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EXTREME AND EXPOSED POINTS IN ORLICZ SPACES

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ABSTRACT. Extreme points of the unit sphere in any Orlicz space over a measure space that contains no atoms of infinite measure are characterized. In the case of a finite-valued Orlicz function and a nonatomic measure space, exposed points of the unit sphere in these spaces are characterized too. Some corollaries and examples are also given.

In most cases in the paper (T, Σ, μ) is a nonatomic positive measure space. $\mathbb{R}$ denotes the reals, $\mathbb{R}_+$ denotes the positive reals and $\mathbb{N}$ denotes the set of natural numbers. By an Orlicz function we understand a nonzero mapping $\Phi\colon \mathbb{R} \longrightarrow [0, +\infty]$ that is convex, even, vanishing and continuous at zero and left-continuous on the whole $\mathbb{R}_+$. By Φ'_- and Φ'_+ we denote the left-hand side and right-hand side derivatives of Φ in $\mathbb{R}_+$, respectively. We denote by $F(\mu)$ the space of all (equivalence classes of) Σ-measurable functions x from T into $\mathbb{R}$. Given an Orlicz function Φ, we define on $F(\mu)$ a convex functional I_Φ by

$$I_\Phi(x) = \int_T \Phi\big(x(t)\big)\, d\mu,$$

and an Orlicz space $L^\Phi(\mu)$ by $L^\Phi(\mu) = \{x \in F(\mu) : I_\Phi(\lambda x) < +\infty \text{ for some } \lambda > 0\}$. This space equipped with the Luxemburg norm

$$\|x\|_\Phi = \inf\{\lambda > 0 : I_\Phi(x/\lambda) \leq 1\}$$

is a Banach space (see [9], [11] and [12]). We shall also consider a subspace $E^\Phi(\mu)$ of $L^\Phi(\mu)$ that is defined to be the set of all x in $F(\mu)$ such that $I_\Phi(\lambda x) < +\infty$ for any $\lambda > 0$. This subspace is equipped with the norm $\|\cdot\|_\Phi$ induced from $L^\Phi(\mu)$. Obviously $E^\Phi(\mu) \neq \{0\}$ if and only if Φ has finite values. By $\mathrm{Ext}(\Phi)$, we define the set of all $u \in \mathbb{R}$ such that there are no numbers $w, v \in \mathbb{R}$, $w \neq v$, satisfying $u = (w+v)/2$ and $\Phi\big((w+v)/2\big) = \frac{1}{2}\big(\Phi(w) + \Phi(v)\big)$. For any Σ-measurable function x, we denote $\mathrm{supp}(x) = \{t \in T : x(t) \neq 0\}$.

We say an Orlicz function Φ satisfies the Δ_2-condition for all $u \in \mathbb{R}$ (at infinity) if there are positive numbers K and a such that $0 < \Phi(a) < +\infty$ and the inequality $\Phi(2u) \leq K\Phi(u)$ holds for all $u \in \mathbb{R}$ (resp. for all $u \in \mathbb{R}$ satisfying $|u| \geq a$).

The statement "the suitable Δ_2-condition" for an Orlicz function Φ means the Δ_2-condition for all $u \in \mathbb{R}$ when μ is nonatomic and infinite and the Δ_2-condition at infinity when μ is nonatomic and finite.

For a Banach space X, we denote by S_X the unit sphere of X. A point $x \in S_X$ is called *an extreme point* if for every $y, z \in S_X$ the equality $x = \frac{1}{2}(y+z)$ implies $y = z$. We say a

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point $x \in S_X$ is an *exposed point* if there is $x^* \in S_{X^*}$—the unit sphere of the dual X^* of X—such that $x^*(x) = 1$ and for every $y \in S_X$, $y \neq x$, we have $x^*(y) < 1$. We say then that x^* exposes the point x.

