing coupling. If $(\mathcal{G}_{\rm in}, \mathcal{G}_{\rm out})$ is good, observe that the law of ω in $\mathcal{G}_{\rm out}$, conditionally on the revealed set, is a linear combination

$$(1-\lambda)\phi_{\mathcal{G}_{\rm out}\cap\Lambda_R}^{\zeta_{\rm out}^0\cup 0} + \lambda\phi_{\mathcal{G}_{\rm out}\cap\Lambda_R}^{\zeta_{\rm out}^1\cup 0} \qquad \text{where} \qquad \lambda := \phi_{\mathcal{F}\cap\Lambda_R}^{\xi\cup 0}[P_1^{\rm in} \overset{\mathcal{G}_{\rm in}}{\longleftrightarrow} P_3^{\rm in} \,|\, (\mathcal{G}_{\rm in}, \mathcal{G}_{\rm out})].$$

Moreover, the conditional law of ω' dominates

$$(1-\lambda)\phi_{\mathcal{G}_{\rm out}\cap\Lambda_R}^{\zeta_{\rm out}^0\cup 1} + \lambda\phi_{\mathcal{G}_{\rm out}\cap\Lambda_R}^{\zeta_{\rm out}^1\cup 1}, \tag{4.35}$$

for the same value λ.

Since $\tilde{A}_4(\mathcal{G}_{\rm out}, R)$ is increasing in ω' and decreasing in ω, we may use equation (4.34) for k (which is our induction hypothesis) applied between $\mathcal{G}_{\rm out}$ and Λ_R, once with the boundary conditions $\zeta_{\rm out}^0$ on $\mathcal{G}_{\rm out}$ and once with the boundary conditions $\zeta_{\rm out}^1$, to produce an increasing coupling of ω and ω' on $\mathcal{G}_{\rm out}$ so that

$$\mathbb{P}[\tilde{A}_4(\mathcal{G}_{\rm out}, R) | \,(\mathcal{G}_{\rm in}, \mathcal{G}_{\rm out}) \text{ good}] \geq (1+\tfrac{\delta}{2})^k\, \mathbb{P}[A_4(\mathcal{G}_{\rm out}, R) | \,(\mathcal{G}_{\rm in}, \mathcal{G}_{\rm out}) \text{ good}]. \tag{4.36}$$

Step 3: Finally, we sample (ω, ω') in $\mathcal{G}_{\rm in} \cap \mathcal{F}$ according to a specific coupling between the measures in this region, with boundary conditions induced by the previously revealed parts of ω and ω', respectively. Recall that this stage is reached only if $(\mathcal{G}_{\rm in}, \mathcal{G}_{\rm out})$ is good and that the revealed configurations suffice to decide whether $\tilde{A}_4(\mathcal{G}_{\rm out}, R)$ occurred or not.

If $\tilde{A}_4(\mathcal{G}_{\rm out}, R)$ did not occur, complete the coupling in an arbitrary increasing way. If $\tilde{A}_4(\mathcal{G}_{\rm out}, R)$ did occur, then the boundary conditions induced by the already revealed parts of ω on $\mathcal{G}_{\rm in}$ are equal to $\zeta_{\rm in}^0$. Indeed, in $\omega \cap \mathcal{G}_{\rm out}$, $P_1^{\rm out}$ and $P_3^{\rm out}$ are disconnected from each other due to the dual arms. On the contrary, the boundary conditions induced by the revealed region of ω' on $\mathcal{G}_{\rm in}$ dominate $\zeta_{\rm in}^1$, since in $\omega' \cap \mathcal{G}_{\rm out}$, $P_1^{\rm out}$ and $P_3^{\rm out}$ are connected to $\partial\Lambda_R$, which is wired. Thus, we may apply Lemma 4.16 to continue the coupling so that

$$\begin{aligned}&\mathbb{P}[\tilde{A}_4(\mathcal{F}, \mathcal{G}_{\rm in}) | \,(\mathcal{G}_{\rm in}, \mathcal{G}_{\rm out}) \text{ good and } \tilde{A}_4(\mathcal{G}_{\rm out}, R)] \\ &\quad \geq (1+\delta)\mathbb{P}[A_4(\mathcal{F}, \mathcal{G}_{\rm in}) | \,(\mathcal{G}_{\rm in}, \mathcal{G}_{\rm out}) \text{ good and } \tilde{A}_4(\mathcal{G}_{\rm out}, R)].\end{aligned} \tag{4.37}$$

This concludes the construction of the coupling $\mathbb{P}$; next we show that $\mathbb{P}$ satisfies equation (4.34).

If $(\mathcal{G}_{\rm in}, \mathcal{G}_{\rm out})$ is good and $\tilde{A}_4(\mathcal{G}_{\rm out}, R)$ and $\tilde{A}_4(\mathcal{F}, \mathcal{G}_{\rm in})$ both occur, then so does $\tilde{A}_4(\mathcal{F}, R)$. By equations (4.36) and (4.37) and the fact that $(\mathcal{G}_{\rm in}, \mathcal{G}_{\rm out})$ are determined by ω alone, we find

$$\begin{aligned}&\mathbb{P}[\tilde{A}_4(\mathcal{F}, R)] \\ &\geq (1+\delta)(1+\tfrac{\delta}{2})^k \sum_{(\mathcal{G}_{\rm in}, \mathcal{G}_{\rm out}) \text{ good}} \phi_{\mathcal{F}}^{\xi\cup 0}[(\mathcal{G}_{\rm in}, \mathcal{G}_{\rm out})]\phi_{\mathcal{F}}^{\xi\cup 0}[A_4(\mathcal{G}_{\rm out}, R) \cap A_4(\mathcal{F}, \mathcal{G}_{\rm in}) | \,(\mathcal{G}_{\rm in}, \mathcal{G}_{\rm out})]\end{aligned}$$

$$= (1+\delta)(1+\tfrac{\delta}{2})^k \sum_{(\mathcal{G}_{\rm in},\mathcal{G}_{\rm out})\text{ good}} \phi_{\mathcal{F}}^{\xi\cup 0}[(\mathcal{G}_{\rm in},\mathcal{G}_{\rm out})]\phi_{\mathcal{F}}^{\xi\cup 0}[A_4(\mathcal{F},R)|\,(\mathcal{G}_{\rm in},\mathcal{G}_{\rm out})]$$
$$= (1+\delta)(1+\tfrac{\delta}{2})^k \phi_{\mathcal{F}}^{\xi\cup 0}[A_4(\mathcal{F},R)\text{ and }(\mathcal{G}_{\rm in},\mathcal{G}_{\rm out})\text{ good}]. \tag{4.38}$$

The first equality is due to the fact that conditionally on a good $(\mathcal{G}_{\rm in},\mathcal{G}_{\rm out})$, $A_4(\mathcal{F},R)$ occurs if and only if both $A_4(\mathcal{G}_{\rm out},R)$ and $A_4(\mathcal{F},\mathcal{G}_{\rm in})$ do. The last equality is obtained directly by summation.

Observe now that for $A_4(\mathcal{F},R)$ to occur, $\mathcal{G}_{\rm in}$ and $\mathcal{G}_{\rm out}$ need to have at least four petals each. Moreover, if they each have exactly four petals, then these need to be connected in such a way that $(\mathcal{G}_{\rm in},\mathcal{G}_{\rm out})$ is good. Thus

$$\phi_{\mathcal{F}}^{\xi\cup 0}[A_4(\mathcal{F},R),(\mathcal{G}_{\rm in},\mathcal{G}_{\rm out})\text{ not good}] \le \phi_{\mathcal{F}}^{\xi\cup 0}[A_4(\mathcal{F},R),\mathcal{G}_{\rm in}\text{ has at least 6 petals}]$$
$$+\phi_{\mathcal{F}}^{\xi\cup 0}[A_4(\mathcal{F},R),\mathcal{G}_{\rm out}\text{ has at least 6 petals}]. \tag{4.39}$$

We will argue that both terms on the right-hand side are small compared to the quantity $\phi_{\mathcal{F}}^{\xi,0}[A_4(\mathcal{F},R)]$, provided ρ is large enough.

Indeed, due to the quasi-multiplicativity of the four- and six-arm events (the former applied to the slightly unusual event $A_4(\mathcal{F},R)$) and the mixing property given by equation (Mix), there exists a universal constant $C<\infty$ (that does not depend on $\mathcal{F}$, p, r or R) such that

$$\phi_{\mathcal{F}}^{\xi\cup 0}[A_4(\mathcal{F},R),\mathcal{G}_{\rm in}\text{ has at least 6 petals}] \le C\phi_{\mathcal{F}}^{\xi\cup 0}[A_4(\mathcal{F},2r)]\phi_{\mathcal{F}}^{\xi\cup 0}[A_4(2\rho r,R)]\pi_6(2r,2\rho r)$$
$$\le C^2\phi_{\mathcal{F}}^{\xi\cup 0}[A_4(\mathcal{F},R)]\frac{\pi_6(2r,2\rho r)}{\pi_4(2r,2\rho r)}.$$

Due to equation (RSW), $\rho>0$ may be chosen independently of $\mathcal{F}$, p, r or R so that

$$C^2\frac{\pi_6(2r,2\rho r)}{\pi_4(2r,2\rho r)} \le \delta/4.$$

Assuming this is the case, and by the same reasoning for the second term of equation (4.39), we find

$$(1+\delta)\phi_{\mathcal{F}}^{\xi,0}\big[A_4(\mathcal{F},R),(\mathcal{G}_{\rm in},\mathcal{G}_{\rm out})\text{ good}\big] \ge (1+\delta/2)\phi_{\mathcal{F}}^{\xi,0}\big[A_4(\mathcal{F},R)\big],$$

which, when inserted in equation (4.38), proves equation (4.34) for $R=(\rho^2+2)^{k+1}r$. □

Proof of (1.17). Fix $\delta>0$ and ρ given by Corollary 4.18. Due to Theorem 1.6(ii) and equation (2.3), it suffices to prove the statement for $R=2(\rho^2+2)^k r$. From now on, fix such values r and R.

We use the same notation as in the proof of Corollary 4.18 and construct a coupling $\mathbb{P}$ between $\phi^0_{\Lambda_R}$ and $\phi^1_{\Lambda_R}$ similar to that of the previous proof.

First, explore the double four-petal flower domain between Λ_r and Λ_{2r}; call it $(\mathcal{G}_{\rm in},\mathcal{G}_{\rm out})$. If no such double flower domain exists, proceed with an arbitrary coupling. If $(\mathcal{G}_{\rm in},\mathcal{G}_{\rm out})$ exists, use the coupling provided by Corollary 4.18 to complete (ω,ω') in $\mathcal{G}_{\rm out}$ so that

$$\mathbb{P}[\tilde{A}_4(\mathcal{G}_{\rm out},R)\,|\,(\mathcal{G}_{\rm in},\mathcal{G}_{\rm out})] \ge (1+\tfrac{\delta}{2})^k\,\mathbb{P}[A_4(\mathcal{G}_{\rm out},R)\,|\,(\mathcal{G}_{\rm in},\mathcal{G}_{\rm out})]. \tag{4.40}$$

Finally, use an arbitrary increasing coupling inside $\mathcal{G}_{\rm in}$.

When $(\mathcal{G}_{\rm in},\mathcal{G}_{\rm out})$ exists and $\tilde{A}_4(\mathcal{G}_{\rm out},R)$ occurs, the boundary conditions imposed by the revealed portions of ω and ω' are equal to $\zeta^0_{\rm in}$ and dominate $\zeta^1_{\rm in}$, respectively. Using Theorem 3.6 and Lemma 3.4, we conclude that

$$\Delta_p(r,R) = \mathbb{P}\big[\omega'\in\mathcal{C}(\Lambda_r),\omega\notin\mathcal{C}(\Lambda_r)\big]$$
$$\gtrsim \mathbb{P}\big[\tilde{A}_4(\mathcal{G}_{\rm out},R),\ (\mathcal{G}_{\rm in},\mathcal{G}_{\rm out})\text{ exists}\big]$$

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$$\gtrsim (1+\tfrac{\delta}{2})^k \mathbb{P}\big[A_4(\mathcal{G}_{\rm out}, R),\ (\mathcal{G}_{\rm in}, \mathcal{G}_{\rm out}) \text{ exists}\big]$$
$$\gtrsim (R/2r)^{\varepsilon} \pi_4(r,R),$$

where $\varepsilon = \log(1+\frac{\delta}{2})/\log(\rho^2+2) > 0$. In the last inequality, we also used the so-called separation of arms for the four-arm event. □

5. Derivatives in terms of Δ_p

5.1. Derivatives for crossing and arm events

In this section, we obtain expressions for the derivatives of probabilities of crossing events and arm events in terms of Δ_p. In addition, we upper-bound the derivative of the mixing rate Δ_p with similar expressions. The relevant results are Propositions 5.1 and 5.9, respectively. These will hold within the critical window and are instrumental in proving the main stability results, Theorem 1.4 and equation (1.16).

We start with a proposition that states a slightly weaker form of Corollary 1.7 but extends the expression to logarithmic derivatives of probabilities of arm events.

Proposition 5.1. *Fix $\eta > 0$. For $p \in (0,1)$ and every η-regular quad $(\mathcal{D}, a, b, c, d)$ at scale $R \le L(p)$,*

$$R^2\Delta_p(R) + \sum_{\ell=R}^{L(p)} \ell\Delta_p(\ell)\Delta_p(R,\ell) \lesssim \tfrac{\mathrm{d}}{\mathrm{d}p}\phi_p[\mathcal{C}(\mathcal{D})] \lesssim \sum_{\ell=1}^{R} \ell\Delta_p(\ell) + \sum_{\ell=R}^{L(p)} \ell\Delta_p(\ell)\Delta_p(R,\ell), \tag{5.1}$$

where the constants in $\lesssim$ depend on η. Moreover, for $k = 1$ or $k \ge 2$ even and any $r \le R \le L(p)$,

$$\big|\tfrac{\mathrm{d}}{\mathrm{d}p} \log \pi_k(p; r, R)\big| \lesssim \sum_{\ell=1}^{R} \ell\Delta_p(\ell) + \sum_{\ell=R}^{L(p)} \ell\Delta_p(\ell)\Delta_p(R,\ell),$$

where the constants in $\lesssim$ depend on k.

After proving Proposition 1.8 in the next section (which states that $\Delta_p(r,R) \gtrsim (r/R)^{2-c}$ inside the critical window), we may use the quasi-multiplicativity of Δ_p to replace the first term $\sum_{\ell\le R} \ell\Delta_p(\ell)$ in the upper bounds above by $R^2\Delta_p(R)$, hence deducing an up-to-constant estimate on the derivative. This is stated in Corollary 7.1.

Remark 5.2. The formula given by equation (5.1) for the derivative of the crossing probability of $\mathcal{D}$ is significantly different than the one for percolation, since edges far from $\mathcal{D}$ – that correspond to the second term in the formula – may contribute substantially. Indeed, if we accept the asymptotic $\Delta(r,R) \asymp (r/R)^{\iota}$, and if $\iota < 1$, then the edges at distance $L(p)$ contribute most to equation (5.1); however, when $\iota > 1$, the derivative is governed by the contributions of edges close to $\mathcal{D}$. See Section 8.2.4 for the consequences of these two types of behaviour.

The proof of Proposition 5.1 is based on the following lemma, which controls the influence of each edge on a crossing event. Below, we call $\Lambda_r(e)$ a *pivotal box* in ω for the crossing event in $(\mathcal{D}, a, b, c, d)$ if $\omega \cup \Lambda_r(e)$ contains a crossing of $\mathcal{D}$ from (ab) to (cd) but $\omega \setminus \Lambda_r(e)$ does not; call this event $\mathrm{Piv}_{r,e}(\mathcal{D})$. Here $\omega \cup \Lambda_r(e)$ and $\omega \setminus \Lambda_r(e)$ are the configurations obtained from ω by opening and closing, respectively, all edges in $\Lambda_r(e)$.

