ON THE ISOMORPHISM OF INTEGRAL GROUP RINGS. I

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1. Introduction. In this note we study the question of automorphisms of the integral group ring $Z(G)$ of a finite group G. We prove that if G is nilpotent of class two, any automorphism of $Z(G)$ is composed of an automorphism of G and an inner automorphism by a suitable unit of $Q(G)$, the group algebra of G with rational coefficients. In § 3, we prove that if two finitely generated abelian groups have isomorphic integral group rings, then the groups are isomorphic. This is an extension of the classical result of Higman (**2**) for the case of finite abelian groups. In the last section we give a new proof of the fact that an isomorphism of integral group rings of finite groups preserves the lattice of normal subgroups. Other proofs are given in (**1**; **4**).

2. Automorphisms of group rings. We shall use the following theorem of Glauberman (a proof can be found in (**4** or **6**)). All groups in this section are finite.

THEOREM 1 (Glauberman). *Let θ: $Z(G) \to Z(H)$ be an isomorphism. Let K_x be a class sum in G, i.e., sum of distinct conjugates of an element x of G. Then $\theta(K_x) = \pm K_y$ for some $y \in H$.*

PROPOSITION 1. *Let θ be an automorphism of $Z(G)$. Let C_i, $1 \leqq i \leqq r$, be the conjugacy classes and K_i the corresponding class sums of G. Suppose that $\theta(K_i) = K_{i'}$, $1 \leqq i, i' \leqq r$, and that there exists an automorphism σ of G such that $\sigma(C_i) = C_{i'}$ for all $1 \leqq i \leqq r$. Then we can find a unit $\gamma \in Q(G)$ such that*

$$\theta(g) = \gamma g^{\sigma} \gamma^{-1} \quad \textit{for all } g \in G.$$

Proof. Extend σ and θ to $Q(G)$ in the natural way. The centre of $Q(G)$ which is generated by the class sums K_i, $1 \leqq i \leqq r$, is kept fixed elementwise by $\sigma^{-1}\theta$. Since $Q(G)$ is semi-simple, we can write

$$Q(G) = S_1 \oplus S_2 \oplus \ldots \oplus S_t,$$

as a direct sum of simple rings S_i. Let $1 = e_1 + e_2 + \ldots + e_t$, where $e_i \in S_i$ are central idempotents. Then

$$S_i = e_i Q(G) \quad \text{and} \quad (\sigma^{-1}\theta)(S_i) = S_i, \qquad 1 \leqq i \leqq t.$$

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As $\sigma^{-1}\theta$ keeps the centre of S_i fixed elementwise, it acts on S_i as an inner automorphism by some $\alpha_i \in S_i$; see (**5**). It follows that $\sigma^{-1}\theta$ is an inner automorphism of $Q(G)$ by $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_t$. Thus, we have for $g \in G$, that

$$(\sigma^{-1}\theta)(g) = \alpha g \alpha^{-1}, \quad \theta(g) = \gamma g^{\sigma} \gamma^{-1}, \quad \text{where } \gamma = \sigma(\alpha).$$

Remark. Let H be a normal subgroup of G. Let $\Delta(H)$ be the kernel of the natural map $Z(G) \to Z(G/H)$. Then $\Delta(H)$ is the two-sided ideal generated by the elements $(1 - h)$, where h runs over H.

PROPOSITION 2. *Let μ be an automorphism of $Z(G)$, where G is a nilpotent group of class two. Let K_i, $1 \leqq i \leqq r$, be the class sums of G. Suppose that $\mu(K_i) = K_{i'}$, $1 \leqq i \leqq r$. Then there exists an automorphism σ of G which, when extended to $Z(G)$, satisfies $\sigma(K_i) = K_{i'}$ for all $1 \leqq i \leqq r$.*

Proof. We first observe that a class sum of a nilpotent group of class two is of the form $g\bar{H}$, where $\bar{H} = \sum_{h \in H} h$ and H is a subgroup of the derived group G'. Let $\mu(g) = \gamma$. Then $\mu(K_g) = \mu(g\bar{H}) = \gamma \bar{H}_1$, where H_1 is another subgroup. Now $\gamma \bar{H}_1 = K_{g_1} = g_1 \bar{H}_2$. We claim that $H_1 = H_2$. Observe that $|H_1| g_1 \bar{H}_2 = |H_1| \gamma \bar{H}_1 = \gamma \bar{H}_1 \cdot \bar{H}_1 = g_1 \bar{H}_2 \cdot \bar{H}_1$ and $|H_1| \bar{H}_2 = \bar{H}_2 \cdot \bar{H}_1$. Therefore, H_1 is contained in H_2, and by symmetry, $H_1 = H_2$. Thus, we have that $\mu(K_g) = \gamma \bar{H}_1 = g_1 \bar{H}_1$. Next, we claim that there exists a $g_2 \in G$ such that $\gamma \equiv g_2 \bmod \Delta(H_1)\Delta(G)$. Since $\gamma \bar{H}_1 = g_1 \bar{H}_1$, we have that