Lemma 5.3. *There exists $c > 0$ such that for any $\eta > 0$, the following holds. For $p \in (0,1)$, $R \le L(p)$, every η-regular quad $(\mathcal{D}, a, b, c, d)$ of size R and every edge e at a distance n from $\partial\mathcal{D}$,*

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$$\Delta_p(R) \lesssim \mathrm{Cov}_p[\omega_e;\mathcal{C}(\mathcal{D})] \lesssim \sum_{r=n/2}^{4R} \frac{\Delta_p(r)}{r}\phi_p[\mathrm{Piv}_{r,e}(\mathcal{D})] \qquad \text{if } n \le 2R, \tag{5.2}$$

$$\mathrm{Cov}_p[\omega_e;\mathcal{C}(\mathcal{D})] \asymp \Delta_p(n)\Delta_p(R,n) \qquad \text{if } 2R \le n \le 2L(p), \tag{5.3}$$

$$\mathrm{Cov}_p[\omega_e;\mathcal{C}(\mathcal{D})] \lesssim \Delta_p(L(p))\Delta_p(R,L(p))e^{-cn/L(p)} \qquad \text{if } n \ge 2L(p), \tag{5.4}$$

where the constants in $\asymp$ and $\lesssim$ depend on η.

For $k = 1$ or $k \ge 2$ even, the same upper bounds hold for $|\mathrm{Cov}_p[\omega_e; A_k(r,R)]|/\phi_p[A_k(r,R)]$ instead of $\mathrm{Cov}_p[\omega_e;\mathcal{C}(\mathcal{D})]$, with n replaced by the distance from e to $\partial\mathrm{Ann}(r,R)$ and with constants that depend on k.

Before proving this lemma and explaining how it implies Theorem 1.6(i), we mention a few important remarks.

Remark 5.4. Due to the choice of $\mathcal{D}$, all edges e inside $\mathcal{D}$ are in case equation (5.2). By equation (RSW), $\phi_p[\mathrm{Piv}_{r,e}(\mathcal{D})]$ is uniformly bounded away from 0 for all $R \le r \le 2R$. These terms of the sum account for a contribution of order at least $\Delta_p(R)$ to the right-hand side of equation (5.2). When e is in the 'bulk' of $\mathcal{D}$ (that is, at a distance of order R from $\mathcal{D}^c$), the lower and upper bounds in equation (5.2) are comparable.

Remark 5.5. The same results hold (the proof below adapts readily) for covariances under a measure ϕ_G^ξ for any subgraph G of $\mathbb{Z}^2$ that contains Λ_{2R}, any boundary conditions ξ and any edge e closer to $\mathcal{D}$ than to G^c. Similarly, it also applies to the measure on a torus of size at least $2R$.

Remark 5.6. The proof of equation (5.3) can be readily adapted to get the following: for every p and $3r \le |x| \le L(p)$,

$$\mathrm{Cov}_p[\mathcal{C}(\mathcal{D}),\mathcal{C}(x+\mathcal{D}')] \asymp \Delta_p(r,|x|)^2, \tag{5.5}$$

where $(\mathcal{D},a,b,c,d)$ and $(\mathcal{D}',a',b',c',d')$ are two η-regular quads of size r.

Remark 5.7. Interestingly, equation (5.2) enables us to deduce some sharp estimates for e near $\partial\mathcal{D}$. For instance, in the case where $e \in \partial\mathcal{D}$ is at a distance ηR of a change of direction of $\partial\mathcal{D}$, one may use that the probability of $\mathrm{Piv}_{r,e}(\mathcal{D})$ is of the order of the scale-to-scale three-arm event on the boundary: that is, of order $(r/R)^2$. Then Proposition 1.8 implies that $\mathrm{Cov}_p[\omega_e;\mathcal{C}(\mathcal{D})] \asymp \Delta_p(R)$. Note that this is quite different from the case of $q = 1$, where the terms involved in the derivative are quite different for edges near $\partial\mathcal{D}$. Also note that we are not claiming that $\mathrm{Cov}_p[\omega_e;\mathcal{C}(\mathcal{D})]$ is always of the order of $\Delta_p(R)$ as e could be close to a corner or a wedge of the domain, in which case the estimate could break.

Theorem 1.6(i) is a particular case of Lemma 5.3.

Proof of Theorem 1.6(i). To get the first estimate, apply equation (5.2), quasi-multiplicativity, and observe that the right-hand side is smaller than $\sum_{r=\eta R}^{R}\frac{1}{r}\Delta_p(r) \lesssim \Delta_p(R)$. To get the second and third estimates, apply equations 5.3–5.4 and the quasi-multiplicativity of Δ_p. □

Proof of Lemma 5.3. We will prove formulas for the crossing of a quad $(\mathcal{D},a,b,c,d)$; the proof for arm events is similar and entails standard adaptations. Write $\mathcal{C}$ instead of $\mathcal{C}(\mathcal{D})$. Recall that

$$\mathrm{Cov}[\omega_e;\mathcal{C}] = \phi[\omega_e](\phi[\mathcal{C}\,|\,\omega_e=1]-\phi[\mathcal{C}]) \asymp \phi[\mathcal{C}\,|\,\omega_e=1]-\phi[\mathcal{C}\,|\,\omega_e=0].$$

We will prove each of the equations separately, starting with the second.

Proof of equation (5.3). Construct an increasing coupling $\mathbb{P}$ between $\phi[\cdot\,|\,\omega_e=0]$ and $\phi[\cdot\,|\,\omega_e=1]$ producing configurations ω and ω' as follows: the coupling contains three stages based on Theorem 4.1(ii), Lemma 3.13 and Theorem 4.9, respectively. We refer to Figure 12 for a picture.

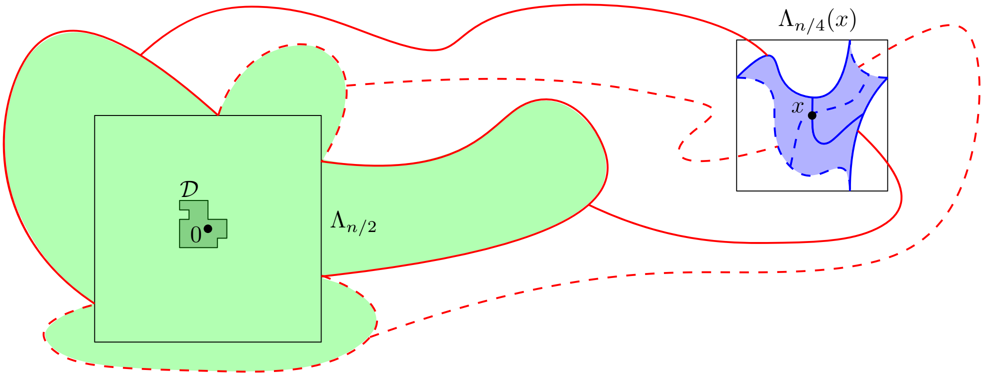

Figure 12. *In blue, the edges discovered until τ_1. In red, the edges discovered between τ_1 and τ_2. Finally, in green, the edges discovered afterwards. Note that the domain $\mathcal{D}$ is a priori much smaller than the box $\Lambda_{n/2}$.*

1. Apply the coupling inside $\Lambda_{n/4}(e)$ between $\phi[\cdot\,|\,\omega_e = 0]$ and $\phi[\cdot\,|\,\omega_e = 1]$ using the procedure of Theorem 4.1(ii), up to the associated stopping time, which we denote by τ_1. If $\tau_1 = \infty$, continue the coupling using an arbitrary increasing coupling. Recall that when $\tau_1 < \infty$, $\mathcal{F}_{\tau_1}$ is a 1/2-well-separated outer flower domain on $\Lambda_{n/4}$ and that $\omega_{[\tau_1]}$ and $\omega'_{[\tau_1]}$ induce a boosting pair of boundary conditions on $\mathcal{F}_{\tau_1}$.
2. Continue the coupling on $\Lambda^c_{n/2}$ between $\phi^{\omega'_{[\tau_1]}}_{\mathcal{F}_{\tau_1}}$ and $\phi^{\omega_{[\tau_1]}}_{\mathcal{F}_{\tau_1}}$ using the procedure of Lemma 3.13, up to the associated stopping time, which we denote by τ_2. If $\tau_2 = \infty$, continue the coupling using an arbitrary increasing coupling. Recall that when $\tau_2 < \infty$, $\mathcal{F}_{\tau_2}$ is a 1/2-well-separated inner flower domain on $\Lambda_{n/2}$ and that $\omega_{[\tau_2]}$ and $\omega'_{[\tau_2]}$ induce a boosting pair of boundary conditions on $\mathcal{F}_{\tau_2}$.
3. Complete the coupling inside $\mathcal{F}_{\tau_2}$ using an arbitrary increasing coupling of $\phi^{\omega'_{[\tau_2]}}_{\mathcal{F}_{\tau_2}}$ and $\phi^{\omega_{[\tau_2]}}_{\mathcal{F}_{\tau_2}}$.

For the upper bound, observe that since $\mathcal{D} \subset \Lambda_{n/2}$ and $\Lambda_{n/2}(e)$ are disjoint, if ζ and ζ' are the boundary conditions induced on $\mathcal{F}_{\tau_1}$, Theorem 4.9 gives

$$\phi[\mathcal{C}\,|\,\omega_e = 1] - \phi[\mathcal{C}\,|\,\omega_e = 0] \le (\phi^1_{\Lambda_{n/2}}[\mathcal{C}] - \phi^0_{\Lambda_{n/2}}[\mathcal{C}])\,\mathbb{P}[\zeta \ne \zeta'] \lesssim \Delta_p(R, n/2)\Delta_p(n). \tag{5.6}$$

Finally, $\Delta_p(R, n/2)$ may be replaced by $\Delta_p(R, n)$ due to equation (4.21). For the lower bound, Theorem 4.9 gives

$$\begin{aligned}\phi[\mathcal{C}\,|\,\omega_e = 1] - \phi[\mathcal{C}\,|\,\omega_e = 0] &= \mathbb{P}[\omega' \in \mathcal{C}, \omega \notin \mathcal{C}]\\ &\ge \mathbb{E}\big[\mathbb{1}_{\{\tau_2<\infty\}}\big(\phi^{\omega'_{[\tau_2]}}_{\mathcal{F}_{\tau_2}}[\mathcal{C}] - \phi^{\omega_{[\tau_2]}}_{\mathcal{F}_{\tau_2}}[\mathcal{C}]\big)\big]\\ &\gtrsim \mathbb{P}[\tau_1 < \infty]\,\mathbb{P}[\tau_2 < \infty\,|\,\tau_1 < \infty]\,\Delta_p(R, n/2).\end{aligned}$$

The second term above is of constant order by Lemma 3.13. The first term is of order $\Delta_p(1, n/4)$ by Theorem 4.9. Since $\Delta_p(1, n/4) \ge \Delta_p(n)$ and $\Delta_p(R, n/2) \ge \Delta_p(R, n)$, we obtain the result. ⋄ □

Proof of equation (5.4). We focus here on the case $p < p_c$; when $p > p_c$, the proof is obtained by duality. Construct the following coupling $\mathbb{P}$ between $\phi_p[\cdot\,|\,\omega_e = 0]$ and $\phi_p[\cdot\,|\,\omega_e = 1]$:

1. Use the coupling given by Theorem 4.1(ii) between $\phi_p[\cdot\,|\,\omega_e = 0]$ and $\phi_p[\cdot\,|\,\omega_e = 1]$ inside $\Lambda_{L(p)}(e)$; let ξ and ξ' be the boundary conditions induced by the revealed configurations ω and ω' on the boundary of $\mathbb{Z}^2 \setminus \Lambda_{L(p)}(e)$.

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2. Continue the coupling in $\mathbb{Z}^2 \setminus \Lambda_{L(p)}$ using the decision tree that explores the connected components of $\Lambda_{L(p)}(e)$ in ω' (see Example 2 of Section 2.3); let ζ, ζ' be the boundary conditions induced by the revealed regions of ω and ω' on $\partial\Lambda_{L(p)}$.
3. Use an arbitrary increasing coupling of $\phi^{\zeta}_{\Lambda_{L(p)},p}$ and $\phi^{\zeta'}_{\Lambda_{L(p)},p}$.

If $\xi = \xi'$, then ω and ω' are identical in the complement of $\Lambda_{L(p)}(e)$. Moreover, as explained in Section 2.3, for ζ to differ from ζ', ω' must contain a connection between $\Lambda_{L(p)}(e)$ and $\Lambda_{L(p)}$. Thus, we find

$$\begin{aligned}\phi_p[\mathcal{C}\,|\,\omega_e=1]-\phi_p[\mathcal{C}\,|\,\omega_e=0] &\le \mathbb{E}\big[\mathbb{1}_{\{\xi\neq\xi'\}}\mathbb{1}_{\omega'\in\{\Lambda_L(e)\longleftrightarrow\Lambda_{L(p)}\}}(\phi^{\zeta'}_{\Lambda_L,p}[\mathcal{C}]-\phi^{\zeta}_{\Lambda_L,p}[\mathcal{C}])\big] \quad (5.7)\\ &\lesssim \Delta_p(L(p))\phi^1_{\mathbb{Z}^2\setminus\Lambda_{L(p)},p}[\Lambda_{L(p)}\longleftrightarrow\Lambda_{L(p)}(e)]\Delta_p(R,L(p)).\end{aligned}$$

In the second inequality, we used the monotonicity of boundary conditions given by equation (CBC) and Theorem 4.9. Finally, due to the mixing property of Corollary 2.10 and Proposition 2.13, the second term of the last product is bounded above by $\exp[-cn/L(p)]$ for some positive constant c. □

Proof of equation (5.2). For the lower bound, translate $\mathcal{D}$ and e such that $e\notin\Lambda_{\eta R/2}$ and $\Lambda_{\eta R/2}\subset\mathcal{D}$. Since $\mathcal{D}$ is assumed η-regular, this is possible. We produce a coupling $\mathbb{P}$ between $\phi_p[\cdot\,|\,\omega_e=0]$ and $\phi_p[\cdot\,|\,\omega_e=1]$ as follows.

1. Explore the double four-petal flower domain $(\mathcal{F}_{\text{in}},\mathcal{F}_{\text{out}})$ in ω between $\Lambda_{\eta R/4}$ and $\Lambda_{\eta R/8}$; if no such double four-petal flower domain exists, reveal the rest of the configurations in arbitrary order.
2. Reveal ω and ω' inside $\mathcal{F}_{\text{in}}$.
3. Reveal the configurations in $\mathcal{F}_{\text{out}}$. □

Now, if $(\mathcal{F}_{\text{in}},\mathcal{F}_{\text{out}})$ exists, the argument of equation (5.3) shows that

$$\phi[P_1^{\text{out}}\xleftrightarrow{\mathcal{F}_{\text{out}}}P_3^{\text{out}}|\omega_e=1,\mathcal{F}_{\text{in}},\mathcal{F}_{\text{out}}]-\phi[P_1^{\text{out}}\xleftrightarrow{\mathcal{F}_{\text{out}}}P_3^{\text{out}}|\omega_e=0,\mathcal{F}_{\text{in}},\mathcal{F}_{\text{out}}]\gtrsim\Delta_p(R).$$

Indeed, e is at a distance comparable to R from $(\mathcal{F}_{\text{in}},\mathcal{F}_{\text{out}})$; for details, see the proof of the point in equation (3.8) of Theorem 3.6. Note that this is simply a statement on the conditional probability of the connections between the petals P_1 and P_3; we do not reveal the configurations ω and ω' in $\mathcal{F}_{\text{out}}$.

Then as in the proof of equation (3.8), the previous estimate gives that

$$\phi[P_1^{\text{in}}\xleftrightarrow{\mathcal{F}_{\text{in}}}P_3^{\text{in}}|\omega_e=1,\mathcal{F}_{\text{in}},\mathcal{F}_{\text{out}}]-\phi[P_1^{\text{in}}\xleftrightarrow{\mathcal{F}_{\text{in}}}P_3^{\text{in}}|\omega_e=0,\mathcal{F}_{\text{in}},\mathcal{F}_{\text{out}}]\gtrsim\Delta_p(R).$$

Let H be the event that P_1^{out} and P_3^{out} are connected in $\omega\cap\mathcal{F}_{\text{out}}\cap\mathcal{D}$ to the arcs (ab) and (cd), respectively, while P_2^{out} and P_4^{out} are connected in $\omega^*\cap\mathcal{F}_{\text{out}}\cap\mathcal{D}$ to the arcs (bc) and (da), respectively. By equation (RSW),

$$\phi[H\,|\,(\mathcal{F}_{\text{in}},\mathcal{F}_{\text{out}})\text{ and }(\omega,\omega')\text{ on }\mathcal{F}^c_{\text{out}}]\gtrsim 1.$$

Finally, when H occurs, $\mathcal{D}$ is crossed if and only if P_1^{in} and P_3^{in} are connected inside $\mathcal{F}_{\text{in}}$. As a consequence, the two displays above and Lemma 3.4 conclude that

$$\begin{aligned}&\phi_p[\mathcal{C}\,|\,\omega_e=1]-\phi_p[\mathcal{C}\,|\,\omega_e=0]\\ &\quad\gtrsim\mathbb{P}[(\mathcal{F}_{\text{in}},\mathcal{F}_{\text{out}})\text{ exists}]\mathbb{P}[P_1^{\text{in}}\xleftrightarrow{\omega'\cap\mathcal{F}_{\text{in}}}P_3^{\text{in}}\text{ but }P_1^{\text{in}}\overset{\omega\cap\mathcal{F}_{\text{in}}}{\not\longleftrightarrow}P_3^{\text{in}}\,|\,(\mathcal{F}_{\text{in}},\mathcal{F}_{\text{out}})\text{ exists}]\\ &\quad\gtrsim\Delta_p(R).\end{aligned}$$

We turn to the upper bound. For $s\ge 1$, let $\mathcal{P}_s := \text{Piv}_{2^s,e}(\mathcal{D})\setminus\text{Piv}_{2^{s-1},e}(\mathcal{D})$. Also set $\mathcal{P}_0:=\text{Piv}_{1,e}(\mathcal{D})$. Observe that $\text{Piv}_{2R,e}(\mathcal{D})$ occurs as such, and therefore the events $\mathcal{P}_s$ for $s=0,\dots,\lceil\log 2R\rceil$ partition the space. Thus

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$$\begin{aligned}\mathrm{Cov}_p(\omega_e,\mathcal{C}) &= \sum_{s=0}^{\lceil \log 2R\rceil} (\phi[\omega_e|\mathcal{C}\cap\mathcal{P}_s]-\phi[\omega_e])\phi_p[\mathcal{C}\cap\mathcal{P}_s]\\ &\lesssim \sum_{s=1}^{\lceil \log 2R\rceil} (\phi^1_{\Lambda_{2^{s-1}}(e)}[\omega_e]-\phi^0_{\Lambda_{2^{s-1}}(e)}[\omega_e])\phi_p[\mathrm{Piv}_{2^s,e}(\mathcal{D})]\\ &\lesssim \sum_{r=1}^{4R} \tfrac{1}{r}\Delta_p(r)\phi_p[\mathrm{Piv}_{r,e}(\mathcal{D})]. \qquad (5.8)\end{aligned}$$

In the first inequality, we use that for $s \geq 1$, $\mathcal{C}\cap\mathcal{P}_s$ is measurable in terms of the edges outside $\Lambda_{2^{s-1}}(e)$, as well as equation (SMP) and the inclusion $\mathcal{P}_s \subset \mathrm{Piv}_{2^s,e}(\mathcal{D})$. The term $s = 0$ of the first line is trivially bounded by $\phi_p[\mathrm{Piv}_{1,e}(\mathcal{D})]$ and may be absorbed by the term $s = 1$ of the second line. The second inequality is a simple reindexing of the sum; it uses equation (4.21) and the monotonicity in r of the events $\mathrm{Piv}_{r,e}(\mathcal{D})$.

Next, we eliminate[5] the terms for $1 \leq r < n/2$ from the last sum of equation (5.8). Notice that if $\mathrm{Piv}_{r,e}(\mathcal{D})$ occurs for $r < n$, then $\mathrm{Piv}_{n,e}(\mathcal{D})$ also occurs, and in addition there exist four arms of alternating type between $\Lambda_r(e)$ and $\partial\Lambda_n(e)$. Due to equation (Mix) and the quasi-multiplicativity of the four-arm event,

$$\phi_p[\mathrm{Piv}_{r,e}(\mathcal{D})]\lesssim\pi_4(p;r,n)\phi_p[\mathrm{Piv}_{n,e}(\mathcal{D})].$$

Now, using equations (1.15) and (1.17), we find

$$\begin{aligned}\sum_{r=1}^{n}\tfrac{1}{r}\Delta_p(r)\phi_p[\mathrm{Piv}_{r,e}(\mathcal{D})] &\lesssim \sum_{r=1}^{n}\tfrac{1}{r}\Delta_p(r)\pi_4(p;r,n)\phi_p[\mathrm{Piv}_{n,e}(\mathcal{D})]\\ &\lesssim \Delta_p(n)\phi_p[\mathrm{Piv}_{n,e}(\mathcal{D})]\tfrac{1}{n^{\varepsilon}}\sum_{r=1}^{n}\tfrac{1}{r^{1-\varepsilon}}\\ &\lesssim \Delta_p(n)\phi_p[\mathrm{Piv}_{n,e}(\mathcal{D})]\lesssim \sum_{r=n/2}^{n}\tfrac{1}{r}\Delta_p(r)\phi_p[\mathrm{Piv}_{r,e}(\mathcal{D})].\end{aligned}$$

In conclusion, the sum in the last line of equation (5.8) may be restricted to $n/2 \leq r \leq 4R$.

Finally, we mention that all upper bounds also hold for $|\mathrm{Cov}_p[\omega_e; A_k(r,R)]|/\phi_p[A_k(r,R)]$ using the same proofs. Indeed, the fact that $\mathcal{C}$ is increasing was only used when proving the lower bound in equation (5.3) – see also Remark 5.8 below. Moreover, when bounding $|\mathrm{Cov}_p[\omega_e; A_k(r,R)]|$, it is standard that we obtain an additional multiplicative term $\phi_p[A_k(r,R)]$ in the upper bound in each case equations 5.2–5.4. □

Remark 5.8. The upper bounds in equations (5.3) and (5.4) may be shown to apply to any event H instead of $\mathcal{C}(\mathcal{D})$, as long as H depends only on the edges in Λ_R. Indeed, for H increasing, the proof above applies mutatis mutandis. When H is not increasing, additional care needs to be taken when bounding $|\phi[H\,|\,\omega_e = 1] - \phi[H\,|\,\omega_e = 0]|$ in equations (5.6) and (5.7). Notice, however, that in both equations, one may produce a coupling of $\phi[\cdot\,|\,\omega_e = 0]$ and $\phi[\cdot\,|\,\omega_e = 1]$ that generates distinct configurations inside Λ_R with a probability bounded from above by the right-hand side of the respective equations. As a consequence, the terms on the right-hand side in equations (5.3) and (5.4) bound from above the distance in total variation between the restrictions of $\phi[\cdot\,|\,\omega_e = 1]$ and $\phi[\cdot\,|\,\omega_e = 0]$ to Λ_R.

[5]This part of the sum may also be eliminated without the use of equation (1.17), but the proof is more complicated. The idea is to consider the measure $\phi^{0\text{ or }1}_{\Lambda_n^c} = \frac{1}{2}\phi^0_{\Lambda_n^c} + \frac{1}{2}\phi^1_{\Lambda_n^c}$ obtained by first choosing a random variable X uniformly in $\{0,1\}$ and then choosing a configuration ω according to $\phi^X_{\Lambda_n^c}$. It may then be shown that $\mathrm{Cov}_p(\omega_e,\mathcal{C}) \asymp \Delta_p(n)(\phi^{0\text{ or }1}_{\Lambda_n^c}[X=1|\mathcal{C}] - \frac{1}{2})$. Then the same computation as in equation (5.8) bounds the last parenthesis by $\sum_{r=n}^{4R}\frac{1}{r}\Delta_p(n,r)\phi_p[\mathrm{Piv}_{r,e}(\mathcal{D})]$.

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Proof of Proposition 5.1. We focus here on the expression for the crossing probability of a quad; the proof for the derivatives of probabilities of arm events is identical to that of the upper bound proved below.

Fix p and some η-regular quad $(\mathcal{D}, a, b, c, d)$ at scale $R \leq L(p)$. By grouping the contributions of different edges depending on their distance to the origin and the boundary of Λ_R and using Lemma 5.3, we have

$$\tfrac{\mathrm{d}}{\mathrm{d}p}\phi_p[\mathcal{C}(\mathcal{D})] = \frac{1}{p(1-p)}\sum_e \mathrm{Cov}_p(\omega_e, \mathcal{C}(\mathcal{D})) \gtrsim R^2\Delta_p(R) + \sum_{n=R}^{L(p)} n\Delta_p(n)\Delta_p(R,n),$$

since there are order R^2 edges in the case equation (5.2) and order n edges at a distance n from $\partial\mathcal{D}$.

For the upper bound, we use the three upper bounds of Lemma 5.3. We split the edges into three categories: those at a distance less than $2R$ from $\partial\mathcal{D}$, those at a distance between $2R$ and $2L(p)$ from $\partial\mathcal{D}$ and those at a distance larger than $2L(p)$ from $\partial\mathcal{D}$.

For the two last categories, keeping in mind that there are $O(n)$ edges at a distance exactly $n \geq 2R$ from $\partial\mathcal{D}$, and applying equations (5.3) and (5.4), we find

$$\begin{aligned}\sum_{e:\,\mathrm{dist}(e,\partial\mathcal{D})\geq 2R} \mathrm{Cov}_p(\omega_e, \mathcal{C}(\mathcal{D})) &\lesssim \sum_{n=2R}^{2L(p)} n\Delta_p(n)\Delta_p(R,n) + \sum_{n\geq 2L(p)} n\Delta_p(L(p))\Delta_p(R, L(p))e^{-cn/L(p)}\\ &\lesssim \sum_{n=2R}^{2L(p)} n\Delta_p(n)\Delta_p(R,n) + L(p)^2\Delta_p(L(p))\Delta_p(R, L(p))\\ &\lesssim R^2\Delta_p(R) + \sum_{n=R}^{L(p)} n\Delta_p(n)\Delta_p(R,n). \end{aligned} \tag{5.9}$$

In the second inequality, we argue that due to the exponential factor, the first $L(p)$ summands account for a positive proportion on the entire second sum. The final inequality may be obtained by considering separately the cases $L(p)/2 \leq R \leq L(p)$ and $R < L(p)/2$. In the former, the second line is dominated entirely by the first term of the last line; in the latter, the second term of the last line dominates the whole expression.

We turn to the edges in the first category: those close to $\partial\mathcal{D}$. We refer to Figure 13 for an illustration. Fix $n \leq 2R$. We start by studying the contribution to the derivative of the edges at a distance n of the arc (ab) (the cases of the other arcs (bc), (cd) and (da) are treated similarly), and we precisely focus on the term on the right-hand side of equation (5.2) for these edges and corresponding to some fixed $n/2 \leq r \leq 3R$. Consider a family of vertices $a = x_0, \dots, x_k$ found counterclockwise along (ab), with $\|x_i - x_{i-1}\|_\infty = r$ for $1 \leq i \leq k$ and with the last vertex x_k at a distance at most r of b (when r is large, k is equal to 0). Consider the event E_i that $(x_{i-4}x_{i-3})$ is dual-connected to (da) and $(x_{i+3}x_{i+4})$ is connected to (cd), with the convention that if $i \leq 4/\eta$ or $i \geq k - 4/\eta$, then E_i is the full event. We claim that for every $e \subset \Lambda_r(x_i)$,

$$\phi_p[\mathrm{Piv}_{r,e}(\mathcal{D})] \lesssim \phi_p[E_i].$$

Indeed, the inequality is trivial when $i \leq 4/\eta$ or $i \geq k - 4/\eta$ since E is the full event. For the remaining values of i (which exist only when $r \ll \eta R$), observe that for $\mathrm{Piv}_{r,e}(\mathcal{D})$ to occur, $\Lambda_{2r}(x_i)$ needs to be connected by a primal path to (cd) and by dual paths to (bc) and (da). In particular, the interface starting from d and delimiting the primal cluster of (cd) in $\mathcal{D}$ necessarily hits $\Lambda_{2r}(x_i)$ before hitting the arc (ab). Let Γ be the section of this interface from d up to the first time it hits $\Lambda_{2r}(x_i)$. Conditionally on Γ, one may use equation (RSW) to construct a dual path connecting Γ to $(x_{i-4}x_{i-3})$ and a primal path connecting Γ to $(x_{i+3}x_{i+4})$ with probability uniformly bounded away from zero. These paths, together with Γ, induce the connections required by E_i.

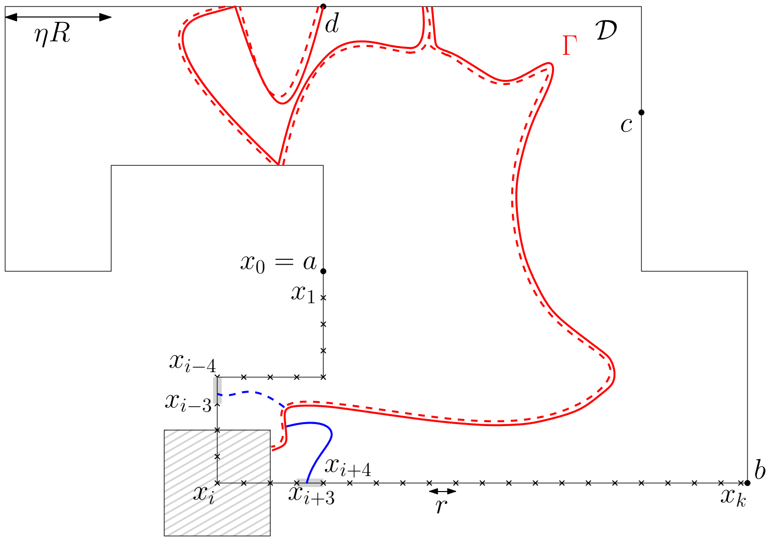

Figure 13. *A depiction of the points $x_0, \ldots, x_k$ (note that x_k is not necessarily equal to b). The occurrence of* $\mathrm{Piv}_{r,e}(\mathcal{D})$ *for some $e \in \Lambda_r(x_i)$ induces the existence of the exploration path Γ (in red). When orienting Γ in the direction of its exploration – that is, from d to $\Lambda_{2r}(x_i)$ – Γ has dual open edges on its right and primal ones on its left. Thus, conditionally on Γ, the blue paths occur with uniformly positive probability and induce the occurrence of E_i.*

Notice now that any edge at distance n from (ab) is contained in at least one $\Lambda_r(x_i)$. Conversely there are $O(r)$ vertices in each $\Lambda_r(x_i)$ at a distance exactly n from (ab). Thus, summing over the edges e at a distance n from (ab) gives

$$\sum_{e:\mathrm{dist}(e,(ab))=n} \phi_p[\mathrm{Piv}_{r,e}(\mathcal{D})] \lesssim r \sum_{i=0}^{k} \phi_p[E_i] \lesssim r,$$

where the second inequality is due to the fact that at most $O(1)$ events E_i can occur simultaneously.

One may do the same with edges at a distance n of (bc), (cd) and (da). Finally, summing the previous displayed equation over r and using equation (5.2), we find

$$\sum_{e:\mathrm{dist}(e,\partial\mathcal{D})\le 2R} \mathrm{Cov}_p(\omega_e, \mathcal{C}(\mathcal{D})) \lesssim \sum_{n=1}^{2R} \sum_{r=n/2}^{4R} \tfrac{1}{r}\Delta_p(r) \sum_{e:\mathrm{dist}(e,\partial\mathcal{D})=n} \phi_p[\mathrm{Piv}_{r,e}(\mathcal{D})]$$
$$\lesssim \sum_{n=1}^{2R} \sum_{r=n/2}^{4R} \Delta_p(r) \lesssim \sum_{r=1}^{R} r\Delta_p(r).$$

The above combined with equation (5.9) implies the upper bound in equation (5.1). □

5.2. *Derivative for the mixing rate*

Proposition 5.9. *For every p and two edges e and f at a distance $R \le L(p)$ from each other,*

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}p} \log \mathrm{Cov}_p(\omega_e, \omega_f)\right| \lesssim \sum_{\ell=1}^{R} \ell\Delta_p(\ell) + \sum_{\ell=R}^{L(p)} \ell\Delta_p(\ell)\Delta_p(R,\ell). \tag{5.10}$$

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Exactly as for crossing events, one may use the next section to replace $\sum_{\ell\le R}\ell\Delta_p(\ell)$ by $R^2\Delta_p(R)$; see Corollary 7.1 of Section 7.