$$\begin{aligned}
\gamma &= g_1 + \sum_{h \in H_1} (1 - h)t(h) \\
&\equiv g_1 + \sum_h (1 - h)n_h \bmod \Delta(H_1)\Delta(G) \quad (\text{where } n_h \in Z) \\
&\equiv g_1 + 1 - \prod_h h^{n_h} \bmod \Delta(H_1)\Delta(G) \\
&\equiv g_1 \prod_h h^{-n_h} \bmod \Delta(H_1)\Delta(G).
\end{aligned}$$

Thus, $\gamma \equiv g_2 \bmod \Delta(H_1)\Delta(G)$, and hence $\gamma \equiv g_2 \bmod \Delta(G')\Delta(G)$. It has been proved independently by Jackson (**3**) and Whitcomb (**6**) that given γ as above, there exists a unique $g_\gamma \in G$ such that $\gamma \equiv g_\gamma \bmod \Delta(G')\Delta(G)$ and $\gamma \to g_\gamma$ is an isomorphism λ: $\mu(G) \to G$. Because of uniqueness, it follows that $g_\gamma = g_2$. Since $g_\gamma \equiv \gamma \bmod \Delta(H_1)\Delta(G)$, we have that $g_\gamma \bar{H}_1 = \gamma \bar{H}_1$. Let $\sigma(g) = \lambda\mu(g)$. Then σ is an automorphism of G and

$$\sigma(K_g) = \lambda\mu(K_g) = \lambda(\gamma \bar{H}_1) = \lambda(\gamma)\bar{H}_1 = g_\gamma \bar{H}_1 = \gamma \bar{H}_1 = \mu(K_g).$$

This completes the proof.

Now we have the following theorem.

THEOREM 2. *Let θ be an automorphism of $Z(G)$, where G is a nilpotent group of class two. Then there exists an automorphism λ of G and a unit γ of $Q(G)$ such that $\theta(g) = \pm \gamma g^{\lambda} \gamma^{-1}$ for all $g \in G$.*

Proof. Let us write $C(\alpha) = \sum_g \alpha_g$ if $\alpha = \sum_g \alpha_g g$ is an element of $Z(G)$. Clearly, $C(\theta(g)) = \pm 1$ for any element g of G. Normalize θ by defining $\mu(g) = C(\theta(g))\cdot\theta(g)$. By linear extension, μ becomes an automorphism of $Z(G)$ which maps class sums to class sums. The theorem now follows from the last two propositions.

3. Group rings of finitely generated abelian groups.

Higman proved (in 1940) that the only units of finite order in the group ring $Z(G)$ of a finite abelian group are $\pm g$, $g \in G$. From this, it follows that only isomorphic finite abelian groups have isomorphic integral group rings. We extend this result to finitely generated abelian groups. We first prove the following lemma.

LEMMA 1. *The only units of finite order in the group ring $Z(G)$ of an arbitrary abelian group G are of the form $\pm t$, where t is a torsion element of G.*

Proof. We can assume without loss of generality that G is finitely generated. Write $G = T \times F$, where T is torsion and F is free. Further, $F = \langle x_1\rangle \times \ldots \times \langle x_l\rangle$. A typical element g of G can be written uniquely as $g = t\cdot x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_l^{\alpha_l}$, where $\alpha_i \in Z$ and $t \in T$. Define $d_i(g) = |\alpha_i|$. Suppose that $\gamma = m_1g_1 + m_2g_2 + \ldots + m_sg_s$ is such that $\gamma^n = 1$. We have to prove that $\gamma = \pm t$. Let

$$n_i = \max_{1 \leqq j \leqq s} d_i(g_j).$$

We shall use the group

$$H = \langle x_1^{2n_1+1}\rangle \times \langle x_2^{2n_2+1}\rangle \times \ldots \times \langle x_l^{2n_l+1}\rangle$$

which is of finite index in G. Clearly, $(\bar{\gamma})^n = 1$, where $\bar{\gamma}$ is the image of γ in the projection $Z(G) \to Z(G/H)$, and by Higman's result, $\bar{\gamma} = \pm xH$, $x \in G$. Since $g_1, \ldots, g_s$ belong to different cosets of H, it must be that $\gamma = \pm g_i$, where $g_i \in T$.