Remark 5.10. Since $\mathrm{Cov}_p(\omega_e,\omega_f)$ was shown in equation (5.3) to be comparable to $\Delta_p(R)^2$, the above should be understood as a bound on the logarithmic derivative of $\Delta_p(R)$. Indeed, equation (5.10) combined with equation (5.3) yields

$$\Big|\log\frac{\Delta_p(R)}{\Delta_{p_c}(R)}\Big|\lesssim\int_{p_c}^{p}\Big(\sum_{\ell=1}^{R}\ell\Delta_u(\ell)+\sum_{\ell=R}^{L(u)}\ell\Delta_u(\ell)\Delta_u(R,\ell)\Big)\,du, \tag{5.11}$$

for all p and $R\le L(p)$. This inequality will be useful when R is of the same order as $L(p)$, in which case the terms with $R\le\ell\le L(p)$ may be absorbed by the first sum on the right-hand side.

Proof. Fix p, e and f as in the statement. Then

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}p}\mathrm{Cov}_p(\omega_e,\omega_f)=\sum_{g\in\mathbb{E}}&\phi_p[\omega_e\omega_f\omega_g]-\phi_p[\omega_e\omega_f]\phi_p[\omega_g]-\phi_p[\omega_f\omega_g]\phi_p[\omega_e]\\&-\phi_p[\omega_e\omega_g]\phi_p[\omega_f]+2\phi_p[\omega_e]\phi_p[\omega_f]\phi_p[\omega_g].\end{aligned} \tag{5.12}$$

We will bound separately the absolute value of each term in the sum above depending on the position of g. First, consider an edge $g\in\Lambda_{2R/3}(e)$; the corresponding term in equation (5.12) may be written as

$$\phi_p[\omega_f]\Big(\phi_p[\omega_e\omega_g|\omega_f]-\phi_p[\omega_g|\omega_f]\phi_p[\omega_e]-\phi_p[\omega_e|\omega_f]\phi_p[\omega_g]-\phi_p[\omega_e\omega_g]+2\phi_p[\omega_e]\phi_p[\omega_g]\Big). \tag{5.13}$$

Write ℓ for the distance between e and g, and set $G:=\Lambda_{4\ell/3}(e)$. Notice that f is outside of G at a distance of at least $R/9$ from it. For any event A depending only on the edges in G, we have

$$\phi_p[A]=\sum_{\xi}\phi_p[B_\xi]\phi^{\xi}_{G,p}[A]\quad\text{ and }\quad\phi_p[A\,|\,\omega_f]=\sum_{\xi}\phi_p[B_\xi\,|\,\omega_f]\phi^{\xi}_{G,p}[A],$$

where the sum is over all boundary conditions ξ imposed on G by the configuration outside of G, and B_ξ is the event that the boundary conditions induced on G are ξ. Thus, equation (5.13) may be written as

$$\phi_p[\omega_f]\sum_{\xi}(\phi_p[B_\xi\,|\,\omega_f]-\phi_p[B_\xi])\underbrace{(\phi^{\xi}_{G,p}[\omega_e\omega_g]-\phi^{\xi}_{G,p}[\omega_g]\phi_p[\omega_e]-\phi^{\xi}_{G,p}[\omega_e]\phi_p[\omega_g]+\phi_p[\omega_e]\phi_p[\omega_g])}_{F(\xi)}.$$

By adding and subtracting $\phi^{\xi}_{G,p}[\omega_e]\phi^{\xi}_{G,p}[\omega_g]$, we find

$$F(\xi)=\mathrm{Cov}^{\xi}_{G,p}(\omega_e,\omega_g)+(\phi^{\xi}_{G,p}[\omega_e]-\phi_p[\omega_e])(\phi^{\xi}_{G,p}[\omega_g]-\phi_p[\omega_g]),$$

where Cov^{ξ}_{G} stands for the covariance under the measure $\phi^{\xi}_{G,p}$. Next, we apply Lemma 5.3 to $\phi^{\xi}_{G,p}$ instead of ϕ_p – notice that the choice of G ensures that both e and g are at a distance of order ℓ from ∂G, and the lemma does indeed apply as explained in Remark 5.5 – to obtain

$$\mathrm{Cov}^{\xi}_{G}(\omega_e,\omega_g)\lesssim\Delta_p(\ell)^2.$$

In addition, for any ξ, Theorem 4.9 yields

$$|\phi^{\xi}_{G,p}[\omega_g]-\phi_p[\omega_g]|\lesssim\Delta_p(\ell)\quad\text{ and }\quad|\phi^{\xi}_{G,p}[\omega_e]-\phi_p[\omega_e]|\lesssim\Delta_p(\ell).$$

Then equation (5.13) may be bounded above by

$$\sum_{\xi} \|F\|_\infty \cdot |\phi_p[B_\xi|\omega_f] - \phi_p[B_\xi]| \lesssim \Delta_p(R)\Delta_p(\ell,R)\Delta_p(\ell)^2 \asymp \Delta_p(\ell)\Delta_p(R)^2, \tag{5.14}$$

since the total variation distance between the restrictions of $\phi_p[\cdot|\omega_f]$ and ϕ_p to G is bounded by a universal multiple of $\Delta_p(R)\Delta_p(\ell,R)$, as shown in Lemma 5.3 (see also Remark 5.8). The second equivalence is given by the quasi-multiplicativity of Δ_p equation (1.15).

Edges $g \in \Lambda_{2R/3}(f)$ have the same contribution to equation (5.13) due to symmetry. Finally, we consider edges g that are outside $\Lambda_{2R/3}(e) \cup \Lambda_{2R/3}(f)$; write ℓ for the distance between g and e. Let G be the domain formed of the edges at a distance at least $R/6$ from the segment uniting e and f; notice that g is at a distance at least $R/6$ from G. Applying the same argument as in the first case, with the roles of ω_f and ω_g inverted, we find that the term corresponding to g in equation (5.12) is bounded by

$$\begin{aligned}\phi_p[\omega_g] \sum_{\xi} \big|\phi_p[\xi \mid \omega_g] - \phi_p[\xi]\big| \cdot \big|\mathrm{Cov}^{\xi}_{G,p}(\omega_e,\omega_f) + (\phi^{\xi}_{G,p}[\omega_e] - \phi_p[\omega_e])(\phi^{\xi}_{G,p}[\omega_f] - \phi_p[\omega_f])\big| \\ \lesssim \Delta_p(\ell \wedge L(p))\Delta_p(R, \ell \wedge L(p))e^{-c\ell/L(p)}\Delta_p(R)^2 \\ \asymp \Delta_p(R)\Delta_p(\ell \wedge L(p))^2 e^{-c\ell/L(p)},\end{aligned} \tag{5.15}$$

where the sum on the left-hand side is over all the boundary conditions ξ on G imposed by the configuration outside of G. Indeed, by Lemma 5.3, the total variation distance between the restrictions of $\phi[\cdot \mid \omega_g]$ and ϕ to G is bounded by a universal multiple of $\Delta_p(\ell \wedge L(p))\Delta_p(R, \ell \wedge L(p))e^{-c\ell/L(p)}$, while each term in the second parenthesis of the left-hand side is bounded by multiples of $\Delta_p(R)^2$.

Summing now over g and using the triangular inequality and equations (5.14) and (5.15), we find

$$\begin{aligned}\Big|\tfrac{\mathrm{d}}{\mathrm{d}p}\mathrm{Cov}_p(\omega_e,\omega_f)\Big| &\lesssim \sum_{\ell=1}^{R/2} \ell\Delta_p(\ell)\Delta_p(R)^2 + \sum_{\ell>R/2} \ell\Delta_p(R)\Delta_p(\ell \wedge L(p))^2 e^{-c\ell/L(p)} \\ &\lesssim \Delta_p(R)^2\Big(\sum_{\ell=1}^{R} \ell\Delta_p(\ell) + \sum_{\ell=R}^{L(p)} \ell\Delta_p(R,\ell)\Delta_p(\ell)\Big).\end{aligned}$$

The second inequality requires some basic algebra: we use the exponential factor to bound the contribution of terms with $\ell > L(p)$ by $L(p)^2\Delta_p(R)\Delta_p(L(p))^2$; the quasi-multiplicativity given by equation (1.15) of Δ_p as well as equations (4.14) and (4.21) allow us to bound the term above as well as the remaining sums over $\ell \le R/2$ and $R/2 < \ell \le L(p)$ by the last line of the display. Finally, divide by $\mathrm{Cov}_p(\omega_e,\omega_f) \asymp \Delta_p(R)^2$ to obtain the desired result. □

6. Lower bound on $\Delta_p(r,R)$: proof of Proposition 1.8

In this section, we prove Proposition 1.8. The section will be divided in two as the case $1 \le q < 4$ is quite different from the case $q = 4$.

6.1. *Lower bound on $\Delta_p(r,R)$ for $1 \le q < 4$*

In this section, we assume that $1 \le q < 4$. We will prove the following stronger statement.

Proposition 6.1. *Fix $1 \le q < 4$. There exists $\delta = \delta(q) > 0$ such that for $p \in (0,1)$ and $r \le R \le L(p)$,*

$$\pi_4(p;r,R) \gtrsim (r/R)^{2-\delta}. \tag{6.1}$$

This implies equation (1.20) since $\Delta_p(r, R) \geq \pi_4(p; r, R)$ by the observation at the beginning of Section 4.5 (one does not even require the polynomial improvement given by equation (1.17)). Let us mention that equation (6.1) is expected to fail for $q = 4$, which explains why the case $q = 4$ needs to be treated separately.

Proof. By symmetry, we only need to treat the case $p \geq p_c$. The case $p = p_c$ was obtained in a recent paper [DMT20, Proposition 6.8]. The argument extends readily to our setting.

Indeed, if E denotes the event that Λ_{3R} contains both an open circuit and a dual open circuit surrounding Λ_{2R}, with the open circuit being connected to $\partial\Lambda_{4R}$, then it is shown in [DMT20] that

$$(R/r)^2\pi_4(p; r, R) \gtrsim \phi_p[E] \inf_{\mathcal{D}} \phi_{\mathcal{D},p}[\mathbf{M}_r(\mathcal{D}, R)],$$

where the infimum is taken over all simply connected domains containing Λ_{2R} and contained in Λ_{3R} and $\mathbf{M}_r(\mathcal{D}, R)$ is an increasing random variable defined in [DMT20, Section 1.5]. When $R \leq L(p)$, equation (RSW) implies that $\phi_p[E] \gtrsim 1$. Moreover, since $\mathbf{M}_r(\mathcal{D}, R)$ is increasing,

$$\inf_{\mathcal{D}} \phi_{\mathcal{D},p}[\mathbf{M}_r(\mathcal{D}, R)] \geq \inf_{\mathcal{D}} \phi_{\mathcal{D},p_c}[\mathbf{M}_r(\mathcal{D}, R)] \gtrsim (R/r)^{\delta_0},$$

for some universal $\delta_0 > 0$, where the last inequality is given by [DMT20, Proposition 1.4]. Combining the above displays produces the desired bound. □

Remark 6.2. The proof looks simple, but we wish to insist that the whole difficulty of the argument is contained in [DMT20].

6.2. Lower bound on $\Delta_p(r, R)$ for $q = 4$

The reasoning of the previous section does not apply for $q = 4$, as it is expected that

$$\pi_4(p; r, R) \asymp (\tfrac{r}{R})^2$$

(see Remark 6.11). Nevertheless, this is not contradictory with the fact that $\Delta_p(r, R) \gtrsim (\frac{r}{R})^{2-\delta}$, as we know from Section 4.5 that $\Delta_p(r, R)$ is polynomially larger than $\pi_4(p; r, R)$. Since we do not currently know how to prove that $\pi_4(p; r, R) \asymp (\frac{r}{R})^2$, we adopt a direct approach to prove Proposition 1.8 that does not involve the comparison to $\pi_4(p; r, R)$.

Proof of Equation (1.20). The lower bound on $\Delta_p(r, R)$ follows directly from the combination of the next two propositions, which, respectively, correspond to the required bound at p_c and the stability of Δ_p below the characteristic length. □

Proposition 6.3 (Lower bound at p_c). *For $q = 4$, there exists $\delta_0 > 0$ such that for every $R \geq 2r > 0$,*

$$\Delta_{p_c}(r, R) \gtrsim \pi_2(r, R)(r/R) \gtrsim (r/R)^{2-\delta_0}. \tag{6.2}$$

Proposition 6.4 (Stability of Δ). *For $q = 4$, every p and $2 \leq r \leq R \leq L(p)$,*

$$\Delta_p(r, R) \asymp \Delta_{p_c}(r, R). \tag{6.3}$$

The section is divided as follows. The proof makes use of the parafermionic observable introduced by Smirnov [Smi10]; we therefore start by recalling this notion. Then we prove each proposition in a separate section, in the order in which they are stated.

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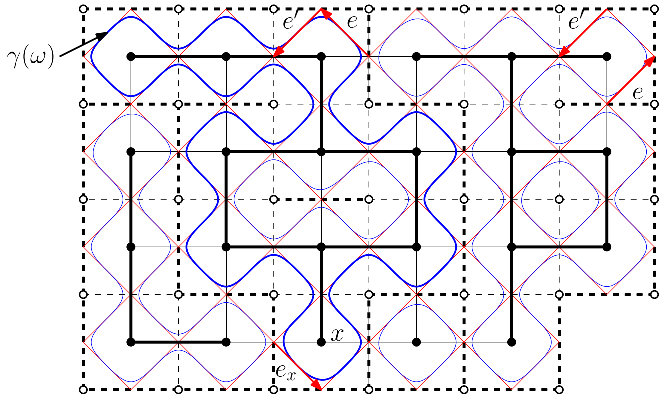

Figure 14. The primal and dual graphs in plain and dashed black lines. In bold (plain and dashed, respectively), the configurations ω and ω^*. The graph $\Omega^\diamond$ is in red, and the configuration $\overline{\omega}$ is in blue. The path $\gamma(\omega)$ is in bold blue. We also presented two examples of edges e and e', one for which the corresponding vertex of the primal graph has degree 3 and one for which it has degree 2.

6.2.1. Parafermionic observable: a crash-course

Since the parafermionic observable is used only in this section, we give minimal details and refer to the literature: for instance, the review article [Dum17] and the research paper [DMT20] (which implements reasoning very similar to the present, with similar notations). We also recommend that the reader look at Figure 14.

Parafermionic observables are usually defined in Dobrushin domains, but in the present study, we limit ourselves to situations where the wired arc is reduced to a single point and therefore the domain has free boundary conditions. We do, however, authorise the domains to be non-simply-connected.

Fix a finite connected subgraph Ω of $\mathbb{Z}^2$ and a vertex $x \in \Omega$ with a neighbour x' in the infinite connected component of $\mathbb{Z}^2 \setminus \Omega$. Recall that $\Omega^\diamond$ is spanned by the edges of $(\mathbb{Z}^2)^\diamond$ bordering the faces of $(\mathbb{Z}^2)^\diamond$ that contain vertices of Ω. As such, the vertices of $\Omega^\diamond$ have degree 2 or 4 in $\Omega^\diamond$ (here, one should be careful when looking at a vertex of $\Omega^\diamond$ that corresponds to an edge outside of Ω with both endpoints in Ω: in this case, we think of this vertex as being split into two 'prime ends' of degree 2 in $\Omega^\diamond$). Let e_x be the first edge of $\Omega^\diamond$ bordering the face of $(\mathbb{Z}^2)^\diamond$ containing x, when going around said face in clockwise order, starting from xx'.

For a configuration ω on Ω, let $\overline{\omega}$ be the loop configuration on $\Omega^\diamond$ associated to ω. In the loop configuration, the loop passing through e_x is called an *exploration path*; it is denoted by $\gamma = \gamma(\omega)$ and is oriented counterclockwise so as to have primal open edges on its left and dual-open edges on its right. For an edge $e \in \gamma$, let $\mathrm{W}_\gamma(e, e_x)$ be the winding of γ between e and e_x: that is, the number of left turns minus the number of right turns taken by γ when going from e to e_x multiplied by $\pi/2$.

Definition 6.5. For (Ω, x) as above, the *parafermionic observable* $F = F_{\Omega,x}$ is defined for any (medial) edge e of $\Omega^\diamond$ by

$$F(e) := \phi^0_{\Omega,p_c,4}[\mathrm{W}_\gamma(e,e_x)\mathrm{e}^{\mathrm{i}\mathrm{W}_\gamma(e,e_x)}\mathbb{1}_{e\in\gamma}].$$

The parafermionic observable satisfies a very special property first observed in [Smi10] (see also [Dum17, Theorem 5.16] for a statement with a similar notation). For every vertex of $\Omega^\diamond$ with four incident edges in $\Omega^\diamond$,

$$\sum_{i=1}^{4} \eta(e_i)F(e_i) = F(e_1) - \mathrm{i}F(e_2) - F(e_3) + \mathrm{i}F(e_4) = 0, \tag{6.4}$$

where e_1, e_2, e_3 and e_4 are the four edges incident to v, indexed in clockwise order, and $\eta(e_i)$ is the complex number of norm one with the same direction as e_i and orientation from v towards the other endpoint of e_i. Summing this relation over all vertices of $\Omega^\diamond$ of degree 4, we obtain that

$$\sum_{e\in\mathcal{C}} \eta(e)F(e) = 0, \tag{6.5}$$

where $\mathcal{C}$ is the set of medial-edges of $\Omega^\diamond$ having exactly one endpoint of degree 2 in $\Omega^\diamond$, and $\eta(e)$ is the complex number of modulus one, collinear with the edge e and oriented towards the outside of Ω.

Remark 6.6. This relation should be understood as stating that the contour integral of the parafermionic observable along the boundary of $\Omega^\diamond$ is 0. While it is common to use the above in simply connected domains, we insist that equation (6.5) is valid for any Ω as above (it is important that x lies on the exterior face).

For random-cluster models with general values of q between 1 and 4 and the loop $O(n)$ models, a similar property was used to obtain estimates for the two-point functions; see, for example, [DS12, DST17, DGPS17, DMT20]. Here, we propose a new use for this property.

For $q = 4$, the complex phase of the observable (that is $e^{i\mathrm{W}_\gamma(e,e_x)}$) is invariant under addition of factors 2π to the winding. Therefore, it is (almost) determined by the orientation of e. Indeed, for any oriented edge e and any ω such that γ passes through e,

$$\eta(e)e^{i\mathrm{W}_\gamma(e,e_x)} = \begin{cases} +1 & \text{if } \gamma \text{ is oriented like } e, \\ -1 & \text{if } \gamma \text{ is oriented opposite to} e. \end{cases} \tag{6.6}$$

Moreover, for $y \in \partial\Omega$ (exclude from this explanation the situation where y has exactly two opposite neighbours in Ω), there exist two edges $e, e' \in \mathcal{C}$ bordering the face of $(\mathbb{Z}^2)^\diamond$ containing y such that γ passes through e if and only if it passes through e'; see Figure 14. When this happens and y is not equal to x, e and e' may be chosen such that γ always passes first through e towards the exterior of Ω and then through e' towards the interior. Finally, γ performs $4 - d_y$ positive $\pi/2$ turns between the passage through e and e', where d_y is the degree of y inside Ω. Thus,

$$\mathrm{W}_\gamma(e, e_x) = \mathrm{W}_\gamma(e', e_x) + (4 - d_y)\pi/2. \tag{6.7}$$

When $y = x$, the two contributions to equation (6.5) of e and e' are 0 and $3\pi/2$, respectively. Inserting equations (6.6) and (6.7) into equation (6.5) yields

$$\sum_{\substack{y\in\partial\Omega \\ y\neq x}} (4 - d_y)\phi^0_\Omega[A(x, y)] = \tfrac{3\pi}{2}, \tag{6.8}$$

where $A(x, y)$ is the event that γ passes between y and its neighbour outside Ω. When y has a neighbour in the infinite component of Ω^c, then $A(x, y) = \{x \longleftrightarrow y\}$; but since we do not ask Ω to be simply connected, this is not always the case for other y.

A more sophisticated analysis also yields equation (6.8) when some $y \in \mathcal{C}$ have exactly two opposite neighbours in Ω.

6.2.2. Lower bound on $\Delta_{p_c}(r, R)$ at p_c: proof of Proposition 6.3

The second inequality follows from equation (2.1); we therefore focus on the first one. By subtracting equation (6.8) for $\Omega = \Lambda_R$ and $\Omega = \mathrm{Ann}(r, R)$ with $x = (0, R)$, we find

$$\sum_{y\in\partial\Lambda_R} (4 - d_y)\big[\phi^0_{\Lambda_R}[A(x, y)] - \phi^0_{\mathrm{Ann}(r,R)}[A(x, y)]\big] = \sum_{y\in\partial\Lambda_r} (4 - d_y)\phi^0_{\mathrm{Ann}(r,R)}[A(x, y)]. \tag{6.9}$$

We will now estimate the terms on the left- and right-hand sides of the above.

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Remark 6.7. It is reasonable to expect that the left-hand side is of order $R\pi_1^+(R)^2\Delta_{p_c}(r,R)$, while the right-hand side is of order $r\pi_1^+(R)\pi_1^+(r)\pi_2(r,R)$, which would produce the desired result. Nevertheless, we are not able to prove this due to boundary terms near corners of the inner box, which we cannot control. Instead, we will use a more sophisticated strategy. Let us mention that the conjectural scaling limit of the model suggests that $\pi_1^+(r,R) \asymp (r/R)$ and $\pi_2(r,R) \asymp \sqrt{r/R}$, which would imply that $\Delta_{p_c}(r,R) \asymp \sqrt{r/R}$.

We start with the right-hand side. For $y \in \partial\Lambda_r$, the event $A(x,y)$ occurs if and only if x and y are connected in ω and Λ_r and Λ_R are connected by a path in ω^*. Quasi-multiplicativity implies that

$$\sum_{y\in\partial\Lambda_r}(4-d_y)\phi^0_{\mathrm{Ann}(r,R)}[A(x,y)] \asymp \pi_1^+(R)\pi_2(r,R)\sum_{k\le r/2}\phi^0_{\mathbb{U}}[(k,0)\longleftrightarrow\partial\Lambda_r], \tag{6.10}$$

where $\mathbb{U} := \{x\in\mathbb{Z}^2 : x_1\le 0 \text{ or } x_2\le 0\}$ is the lower-left three-quarter plane.

We turn to the left-hand side. For $y\in\partial\Lambda_R$, $A(x,y)$ corresponds to the event that x and y are connected. Also, the measure $\phi^0_{\mathrm{Ann}(r,R)}$ may be viewed as the measure in Λ_R conditioned on the event $\{\Lambda_r\equiv 0\}$ that every edge with at least one endpoint in Λ_{r-1} is closed. The previous study of $\Delta_{p_c}(r,R)$ thus implies that for any $r\le R/2$,

$$\phi^0_{\Lambda_R}[A(x,y)]-\phi^0_{\Lambda_R}[A(x,y)\,|\,\Lambda_r\equiv 0]\lesssim\Delta_{p_c}(r,R)\big[\phi^0_{\Lambda_R}[A(x,y)]-\phi^0_{\Lambda_R}[A(x,y)\,|\,\Lambda_{R/2}\equiv 0]\big].$$

Indeed, consider the coupling constructed in Theorem 4.1(ii) between $\phi^0_{\Lambda_R}$ and $\phi^0_{\Lambda_R}[.\,|\,\Lambda_r\equiv 0]$ inside $\Lambda_{R/2}$ (see also Remark 4.3 for the application of the theorem to $\phi^0_{\Lambda_R}$ rather than ϕ). To observe a difference for the event $A(x,y)$, it is necessary that the boundary conditions induced on $\Lambda^c_{R/2}$ by the two configurations are distinct. Theorem 4.9 states that this occurs with a probability of order $\Delta_{p_c}(r,R)$. Finally, when the boundary conditions are indeed distinct, the difference of probabilities of $A(x,y)$ is bounded from above by $\phi^0_{\Lambda_R}[A(x,y)]-\phi^0_{\Lambda_R}[A(x,y)\,|\,\Lambda_{R/2}\equiv 0]$.

Summing the above display, we deduce that

$$\sum_{y\in\partial\Lambda_R}(4-d_y)\big[\phi^0_{\Lambda_R}[A(x,y)]-\phi^0_{\mathrm{Ann}(r,R)}[A(x,y)]\big]\lesssim\Delta_{p_c}(r,R)\Sigma_R, \tag{6.11}$$

where

$$\Sigma_R := \sum_{y\in\partial\Lambda_R}(4-d_y)\big[\phi^0_{\Lambda_R}[A(x,y)]-\phi^0_{\Lambda_R}[A(x,y)\,|\,\Lambda_{R/2}\equiv 0]\big] \tag{6.12}$$

is a constant that depends on R only. Inserting equations (6.10) and (6.11) into equation (6.9) gives

$$\Delta_{p_c}(r,R)\Sigma_R\gtrsim\pi_1^+(R)\pi_2(r,R)\sum_{k\le r/2}\phi^0_{\mathbb{U}}[(k,0)\longleftrightarrow\partial\Lambda_r],$$

for any $r\le R/2$. Furthermore, equations (6.11) and (6.9) applied to $r=R/2$ show that

$$\Sigma_R\asymp\pi_1^+(R)\sum_{k\le R/4}\phi^0_{\mathbb{U}}[(k,0)\longleftrightarrow\partial\Lambda_R].$$

From the two displays above, we conclude that

$$\Delta_{p_c}(r,R)\gtrsim\pi_2(r,R)\frac{\sum_{k\le r/2}\phi^0_{\mathbb{U}}[(k,0)\longleftrightarrow\partial\Lambda_r]}{\sum_{k\le R/4}\phi^0_{\mathbb{U}}[(k,0)\longleftrightarrow\partial\Lambda_R]}\gtrsim\pi_2(r,R)(r/R). \tag{6.13}$$

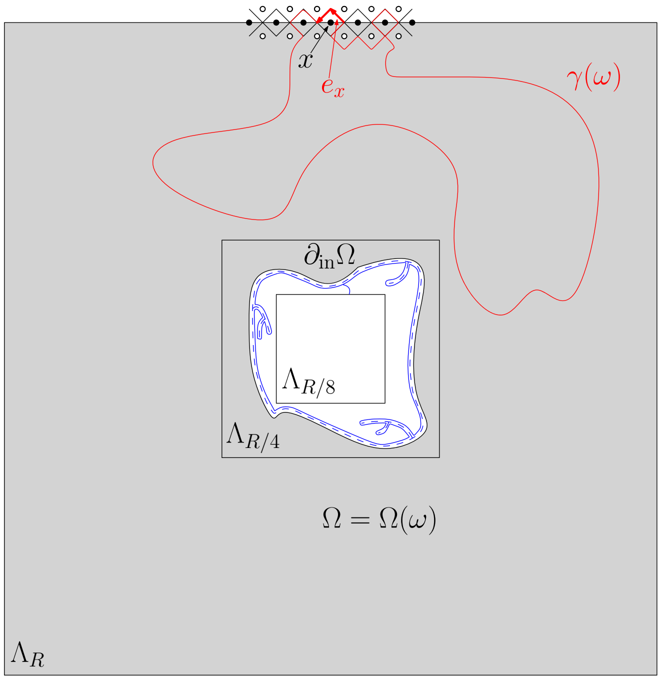

Figure 15. *In grey, the domain $\mathcal{H}$ (which is a topological R-annulus) carved by the event F (in blue). In red, the exploration path $\gamma = \gamma(\omega)$ going around the boundary vertex x.*

The inequality above is obtained by summing over $j = 0, \ldots, R/r$ and $k = 1, \ldots, r/2$ the inequality below:

$$\phi^0_{\mathbb{U}}[(k + rj/2, 0) \longleftrightarrow \partial\Lambda_R] \le \phi^0_{\mathbb{U}}[(k, 0) \longleftrightarrow \partial\Lambda_r],$$

which is a direct consequence of the comparison between boundary conditions in equation (CBC). □

Remark 6.8. One can improve equation (6.2) when $r = 1$. Applying the left of equation (6.13) with $r = 1$, the comparison between boundary conditions in equations (CBC) and (2.1) implies that

$$\Delta_{p_c}(R) \gtrsim \frac{\pi_2(R)}{\sum_{k \le R/4} \phi^0_{\mathbb{U}}[(k, 0) \longleftrightarrow \partial\Lambda_R]} \ge \frac{\pi_2(R)}{\frac{1}{4} R\pi_1(3R/4)} \gtrsim \frac{\pi_2(R)}{R\pi_1(R)} \gtrsim \frac{R^{c-2}}{\pi_1(R)}.$$

6.2.3. Stability of $\Delta_p(r, R)$: proof of Proposition 6.4

The proof is based on the following quantity. Call a subgraph Ω of $\mathbb{Z}^2$ a *topological R-annulus* if it is of the form $\Lambda_R \setminus H$ with H a simply connected domain such that $\Lambda_{R/8} \subset H \subset \Lambda_{R/4}$; see Figure 15. The boundary of Ω is split into $\partial\Lambda_R$ and $\partial_{\text{in}}\Omega := \partial\Omega \setminus \partial\Lambda_R$. Set

$$\mathbf{N}_\Omega(p) := \sum_{y \in \partial_{\text{in}}\Omega} \phi^0_{\Omega,p}[y \longleftrightarrow \partial\Lambda_{R/2}].$$

The first observation is the following lemma stating that for $q = 4$, the quantity $\mathbf{N}_\Omega(p)$ does not depend on the choice of the R-annulus Ω or the choice of p such that $R \le L(p)$.

Lemma 6.9. *For every $p \ge p_c$ and $R \le L(p)$,*

$$\sup_\Omega \mathbf{N}_\Omega(p^*) \asymp \sup_\Omega \mathbf{N}_\Omega(p_c) \asymp \inf_\Omega \mathbf{N}_\Omega(p_c) \asymp \inf_\Omega \mathbf{N}_\Omega(p), \tag{6.14}$$

where the infimum and supremum are taken over topological R-annuli.

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Proof. We start by proving that

$$\sup_\Omega \mathbf{N}_\Omega(p_c) \asymp \Sigma_R/\pi_1^+(R) \asymp \inf_\Omega \mathbf{N}_\Omega(p_c), \tag{6.15}$$

where Σ_R was introduced in equation (6.12) in the previous section.

Fix Ω a topological R-annulus, and set $x := (0, R)$. Then due to equation (2.1), for every $y \in \partial_{\text{in}}\Omega$,

$$\phi^0_{\Omega,p_c}[y \longleftrightarrow x \text{ and } \partial_{\text{in}}\Omega \overset{*}{\longleftrightarrow} \partial\Lambda_R] \asymp \pi_1^+(R)\phi^0_{\Omega,p_c}[y \longleftrightarrow \partial\Lambda_{R/2}].$$

In particular, we have that

$$\mathbf{N}_\Omega(p_c) \asymp \frac{1}{\pi_1^+(R)} \sum_{y\in\partial_{\text{in}}\Omega} \phi^0_{\Omega,p_c}[A(x,y)]. \tag{6.16}$$

Subtracting equation (6.8) for the domains Λ_R and Ω, and following the proof of Proposition 6.3, we find that

$$\sum_{y\in\partial_{\text{in}}\Omega} \phi^0_{\Omega,p_c}[A(x,y)] \asymp \Sigma_R,$$

which concludes the proof of equation (6.15).

We now prove that for every p and $R \le L(p)$,

$$\inf_\Omega \mathbf{N}_\Omega(p) \lesssim R^2\pi_4(p;R) \lesssim \sup_\Omega \mathbf{N}_\Omega(p), \tag{6.17}$$

where the infimum and supremum are taken over topological R-annuli. Note that this inequality implies the result. Indeed, since $\mathbf{N}_\Omega(p)$ is increasing in p and $\pi_4(p^*;R) = \pi_4(p;R)$ (by duality), the previous inequality implies that for $p \ge p_c$ and $R \le L(p) = L(p^*)$,

$$\sup_\Omega \mathbf{N}_\Omega(p_c) \ge \sup_\Omega \mathbf{N}_\Omega(p^*) \gtrsim R^2\pi_4(p^*,R) = R^2\pi_4(p;R) \gtrsim \inf_\Omega \mathbf{N}_\Omega(p) \ge \inf_\Omega \mathbf{N}_\Omega(p_c),$$

which combines with equation (6.15) to give the result. We now focus on the proof of equation (6.17). We proceed similarly to [DMT20, Proposition 6.8], but from inside out.