THEOREM 3. *Suppose that G and H are finitely generated abelian groups. Then $Z(G) \simeq Z(H)$ implies $G \simeq H$.*

Proof. Write $G = T \times \langle x_1\rangle \times \ldots \times \langle x_l\rangle$ and $H = T_1 \times \langle y_1\rangle \times \ldots \times \langle y_m\rangle$, as in the lemma. Let θ: $Z(G) \to Z(H)$ be the given isomorphism. For $t \in T$, set $\mu(t) = t_1 \in T_1$ if $\theta(t) = \pm t_1$ and set $\mu(x_i) = \theta(x_i)$. By linear extension we obtain an isomorphism μ: $Z(G) \to Z(H)$ such that $\mu(T) = T_1$. We conclude that $Z(G/T) \cong Z(H/T_1)$. Since $Z(G/T)$ is free of zero divisors and the degree of transcendency over Q of its field of quotients is l, it follows that $l = m$ and that $G \simeq H$.

4. Central idempotents and normal subgroups.

In this section, all groups are finite. Let N be a normal subgroup of G; then $\bar{N} = \sum_{n\in N} n$ is a central element with the property that $(\bar{N})^2 = |N|\cdot\bar{N}$. We characterize all $\bar{N}$, where N is a normal subgroup of G as certain elements of $Z(G)$ with this property. More precisely, we prove the following proposition.

PROPOSITION 3. *Let* $\gamma = \sum_{g \in G} \gamma_g g$ *be a central element of* $Z(G)$ *such that* $\gamma_e = 1, \sum \gamma_g \neq 0$ *and* $\gamma^2 = m\gamma$, *where* m *is a natural number. Then* $\gamma = \sum_{h \in H} h$, *where* H *is a normal subgroup of* G.

Proof. For a typical element $\alpha = \sum_{g \in G} \alpha_g g$ of $Z(G)$, write $\alpha^* = \sum_{g \in G} \alpha_g g^{-1}$. It is clear that $(\alpha + \beta)^* = \alpha^* + \beta^*$ and $(\alpha\beta)^* = \beta^*\alpha^*$, and thus we have that $(\gamma^*)^2 = m\gamma^*$ and $(\gamma\gamma^*)^2 = m^2\gamma\gamma^*$. We also remark that $\sum_g \gamma_g = m$. By the eigenvalues (trace) of an element α of $Z(G)$ we mean the eigenvalues (trace) of $R(\alpha)$, where $\alpha \to R(\alpha)$ is the regular representation of $Z(G)$. Clearly, γ is diagonalizable. The only possible eigenvalues of γ are 0 and m. Since $\gamma_e = 1$, we have that $\operatorname{trace}(\gamma) = (G:1)$, and therefore exactly $(G:1) - (G:1)/m$ eigenvalues of γ are zero. Thus, at least $(G:1) - (G:1)/m$ eigenvalues of $\gamma\gamma^*$ are zero, and therefore

$$(G:1)\left(\sum_g {\gamma_g}^2\right) = \operatorname{trace}(\gamma\gamma^*) \leqq m^2 \cdot \frac{(G:1)}{m}.$$

We conclude that $(\sum_g \gamma_g^2) \leqq m = (\sum_g \gamma_g)$, and therefore $\gamma_g = 0$ or 1. Since $\gamma^2 = m\gamma$, it follows that $\gamma = \sum_{h \in H} h$, where H is a subgroup. Since γ is central, H is normal and the proof is complete.

THEOREM 4. *Let* θ: $Z(G) \to Z(H)$ *be a normalized isomorphism* (*i.e.*, $C(\theta(g)) = 1$ *for all* $g \in G$). *Then there exists a one-to-one correspondence between the normal subgroups of* G *and* H *which preserves order*, *union*, *and intersection*.

Proof. Let N be a normal subgroup of G. Then $\bar{N} = \sum_{n \in N} n$ satisfies $(\bar{N})^2 = |N|\bar{N}$. Let $\theta(\bar{N}) = \gamma = \sum_g \gamma_g g$; then $\gamma^2 = |N|\gamma$ and $C(\gamma) \neq 0$. Also, $\gamma_e = 1$, due to the fact that for a unit of finite order $\beta = \sum \beta_g g$, $\beta_e \neq 0$ implies $\beta = \pm e$ (see (**1**) for the proof); therefore, $\beta_e = 0$ whenever $\beta = \theta(x)$ for some $x \neq e$ in G. Thus, by the last proposition, we have that $\gamma = \bar{M}$, where M is a normal subgroup of H and $|M| = |N|$. The remainder of the result follows from the fact that for two normal subgroups N_1 and N_2 of G, $\bar{N}_1 \cdot \bar{N}_2 = m\bar{N}_2$ if and only if N_1 is a subgroup of N_2.

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