Let F be the event that in $\text{Ann}(R/8, R/4)$, there exists an open circuit surrounding $\Lambda_{R/8}$ that is connected to $\Lambda_{R/8}$, as well as a dual-open circuit, which is necessarily outside the open one; see Figure 15. If F occurs, let $\mathcal{H} = \mathcal{H}(\omega)$ be the graph formed of the union of all open clusters that intersect $\Lambda_{R/8}$, along with all finite components of $\mathbb{Z}^2$ minus the said union. Then due to the definition of F, $\Lambda_{R/8} \subset \mathcal{H} \subset \Lambda_{R/4}$. Write $\Omega(\omega) := \Lambda_R \setminus \mathcal{H}$ for the random topological R-annulus formed by removing $\mathcal{H}$.

When the measure $\phi^0_{\Lambda_R,p}$ is conditioned on $F \cap \{\Omega(\omega) = \Omega\}$, its restriction to Ω is $\phi^0_{\Omega,p}$. Notice that if $y \in \partial_{\text{in}}\Omega$ is connected to $\partial\Lambda_{R/2}$ by an open path, then a four-arm event to distance $R/8$ occurs around y. Thus, using the quasi-multiplicativity and equations (Mix) and (RSW) (to bound the probability of F from below), we find that

$$R^2\pi_4(p;R) \gtrsim \phi_{\Lambda_R,p}[\mathbf{N}_{\Omega(\omega)}(p)\, \mathbb{1}_F] \gtrsim \inf_\Omega \mathbf{N}_\Omega(p).$$

Conversely, the separation of arms for the four-arm event and equation (RSW) show that for each $y \in \text{Ann}(5R/32, 7R/32)$,

$$\phi_{\Lambda_R,p}[F \cap \{y \in \partial_{\text{in}}\Omega(\omega)\} \cap \{y \longleftrightarrow \partial\Lambda_{R/2}\}] \gtrsim \pi_4(p;R).$$

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Summing over all y, we find

$$R^2\pi_4(p;R)\lesssim\phi_{\Lambda_R,p}[\mathbf{N}_{\Omega(\omega)}(p)\mathbb{1}_F]\leq\sup_{\Omega}\mathbf{N}_\Omega(p).$$ □

Remark 6.10. The previous proof shows as a byproduct that $\pi_4(p;R)\asymp\pi_4(R)$ for every $R\leq L(p)$. It is somehow surprising that the stability of π_4 below the correlation length can be directly extracted from the parafermionic observable. Recall, however, that this is only valid for $q=4$.

Remark 6.11. The previous proof also implies that $R^2\pi_4(R)$ is of the order of $\mathbf{N}_\Omega(p_c)$ for every topological R-annulus Ω. In particular, taking $\Omega=\Lambda_{R/8}$ gives that

$$R^2\pi_4(R)\asymp R\pi_1^+(R). \tag{6.18}$$

The conjectural scaling limit of the model suggests that $\pi_1^+(R)\asymp R^{-1}$, which would imply that $\pi_4(R)\asymp R^{-2}$. We see that in this case, the fact that $\Delta_{p_c}(R)\gg\pi_4(R)$ is crucial for the bound in equation (6.2).

Proof of Proposition 6.4. We treat the case $p>p_c$; the case $p<p_c$ can be done similarly. Fix $r\leq R\leq L(p)$. The inequality in equation (5.11) implies that

$$\Big|\log\frac{\Delta_p(r,R)}{\Delta_{p_c}(r,R)}\Big|\lesssim\underbrace{\int_{p_c}^{p}\Big(\sum_{\ell=1}^{R}\ell\Delta_u(\ell)\Big)du}_{(A)}+\underbrace{\int_{p_c}^{p}\Big(\sum_{\ell=R}^{L(u)}\ell\Delta_u(\ell)\Delta_u(R,\ell)\Big)du}_{(B)},$$

and we need to prove that (A) and (B) are bounded by constants that are independent of r, R and $p>p_c$. The bound on (B) is easy to obtain since equation (5.1) shows that $(B)\lesssim\phi_p[\mathcal{C}(\Lambda_R)]-\phi_{p_c}[\mathcal{C}(\Lambda_R)]\leq 1$. We therefore focus on (A).

Choose the topological R-annulus Ω minimising $\mathbf{N}_\Omega(p)$, and observe that by Lemma 6.9, $\mathbf{N}_\Omega(p)\asymp\mathbf{N}_\Omega(p_c)$. We claim that

$$\tfrac{d}{du}\log\mathbf{N}_\Omega(u)\gtrsim\sum_{\ell=1}^{R}\ell\Delta_u(\ell). \tag{6.19}$$

Observe that equation (6.19) implies that $(A)\lesssim\log\mathbf{N}_\Omega(p)-\log\mathbf{N}_\Omega(p_c)\lesssim 1$, which concludes the proof of the proposition. Thus, we only need to prove equation (6.19), which we do next.

Start by observing that

$$\tfrac{d}{du}\log\mathbf{N}_\Omega(u)=\sum_{y\in\partial_{\rm in}\Omega}\tfrac{d}{du}\phi^0_{\Omega,u}[y\longleftrightarrow\partial\Lambda_{R/2}]=\tfrac{1}{u(1-u)}\sum_{y\in\partial_{\rm in}\Omega}\sum_{e\in\Omega}\mathrm{Cov}[y\longleftrightarrow\partial\Lambda_{R/2},\omega_e]. \tag{6.20}$$

Fix some $y\in\partial_{\rm in}\Omega$, and set $N:=\lfloor\log_8 R\rfloor$. Then there exist points $z_1,\dots,z_{N-2}$ in Ω such that for each k, $\Lambda_{8^k}(z_k)\subset\Omega$ but y is not connected to $\partial\Lambda_R$ in the subgraph $\Omega\setminus\Lambda_{2\cdot 8^k}(z_k)$; the latter condition includes the situation where $y\in\Lambda_{2\cdot 8^k}(z_k)$. See Figure 16 and its caption for an explanation of this elementary fact. For $k<\ell$, since $\Lambda_{2\cdot 8^k}(z_k)$ intersects the boundary of Ω, but $\Lambda_{8^\ell}(z_\ell)\subset\Omega$, we conclude that $\Lambda_{8^k/2}(z_k)$ and $\Lambda_{8^\ell/2}(z_\ell)$ do not intersect. Thus, the boxes $(\Lambda_{8^k/2}(z_k))_{k\leq N-2}$ are pairwise disjoint.

Now, for any such k and $e\in\Lambda_{8^k/2}(z_k)$, a similar argument as that used for the lower bound in equation (5.2) shows that

$$\mathrm{Cov}[y\longleftrightarrow\partial\Lambda_{R/2},\omega_e]\gtrsim\Delta_u(8^k)\phi^0_{\Omega,u}[y\longleftrightarrow\partial\Lambda_{R/2}]. \tag{6.21}$$

Indeed, consider the coupling constructed in Theorem 4.1(ii) between $\phi^0_{\Omega,u}[\,\cdot\,|\omega_e=0]$ and $\phi^0_{\Omega,u}[\,\cdot\,|\omega_e=1]$ inside the box $\Lambda_{8^k/4}(e)$ (see also Remark 4.3 for the application of the theorem to $\phi^0_{\Omega,u}$

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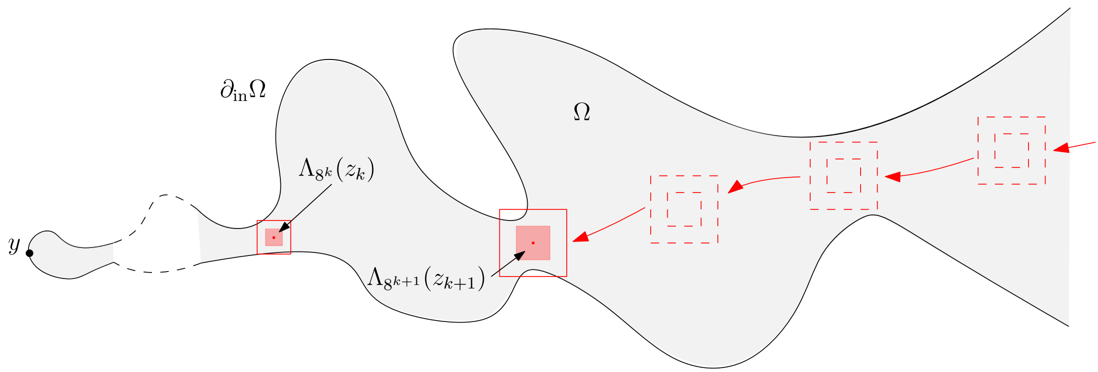

Figure 16. *By sliding the boxes along a given path one by one from the exterior towards y, one finds a first time that the twice larger box $\Lambda_{2\cdot 8^k}(z_k)$ disconnects y from the boundary. Note that by construction, the first time it stops cannot be quite the same for different k. In particular, the box $\Lambda_{8^k/2}(z_k)$ being at a distance $8^k/2$ of the boundary of $\Lambda_{8^k}(z_k)$, it is also at such a distance of $\partial_{\text{in}}\Omega$. Since, on the contrary, $\Lambda_{8^\ell/2}(z_\ell)$ is within a distance $8^\ell \times 3/2$ of it, we immediately deduce that the two boxes do not intersect as soon as $\ell < k$.*

rather than ϕ). With a probability of order $\Delta_u(8^k)$, we have $\tau < \infty$ and the boundary conditions induced on $\mathcal{F}_\tau$ by the two configurations form a boosting pair. When this occurs, by equation (RSW), the configuration inside $\mathcal{F}_\tau$ has a probability of at least

$$\phi^0_{\Omega,u}[y \longleftrightarrow \Lambda_{8^k/2}(z_k) \text{ and } \Lambda_{8^k/2}(z_k) \longleftrightarrow \partial\Lambda_{R/2}] \asymp \phi^0_{\Omega,u}[y \longleftrightarrow \partial\Lambda_{R/2}]$$

to be such that the connection between y and $\partial\Lambda_{R/2}$ only occurs for the higher boundary condition.

Summing equation (6.21) over e and keeping in mind that all covariances in equation (6.20) are non-negative, we find

$$\begin{aligned}\tfrac{d}{du}\phi^0_{\Omega,u}[y \longleftrightarrow \partial\Lambda_{R/2}] &\gtrsim \sum_{k=1}^{N-2} \sum_{e\in\Lambda_{8^k/2}(z_k)} \mathrm{Cov}[y \longleftrightarrow \partial\Lambda_{R/2}, \omega_e] \\ &\gtrsim \sum_{k=1}^{N-2} 8^{2k}\Delta_u(8^k)\phi^0_{\Omega,u}[y \longleftrightarrow \partial\Lambda_{R/2}] \\ &\gtrsim \Big(\sum_{\ell=1}^{R} \ell\Delta_u(\ell)\Big)\phi^0_{\Omega,u}[y \longleftrightarrow \partial\Lambda_{R/2}].\end{aligned}$$

For the last inequality, we used the fact that $\Delta_u(\ell) \asymp \Delta_u(8^k)$ for any $8^k \leq \ell \leq 8^{k+1}$. Finally, summing over y, we find

$$\tfrac{d}{du}\mathbf{N}_\Omega(u) \gtrsim \mathbf{N}_\Omega(u)\sum_{\ell=1}^{R}\ell\Delta_u(\ell),$$

which concludes the proof of equation (6.19) and the whole proposition. □

7. Proofs of the stability theorems

In this section, we prove stability results for crossing probabilities, arm events and Δ_p. First, observe that due to the previous section, Propositions 5.1 and 5.9 immediately lead to the following corollary.

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Corollary 7.1.

(i) *Fix* $\eta > 0$. *For every* $p \in (0,1)$ *and every* η*-regular quad* $(\mathcal{D}, a, b, c, d)$ *at scale* $R \leq L(p)$,

$$\tfrac{\mathrm{d}}{\mathrm{d}p}\phi_p[\mathcal{C}(\mathcal{D})] \asymp R^2\Delta_p(R) + \sum_{\ell=R}^{L(p)} \ell\Delta_p(\ell)\Delta_p(R,\ell), \tag{7.1}$$

where the constants in $\asymp$ *depend on* η.

(ii) *For every* $p \in (0,1)$, $k = 1$ *or* $k \geq 2$ *even and* $r \leq R \leq L(p)$,

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}p}\log \pi_k(p;r,R)\right| \lesssim R^2\Delta_p(R) + \sum_{\ell=R}^{L(p)} \ell\Delta_p(\ell)\Delta_p(R,\ell), \tag{7.2}$$

where the constants in $\lesssim$ *depend on* k.

(iii) *For every* $p \in (0,1)$ *and two edges* e *and* f *at a distance* $R \leq L(p)$ *from each other,*

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}p}\log \mathrm{Cov}_p(\omega_e,\omega_f)\right| \lesssim R^2\Delta_p(R) + \sum_{\ell=R}^{L(p)} \ell\Delta_p(\ell)\Delta_p(R,\ell). \tag{7.3}$$

Proof. By equations (1.20) and (1.15), for any $p \in (0,1)$ and $R \leq L(p)$, $\sum_{\ell=1}^{R} \ell\Delta_p(\ell) \lesssim R^2\Delta_p(R)$. Insert this into Propositions 5.1 and 5.9 to obtain the desired results. □

We next prove the stability of crossing probabilities and arm event probabilities stated in Theorem 1.4.

Proof of Theorem 1.4. Fix $p \neq p_c$. We prove equation (1.7); the stability for arm events can be deduced similarly. Fix some η-regular discrete quad $(\mathcal{D}, a, b, c, d)$ at some scale $R \geq 1$. The inequality given by equation (1.7) is trivial when $R > L(p)$, and we focus on the case where $R \leq L(p)$. Applying equation (7.1) to u between p_c and p, we find

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}u}\phi_u[\mathcal{C}(\mathcal{D})] &\asymp R^2\Delta_u(R) + \sum_{\ell=R}^{L(p)} \ell\Delta_u(\ell)\Delta_u(R,\ell) + \sum_{\ell=L(p)}^{L(u)} \ell\Delta_u(\ell)\Delta_u(R,\ell) \\ &\lesssim \left(\tfrac{R}{L(p)}\right)^{\delta}\left[L(p)^2\Delta_u(L(p)) + \sum_{\ell=L(p)}^{L(u)} \ell\Delta_u(\ell)\Delta_u(L(p),\ell)\right],\end{aligned} \tag{7.4}$$

where the inequality is due to a simple computation based on the quasi-multiplicativity of Δ_p and equation (1.20). The terms for $\ell \geq L(p)$ are dominated by the corresponding sum in the second line.

Corollary 7.1 applied to $\mathcal{C}(\Lambda_{L(p)})$ together with equation (7.4) imply that

$$\tfrac{\mathrm{d}}{\mathrm{d}u}\phi_u[\mathcal{C}(\mathcal{D})] \lesssim \left(\tfrac{R}{L(p)}\right)^{\delta} \tfrac{\mathrm{d}}{\mathrm{d}u}\phi_u[\mathcal{C}(\Lambda_{L(p)})].$$

Integrate the above between p_c and p to find

$$|\phi_p[\mathcal{C}(\mathcal{D})] - \phi_{p_c}[\mathcal{C}(\mathcal{D})]| \lesssim \left(\tfrac{R}{L(p)}\right)^{\delta} |\phi_p[\mathcal{C}(\Lambda_{L(p)})] - \phi_{p_c}[\mathcal{C}(\Lambda_{L(p)})]| \leq \left(\tfrac{R}{L(p)}\right)^{\delta}.$$ □

Remark 7.2. One may also obtain the following improvement of equation (1.8) (similar to equation (1.7)):

$$\exp[-C(R/L(p))^{\delta}] \leq \frac{\pi_1(p;R)}{\pi_1(p_c,R)} \leq \exp[C(R/L(p))^{\delta}] \qquad \text{for all } R \leq L(p), \tag{7.5}$$

where $\delta, C > 0$ are universal constants depending only on q.

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Furthermore, since Corollary 7.1 applies to logarithmic derivatives of certain arm event probabilities, (7.5) may also be shown for $\pi_k(p;R)$ (with $k=1$ or $k\geq 2$ even) and $\pi_k^+(p;R)$ (for any $k\geq 1$), with C depending on k. Notice, however, that we do not claim to prove stability for probabilities of arm events in the half-plane *with boundary conditions on the half-plane*.

We now turn to the proof of the stability of Δ_p. Let us note that for $q=4$, the stability of Δ_p was also proved in Section 6.2, as a step towards equation (1.20).

Proof of Theorem 1.6(iii). Fix $p\neq p_c$, $R\leq L(p)$ and two edges e and f at a distance R from each other. For u between p and p_c, Corollary 7.1 gives

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}u}\log \mathrm{Cov}_u(\omega_e,\omega_f)\right|\lesssim R^2\Delta_u(R)+\sum_{\ell=R}^{L(u)}\ell\Delta_u(\ell)\Delta_u(R,\ell)\lesssim\tfrac{\mathrm{d}}{\mathrm{d}u}\phi_u[\mathcal{C}(\Lambda_R)].$$

By integrating the above between p_c and p, we find

$$\frac{\mathrm{Cov}_p(\omega_e,\omega_f)}{\mathrm{Cov}_{p_c}(\omega_e,\omega_f)}\asymp 1. \tag{7.6}$$

Now apply equation (5.3) to deduce that

$$\Delta_p(R)\asymp\sqrt{\mathrm{Cov}_p(\omega_e,\omega_f)}\asymp\sqrt{\mathrm{Cov}_{p_c}(\omega_e,\omega_f)}\asymp\Delta_{p_c}(R). \qquad \square$$

Remark 7.3. It will not surprise the reader that the same type of improved stability as in equation (7.5) may be shown for the covariance. Getting the same result for Δ_p itself seems more difficult as we crucially rely on an up-to-constant relation between Δ_p and the covariance and that the derivative of Δ_p itself is less obvious.

8. Derivation of the scaling relations

This section is dedicated to proving the scaling relations (Theorems 1.9, 1.10 and 1.11). The proof of the near-critical scaling relations (Theorem 1.11) is based on the stability below the characteristic length (Theorem 1.4 and Theorem 1.6(iii)). With the latter results at our disposal, the proofs of the critical and near-critical scaling relations given by equations 1.22–1.27 are very close to those for Bernoulli percolation and contain no significant innovation. For this reason, we are voluntarily quick on these proofs, trying to merely recall the crucial ingredients.

The main novelties in this section are the independent proof of the scaling relation involving the magnetic field (Section 8.3) and the derivation of the scaling relation involving α.

8.1. *Scaling relations at criticality: proof of Theorem 1.9*

In this section, we work with $p=p_c$, and we drop it from the notation. We will prove a stronger result, which implies equation (1.22) when taking $x\in\partial\Lambda_R$.

Lemma 8.1. *Fix $1\leq q\leq 4$. For every $R\geq 1$ and every $x\in\Lambda_R$,*

$$\pi_1(R)^2\lesssim\phi_{p_c}[0\overset{\Lambda_{2R}}{\longleftrightarrow}x,0\longleftrightarrow\partial\Lambda_R]\leq\phi_{p_c}[0\longleftrightarrow x]\lesssim\pi_1(|x|)^2. \tag{8.1}$$

Proof. The middle inequality is obvious. For the right one, observe that if 0 and x are connected, then $0\longleftrightarrow\partial\Lambda_r$ and $x\longleftrightarrow\partial\Lambda_r(x)$, where $r:=\lfloor|x|/C_{\mathrm{mix}}\rfloor$. The invariance under translations of ϕ_{p_c}, the mixing property given by equation (Mix) and the quasi-multiplicativity of the one-arm event in equation (2.3) give that

$$\phi_{p_c}[0 \longleftrightarrow x] \lesssim \pi_1(r)^2 \lesssim \pi_1(|x|)^2.$$

We now prove the left inequality. The FKG inequality in equation (FKG) implies that

$$\begin{aligned}\phi_{p_c}[0 \xleftrightarrow{\Lambda_{2R}} x, 0 \longleftrightarrow \partial\Lambda_R] &\geq \phi_{p_c}[A_R, 0 \longleftrightarrow \partial\Lambda_{2R}, x \longleftrightarrow \partial\Lambda_{3R}(x)] \\ &\geq \phi_{p_c}[A_R]\pi_1(2R)\pi_1(3R) \\ &\gtrsim \pi_1(R)^2,\end{aligned}$$

where we used equations (RSW′) and (2.3). □

Recall that C is the cluster of 0. Let $\mathrm{rad}(\mathsf{C}) := \max\{r : \mathsf{C} \text{ intersects } \partial\Lambda_r\}$ be the radius of C.

Lemma 8.2. *For every $R \geq 1$,*

$$R^2\pi_1(R) \lesssim \phi_{p_c}[|\mathsf{C}|\,|R \leq \mathrm{rad}(\mathsf{C}) < 4R] \leq \sqrt{\phi_{p_c}[|\mathsf{C}|^2\,|R \leq \mathrm{rad}(\mathsf{C}) < 4R]} \lesssim R^2\pi_1(R). \tag{8.2}$$

Proof. Fix $n \geq 1$. The inequality in the middle is the Cauchy-Schwarz inequality. For the first inequality, observe that equations (8.1) and (RSW′) imply that

$$\phi_{p_c}[|\mathsf{C}|\mathbb{1}_{R \leq \mathrm{rad}(\mathsf{C}) < 4R}] \geq \sum_{x \in \Lambda_R} \phi_{p_c}[0 \xleftrightarrow{\Lambda_{2R}} x, \Lambda_{2R} \not\longleftrightarrow \partial\Lambda_{4R}] \gtrsim |\Lambda_R|\pi_1(R)^2.$$

Divide the above by $\phi_{p_c}[R \leq \mathrm{rad}(\mathsf{C}) < 4R] \leq \pi_1(R)$ to obtain the first inequality in equation (8.2).

We turn to the last inequality of equation (8.2). We have

$$\phi_{p_c}[|\mathsf{C}|^2\mathbb{1}_{R \leq \mathrm{rad}(\mathsf{C}) < 4R}] \leq \sum_{x,y \in \Lambda_{4R}} \phi_{p_c}[0 \longleftrightarrow x, 0 \longleftrightarrow y, 0 \longleftrightarrow \partial\Lambda_R].$$

Fix $x, y \in \Lambda_{4R}$, and assume first that $|x| \leq |y| \leq |x-y|$. Set $\ell := |x|$ and $k := |y|$. Observe that the event on the right induces the following events, which are listed along with the order – up to constants – of their probabilities (which are obtained thanks to equation (2.3)):

- $0 \longleftrightarrow \partial\Lambda_{\ell/C_{\mathrm{mix}}}$ – of probability of order $\pi_1(\ell)$;
- $x \longleftrightarrow \partial\Lambda_{\ell/C_{\mathrm{mix}}}(x)$ – of probability of order $\pi_1(\ell)$;
- $y \longleftrightarrow \partial\Lambda_{k/C_{\mathrm{mix}}}(y)$ – of probability of order $\pi_1(k)$;
- $\partial\Lambda_{\min\{\ell,k\}} \longleftrightarrow \partial\Lambda_k$ – of probability of order $\pi_1(k)/\pi_1(\ell)$;
- $\partial\Lambda_{\min\{C_{\mathrm{mix}}k,R\}} \longleftrightarrow \partial\Lambda_R$ – of probability of order $\pi_1(R)/\pi_1(k)$.

Several iterations of the mixing property in equations (Mix) and (2.3) imply that

$$\phi_{p_c}[0 \longleftrightarrow x, 0 \longleftrightarrow y, 0 \longleftrightarrow \partial\Lambda_R] \lesssim \pi_1(\ell)\pi_1(k)\pi_1(R). \tag{8.3}$$

Observe now that for $1 \leq \ell \leq k \leq 4R$, there are 8ℓ vertices $x \in \mathbb{Z}^2$ with $|x| = \ell$ and $8k$ vertices y with $|y| = k$. Thus,

$$\sum_{\substack{x,y \in \Lambda_{4R} \\ |x| \leq |y| \leq |x-y|}} \phi_{p_c}[0 \longleftrightarrow x, 0 \longleftrightarrow y, 0 \longleftrightarrow \partial\Lambda_R] \lesssim \sum_{1 \leq \ell \leq k \leq 4R} \ell\pi_1(\ell)k\pi_1(k)\pi_1(R) \lesssim R^4\pi_1(R)^3,$$

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where in the last line, we used that

$$\sum_{k=1}^{R} k\pi_1(k) \lesssim R^2\pi_1(R) \tag{8.4}$$

which is a consequence of equations (2.2) and (2.3). The same upper bound may be obtained for any of the other five possible orderings of $|x|, |y|, |x-y|$. Overall, we conclude that

$$\phi_{p_c}[|\mathsf{C}|^2 \mathbb{1}_{R \le \mathrm{rad}(\mathsf{C}) < 4R}] \lesssim R^4 \pi_1(R)^3,$$

which gives the result by dividing by

$$\phi_{p_c}[R \le \mathrm{rad}(\mathsf{C}) < 4R] \ge \phi_{p_c}[0 \longleftrightarrow \partial\Lambda_R, \Lambda_R \not\longleftrightarrow \partial\Lambda_{2R}] \gtrsim \pi_1(R), \tag{8.5}$$

where in the last inequality, we used the mixing property in equations (Mix) and (RSW′). □

Proof of Theorem 1.9. Lemma 8.1 applied with $R = 2|x|$ and equation (2.3) directly imply equation (1.22). We turn to the proof of equation (1.23). Fix $R \ge 1$ and $r := \varphi(R)$. Let us start with the lower bound on $\phi_{p_c}[|\mathsf{C}| \ge R]$. Let $c \in (0,1)$ be the constant appearing in the first bound $\lesssim$ of equation (8.2). Then using the definition of φ, equation (8.2), the Paley–Zygmund inequality and equation (8.5), we find

$$\begin{aligned}
\phi_{p_c}[|\mathsf{C}| \ge \tfrac{c}{2}R] &\ge \phi_{p_c}[|\mathsf{C}| \ge \tfrac{c}{2}r^2\pi_1(r)] \\
&\ge \phi_{p_c}[|\mathsf{C}| \ge \tfrac{1}{2}\phi_{p_c}(|\mathsf{C}| \mid r \le \mathrm{rad}(\mathsf{C}) < 4r)] \\
&\gtrsim \frac{\phi_{p_c}[|\mathsf{C}| \mid r \le \mathrm{rad}(\mathsf{C}) < 4r]^2}{\phi_{p_c}[|\mathsf{C}|^2 \mid r \le \mathrm{rad}(\mathsf{C}) < 4r]}\,\phi_{p_c}[r \le \mathrm{rad}(\mathsf{C}) < 4r] \\
&\gtrsim \phi_{p_c}[r \le \mathrm{rad}(\mathsf{C}) < 4r] \\
&\gtrsim \pi_1(r).
\end{aligned}$$

This concludes the proof of the lower bound since equation (2.3) implies that $\phi(\frac{c}{2}R) \asymp \phi(R) = r$.

We turn to the complementary upper bound. Using the Markov inequality in the third line and the definition of φ and equation (8.1) in the fourth, we obtain that

$$\begin{aligned}
\phi_{p_c}[|\mathsf{C}| \ge R] &= \phi_{p_c}[|\mathsf{C}| \ge R, \mathrm{rad}(\mathsf{C}) > r] + \phi_{p_c}[|\mathsf{C}| \ge R, \mathrm{rad}(\mathsf{C}) \le r] \\
&\le \pi_1(r) + \phi_{p_c}[|\mathsf{C} \cap \Lambda_r| \ge R] \\
&\le \pi_1(r) + \frac{1}{R}\sum_{u \in \Lambda_r} \phi_{p_c}[0 \longleftrightarrow u] \\
&\lesssim \pi_1(r) + \frac{1}{r^2\pi_1(r)}\sum_{u \in \Lambda_r} \pi_1(|u|)^2 \lesssim \pi_1(r),
\end{aligned}$$

where the last inequality follows from equation (2.2) via the following computation:

$$\frac{1}{k}\sum_{k=1}^{r} \frac{k\pi_1(k)^2}{r\pi_1(r)^2} \lesssim \frac{1}{r}\sum_{k=1}^{r} (k/r)^{1-2c} \lesssim 1.$$

□

8.2. Scaling relations in the near-critical regime: proof of Theorem 1.11

By Theorem 1.3 and equation (2.3), we have that $L(p) \asymp \xi(p)$ and $\pi_1(L(p)) \asymp \pi_1(\xi(p))$, so we only need to show equations 1.25–1.28 with $L(p)$ instead of $\xi(p)$.

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8.2.1. Proof of equation (1.25) (scaling relation between β, ν and ξ_1)
On the one hand, the stability below the characteristic length (Theorem 1.4) gives

$$\theta(p) \le \pi_1(p; L(p)) \lesssim \pi_1(L(p)).$$

On the other hand, the FKG inequality given by equations (FKG) and (RSW′) and Corollary 2.15 imply that

$$\theta(p) \ge \phi_p[0 \longleftrightarrow \partial\Lambda_{2L(p)}, A_{L(p)}, \Lambda_{L(p)} \longleftrightarrow \infty] \gtrsim \pi_1(p; 2L(p)) \ge \pi_1(2L(p)) \gtrsim \pi_1(L(p)). \qquad \square$$

8.2.2. Proof of equation (1.26) (scaling relation between γ, ν and ξ_1)
We start with the case $p < p_c$. Due to equations (Mix) and (RSW), Theorem 1.4 and Proposition 2.13, we have

$$c\,\pi_1(R)^2 \exp[-C\,|x|/L(p)] \le \phi_p[0 \longleftrightarrow x] \le C\,\pi_1(R)^2 \exp[-c\,|x|/L(p)], \tag{8.6}$$

where $R := \min\{|x|, L(p)\}$ and $c, C > 0$ are uniform constants. Summing equation (8.6) over $x \in \mathbb{Z}^2$, we find

$$\sum_{R=1}^{L(p)} R\pi_1(R)^2 \lesssim \chi(p) \lesssim \sum_{R=1}^{L(p)} R\pi_1(R)^2 + \pi_1(L(p))^2 \sum_{R>L(p)} R\exp[-c\,R/L(p)].$$

Due to the exponential factor, the second term on the right-hand side of the above is of order $L(p)^2\pi_1(L(p))^2$. Finally, equation (1.26) follows from the observation that due to equations (2.2) and (2.3),

$$\pi_1(R) \lesssim (L(p)/R)^{1/2-c}\pi_1(L(p)) \qquad \text{for } R \le L(p),$$

and from the equivalence in equation (1.6) between $L(p)$ and $\xi(p)$.

We now turn to the case $p > p_c$. The only additional difficulty comes from the fact that we need to force 0 and x not to be connected to infinity. For the lower bound, sum over every $x \in \Lambda_{L(p)/2}$ the following inequality

$$\phi_p[0 \longleftrightarrow x, 0 \not\longleftrightarrow \infty] \ge \phi_p[\Lambda_{L(p)} \not\longleftrightarrow \infty]\phi^0_{\Lambda_{L(p)},p}[0 \longleftrightarrow x] \gtrsim \pi_1(R)^2,$$

where the first inequality is due to the spatial Markov property in equation (SMP), and the second inequality follows from equations (RSW) and (p-MON) and an argument similar to Lemma 8.1.

We turn to the upper bound. Let A^*_x be the event that there exists a circuit in ω^* surrounding 0 and x. By considering four translations and rotations of A^*_x and applying the FKG inequality, we find that

$$\phi_p[A^*_x]^4 \le \phi_p[\Lambda_{x/2} \not\longleftrightarrow \infty] \lesssim \exp(-4c|x|/L(p)) \tag{8.7}$$

for some constant $c > 0$, due to Corollary 2.15. Applying the FKG inequality again, we have

$$\begin{aligned}\phi_p[0 \longleftrightarrow x, 0 \not\longleftrightarrow \infty] &\le \phi_p[0 \longleftrightarrow x]\phi_p[A^*_x] \\ &\lesssim \pi_1(R)^2 \exp(-c|x|/L(p)).\end{aligned}$$

For the second inequality, we use equation (8.7) and bound the connection probability between 0 and x by the argument of Lemma 8.1 together with Theorem 1.4. Summing this inequality over $x \in \mathbb{Z}^2$ gives the upper bound. $\square$

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8.2.3. Proof of equation (1.27) (scaling relation between ι and ν)
Assume $p > p_c$; the case $p < p_c$ is identical. Write $L = L(p)$. Use Corollary 7.1 and integrate the derivative of $g(p) := \phi_p[\mathcal{C}(\Lambda_L)]$ between p_c and p to get

$$\int_{p_c}^{p} L^2\Delta_u(L)du \lesssim g(p) - g(p_c) \lesssim 1. \tag{8.8}$$

In the other direction, let $p_0 \in [p_c, p]$ be such that $L(p_0) = RL(p)$ for some $R > 1$. Theorem 1.4 and the definition of $L(p)$ imply that

$$g(p) - g(p_0) \geq g(p) - g(p_c) - CR^{-\varepsilon} \gtrsim 1 \tag{8.9}$$

provided that R is large enough. For $u \in [p_0, p]$, Corollary 7.1 together with the quasi-multiplicativity property Theorem 1.6(ii) imply

$$g'(u) \lesssim L(u)^2\Delta_u(L(u)) \lesssim L^2\Delta_u(L).$$

(Note that the constant in $\lesssim$ depends on R.) Integrating the previous displayed equation between p_0 and p and then using equation (8.9) gives

$$1 \lesssim g(p) - g(p_0) \lesssim \int_{p_0}^{p} L^2\Delta_u(L)du \leq \int_{p_c}^{p} L^2\Delta_u(L)du.$$

Together with equation (8.8), the previous displayed equation and the stability of $\Delta_u(L)$ given by Theorem 1.6(iii) show that

$$1 \asymp \int_{p_c}^{p} L^2\Delta_u(L)du \asymp (p - p_c)L^2\Delta_{p_c}(L),$$

which concludes the proof. □

8.2.4. Proof of equation (1.28) (scaling relation between ι and α)
A straightforward computation involving equation (1.9) shows that

$$f''(p) = 2\frac{\mathrm{d}}{\mathrm{d}p}\phi_p[\omega_e] = 2\sum_f \mathrm{Cov}_p(\omega_e, \omega_f),$$

where e is a fixed edge of $\mathbb{Z}^2$ and the sum is over all edges f. By Lemma 5.3 applied to $\mathcal{D}$ formed of the single edge e, we find (for the second equivalence, we use Theorem 1.6(iii))

$$\mathrm{Cov}_p(\omega_e; \omega_f) \asymp \Delta_p(\ell \wedge L(p))^2 e^{-c\ell/L(p)} \asymp \Delta_{p_c}(\ell \wedge L(p))^2 e^{-c\ell/L(p)},$$

where ℓ is the distance between e and f. Summing the displayed equation above over all edges f, we conclude that

$$f''(p) \asymp \sum_{\ell=1}^{L(p)} \ell\Delta_{p_c}(\ell)^2,$$

as required. □

Remark 8.3. The above shows that if the phase transition is of second order (meaning that $f''(p)$ diverges as p tends to p_c), then $\sum_\ell \ell\Delta_p(\ell)^2$ diverges, which, using the interpretation of crossing probabilities in terms of $\Delta_p(\ell)$, implies that the crossing probabilities of quads for the infinite-volume measure are not differentiable at p_c.

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Remark 8.4. When $\sum_{\ell\geq 1}\ell\Delta(\ell)^2$ converges, the computation above simply proves that $f''(p)$ remains bounded uniformly in p. Nevertheless, it is easy to deduce by differentiating one more time that for $p\neq p_c$ (we drop p from the notation),

$$f'''(p)\leq\sum_{f,g}\phi[\omega_e\omega_f\omega_g]-\phi[\omega_e\omega_f]\phi[\omega_g]-\phi[\omega_f\omega_g]\phi[\omega_e]-\phi[\omega_e\omega_g]\phi[\omega_f]+2\phi[\omega_e]\phi[\omega_f]\phi[\omega_g]$$
$$\lesssim\sum_{\ell\leq L(p)}\ell\Delta(\ell)\Delta(\ell',\ell)\sum_{\ell'\leq\ell}\ell'\Delta(\ell')^2\lesssim L(p)^4\Delta(L(p))^3\lesssim\frac{1}{|p-p_c|^3L(p)^2}.$$

If one defines α in this framework by the formula $f'''(p)=|p-p_c|^{-\alpha-1-o(1)}$, we deduce that $\alpha\leq 2-2\nu$. The converse bound does not follow by the same computation since the summand on the first line is not of definite sign; at the time of writing, the matching lower bound on $f'''(p)$ remains unproven.

8.3. Scaling relation with magnetic field: proof of Theorem 1.10

We insist again on the fact that this section is independent of the rest of the paper. Below, we work with the graph $\mathbb{Z}^2$ with the addition of the ghost vertex. We drop p_c and q but keep h in the notation except when it is equal to 0, in which case we omit it as well. We start with a lemma relating certain quantities at $h=0$ with the corresponding quantities at $h\geq 0$.

Set $\tilde\pi_1(h,R):=\phi^1_{\Lambda_R,h}[0\longleftrightarrow\partial\Lambda_R]$, where connections need to occur in $\mathbb{Z}^2$. Additionally, let A_R^* be the event that there exists a dual circuit in $\mathrm{Ann}(R,2R)$ surrounding Λ_R and

$$\beta(h,R):=\phi^1_{\mathrm{Ann}(R,2R),h}[A_R^*].$$

Finally, for $C>1$, define

$$h_c(R)=h_c(R,C):=\inf\big\{h>0:\exists\, r\leq R\text{ such that }\tilde\pi_1(h,r)>C\tilde\pi_1(r)\text{ or }\beta(h,r)<C^{-1}\beta(r)\big\}.$$

Lemma 8.5. *For any $C>1$, there exists $\varepsilon>0$ such that for every R,*

$$h_c(R)R^2\pi_1(R)\geq\varepsilon. \tag{8.10}$$

Due to the definition of h_c, crossing estimates as in equation (RSW) apply in the regime of (r,h) with $r\leq R$ and $h\leq h_c(R)$. Indeed, the crossing probabilities in the primal model increase with h, which ensures the lower bounds. For the upper bounds, observe that $\beta(h,r)$ involves the boundary conditions that render dual crossings least likely. (RSW) applied at p_c combined with the definition of $h_c(R)$ implies the uniform positivity of $\beta(r,h)$ for $r\leq R$ and $h\leq h_c(R)$. The FKG inequality and the monotonicity of boundary conditions imply lower bounds for crossing probabilities in the dual model, as claimed.

As a consequence, a similar proof to that of Lemma 8.1 applies for all $r\leq R$ and $0\leq h\leq h_c(R)$ and yields

$$\phi^1_{\mathrm{Ann}(r,2r),h}[y\longleftrightarrow\partial\Lambda_{2r}]\lesssim\pi_1(\mathrm{dist}(y,\partial\mathrm{Ann}(r,2r)))\qquad\text{for } y\in\mathrm{Ann}(r,2r)\text{ and}$$
$$\phi^1_{\Lambda_r,h}[0\longleftrightarrow y,0\longleftrightarrow\partial\Lambda_r]\lesssim\pi_1(r)\pi_1(\|y\|\wedge\mathrm{dist}(y,\partial\Lambda_r))\qquad\text{for } y\in\Lambda_r, \tag{8.11}$$

where the constants in $\lesssim$ depend on C.

Proof of Lemma 8.5. Fix $C>1$ and $r\leq R$. All constants in the signs $\lesssim$ below are allowed to depend on C but not on r or R. The differential formula [Gri06, Theorem 3.12] reads

$$\tfrac{d}{dh}\tilde\pi_1(h,r)=\tfrac{1}{1-e^{-h}}\sum_{y\in\Lambda_r}\phi^1_{\Lambda_r,h}[0\leftrightarrow\partial\Lambda_r,\omega_{y\mathfrak{g}}=1]-\phi^1_{\Lambda_r,h}[0\leftrightarrow\partial\Lambda_r]\phi^1_{\Lambda_r,h}[\omega_{y\mathfrak{g}}=1]. \tag{8.12}$$

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Let us analyse the right-hand side of the above. Recall that C is the cluster of the origin for the connectivity in $\mathbb{Z}^2$. First, we show that the vertices $y \notin \mathsf{C}$ have a negative contribution. Fix some $y \in \Lambda_r$. For a set $\mathcal{C} \subset \Lambda_r$ containing 0 and $y \notin \mathcal{C}$,

$$\begin{aligned}\phi^1_{\Lambda_r,h}[0 \leftrightarrow \partial\Lambda_r, \omega_{y\mathfrak{g}} = 1, \mathsf{C} = \mathcal{C}] &= \phi^1_{\Lambda_r,h}[\phi^1_{\Lambda_r,h}(\omega_{y\mathfrak{g}} = 1 \,|\, \mathsf{C} = \mathcal{C})\, \mathbb{1}_{0\leftrightarrow\partial\Lambda_r} \mathbb{1}_{\mathsf{C}=\mathcal{C}}]\\ &\le \phi^1_{\Lambda_r,h}[\omega_{y\mathfrak{g}} = 1]\phi^1_{\Lambda_r,h}[0 \leftrightarrow \partial\Lambda_r, \mathsf{C} = \mathcal{C}].\end{aligned}$$

The first inequality is due to equations (SMP) and (CBC) since $\phi^1_{\Lambda_r,h}[\cdot\,|\,\mathsf{C} = \mathcal{C}]$ is a random-cluster measure with free boundary conditions on $(\Lambda_r\setminus\mathcal{C})\cup\{\mathfrak{g}\}$ and is stochastically dominated by the restriction of $\phi^1_{\Lambda_r,h}$ to $(\Lambda_r \setminus \mathcal{C}) \cup \{\mathfrak{g}\}$.

Summing the above over every $\mathcal{C} \subset \Lambda_r$, we find that

$$\phi^1_{\Lambda_r,h}[0 \leftrightarrow \partial\Lambda_r, \omega_{y\mathfrak{g}} = 1] - \phi^1_{\Lambda_r,h}[\omega_{y\mathfrak{g}} = 1]\phi^1_{\Lambda_r,h}[0 \leftrightarrow \partial\Lambda_r] \le \phi^1_{\Lambda_r,h}[0 \leftrightarrow y, 0 \leftrightarrow \partial\Lambda_r, \omega_{y\mathfrak{g}} = 1].$$

Plugging this inequality in equation (8.12) gives

$$\tfrac{d}{dh}\tilde\pi_1(h,r) \le \tfrac{1}{1-e^{-h}} \sum_{y\in\Lambda_r} \phi^1_{\Lambda_r,h}[0 \leftrightarrow y, 0 \leftrightarrow \partial\Lambda_r, \omega_{y\mathfrak{g}} = 1] \le \sum_{y\in\Lambda_r} \phi^1_{\Lambda_r,h}[0 \leftrightarrow y, 0 \leftrightarrow \partial\Lambda_r],$$

where the second inequality comes from equation (SMP), which implies that

$$\phi^1_{\Lambda_r,h}[\omega_{v\mathfrak{g}} = 1\,|0 \leftrightarrow y, 0 \leftrightarrow \partial\Lambda_r] \le 1 - e^{-h}.$$

Now, assuming that $h \le h_c(R)$, equation (8.11) applies, and we conclude that

$$\tfrac{d}{dh}\tilde\pi_1(h,r) \lesssim \sum_{y\in\Lambda_r} \pi_1(r)\pi_1(\|y\| \wedge \mathrm{dist}(y,\partial\Lambda_r)) \lesssim r\pi_1(r) \sum_{k=1}^{r/2} \pi_1(k) \lesssim r^2\pi_1(r)^2.$$

The second inequality uses the fact that there are at most order r vertices at distance k from 0 or $\partial\Lambda_r$; the third one is a standard consequence of the polynomial decay of π_1 in equation (2.2) and its quasi-multiplicativity in equation (2.3). Keeping in mind that $\pi_1(r) \le \tilde\pi_1(h,r)$, the above implies

$$\tfrac{d}{dh}\log\tilde\pi_1(h,r) \lesssim r^2\pi_1(r). \tag{8.13}$$

A similar computation, where C is replaced by the cluster of $\partial\Lambda_{2r}$, implies that

$$-\tfrac{d}{dh}\log\beta(h,r) \lesssim \sum_{y\in\mathrm{Ann}(r,2r)} \phi^1_{\mathrm{Ann}(r,2r),h}[y \leftrightarrow \partial\mathrm{Ann}(r,2r)] \lesssim r^2\pi_1(r). \tag{8.14}$$

We are now in a position to conclude. Let c_0 be a constant larger than the constants involved in the inequalities $\lesssim$ in equations (8.13) and (8.14). Then for $\varepsilon > 0$, integrate these two inequalities for h between 0 and $h' \le \min\{h_c(R), \varepsilon/(R^2\pi_1(R))\}$. We find

$$\left.\begin{array}{r}\log\tilde\pi_1(h',r) - \log\tilde\pi_1(r)\\ \log\beta(r) - \log\beta(h',r)\end{array}\right\} \le c_0h'r^2\pi_1(r) \le \varepsilon c_0.$$

Now, for $\varepsilon < (\log C)/c_0$, the above shows that $h' < h_c(R)$, which implies equation (8.10). □

Proof of Theorem 1.10. Fix $h > 0$. Again, C is the cluster of the origin when considering connections in $\mathbb{Z}^2$ only. We start with the lower bound. We have

$$\phi_h[0 \longleftrightarrow \mathfrak{g}] \ge \phi_h[0 \longleftrightarrow \mathfrak{g}, |\mathsf{C}| \ge 1/h] \gtrsim \phi_h[|\mathsf{C}| \ge 1/h] \ge \phi_0[|\mathsf{C}| \ge 1/h] \gtrsim \pi_1(\varphi(1/h)),$$

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where the fourth inequality is due to equation (1.23), the third to monotonicity in h equation (h-MON) and the second to the fact that conditioned on $|\mathsf{C}| \geq 1/h$, there is a positive probability for 0 to be connected to $\mathfrak{g}$. This last property can be easily deduced from the finite energy property, which states that every edge connecting $\mathbb{Z}$ to $\mathfrak{g}$ has a probability larger than $(1-e^{-h})/(1+(q-1)e^{-h})$ and smaller than h of being open, regardless of the states of other edges.

Let us now derive the upper bound. Let ε be the quantity given by Lemma 8.5 for some fixed $C > 1$. Choose R to be the largest integer such that $hR^2\pi_1(R) \leq \varepsilon$. Notice that this implies that $h \leq h_c(R)$. We deduce that

$$\begin{aligned}\phi_h[0 \longleftrightarrow \mathfrak{g}] &\leq \pi_1(h,R) + \phi_h[0 \longleftrightarrow \mathfrak{g},\, 0 \not\longleftrightarrow \partial\Lambda_R]\\ &\leq \pi_1(h,R) + \sum_{y\in\Lambda_R} \phi_h[0 \longleftrightarrow y,\, \omega_{y\mathfrak{g}} = 1]\\ &\leq \pi_1(h,R) + h\sum_{y\in\Lambda_R} \phi_h[0 \longleftrightarrow y],\end{aligned}$$

where the last inequality uses the finite energy property. Now, $h \leq h_c(R)$ implies that $\pi_1(h,R) \lesssim \pi_1(R)$ and that similarly to Lemma 8.1, $\phi_h[0 \longleftrightarrow y] \lesssim \pi_1(|y|)^2$. We deduce from the above that

$$\phi_h[0 \longleftrightarrow \mathfrak{g}] \lesssim \pi_1(R) + h\sum_{y\in\Lambda_R} \pi_1(|y|)^2 \lesssim \pi_1(R) + hR^2\pi_1(R)^2 \lesssim \pi_1(R),$$

where in the second inequality, we used a computation similar to equation (1.26). Recall the definition in equation (1.21) of φ. Due to the quasi-multiplicativity of π_1 in equation (2.3), we have that $R = \varphi(\varepsilon/h) \gtrsim \varphi(1/h)$. Applying equation (2.3) again, we find

$$\phi_h[0 \longleftrightarrow \mathfrak{g}] \lesssim \pi_1(\varphi(1/h)),$$

which concludes the proof. □

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