It is easy to see that exposed points are extreme and the converse is true *e.g.* for strictly convex (rotund) Banach spaces. In Corollary 5 it will be seen that for Orlicz spaces, not strictly convex, there may exist extreme points which are not exposed.

To characterize exposed points of S_{L^Φ} or S_{E^Φ} we shall need a general (integral) form of linear order continuous functionals on $L^\Phi(\mu)$ or $E^\Phi(\mu)$ (see [1], [13], [14]), especially those which attain their norms at some element of the unit sphere (see [4], [7], [8], [9] and [10] for characterizations of such functionals over $E^\Phi(\mu)$ and partially over $L^\Phi(\mu)$).

Exposed points of the unit sphere in the Lebesgue-Bochner space $L^p(X)$, $1 < p \leq \infty$, were characterized in [2]. In [17] and [18] some characterizations of extreme points of the unit sphere in Musielak-Orlicz spaces over nonatomic and purely atomic (counting) measure spaces, respectively, are given. The first paper is written in Chinese. Since the authors of this paper have obtained a characterization of such points independently and do not know any characterization of these points in English, they decided to give the respective proof.

RESULTS. We start with the following:

THEOREM 1. *(i) Assume that Φ is continuous and (T, Σ, μ) does not contain atoms of infinite measure. Then x is an extreme point of S_{L^Φ} if and only if $I_\Phi(x) = 1$ and either*

(a) $x(t)$ are in $\mathrm{Ext}(\Phi)$ *for μ- a.e. $t \in T$, or*

(b) there exists an atom A such that $x(t)$ are in $\mathrm{Ext}(\Phi)$ *for μ- a.e. $t \in T \setminus A$ and $x\chi_A = u_0$, where $\Phi(u_0) \neq 0$.*

(ii) If Φ is discontinuous and $I_\Phi(x) = 1$, x is an extreme point of S_{L^Φ} if and only if either (a) or (b) written above holds.

(iii) If Φ is discontinuous and $I_\Phi(x) < 1$, x is an extreme point of S_{L^Φ} if and only if $|x(t)| = a(\Phi)$ for μ- a.e. $t \in T$.

PROOF. We restrict ourselves only to a nonatomic measure.

SUFFICIENCY. Assume first that $I_\Phi(x) = 1$ and $x(t) \in \mathrm{Ext}(\Phi)$ for μ- a.e. $t \in \mathrm{supp}(x)$. Take an arbitrary y and z in S_{L^Φ} such that $x = (y+z)/2$. We shall prove that $y = z$. Assume for a contrary that this is not true. Then, in virtue of our assumption, we get

$$1 = I_\Phi(x) = I_\Phi\Big(\frac{1}{2}(y+z)\Big) < \frac{1}{2}\Big(I_\Phi(y) + I_\Phi(z)\Big) \leq 1,$$

i.e. a contradiction.

Now, assume that $a(\Phi) < +\infty$ and $\mu\Big(\big\{t \in T : |x(t)| = a(\Phi)\big\}\Big) = \mu(T)$. Assume that $x \in S_{L^\Phi}$ and $x = \frac{1}{2}(y+z)$, where $y, z \in S_{L^\Phi}$. We need to prove that $y = z$. We have $I_\Phi(y) \leq 1$ and $I_\Phi(z) \leq 1$, whence it follows that $|y(t)| \leq a(\Phi)$ and $|z(t)| \leq a(\Phi)$ for μ- a.e. $t \in T$. Note that it must be $|y(t)| = a(\Phi)$ for μ- a.e. $t \in T$. In fact, in the opposite case it would be $|y(t)| < a(\Phi)$ for $t \in A$, where $\mu(A) > 0$. Then it must be $|z(t)| > a(\Phi)$ for $t \in A$, which yields $I_\Phi(z) = +\infty$, a contradiction. In the same way we can prove that

$|z(t)| = a(\Phi)$ for μ- a.e. $t \in T$. Therefore, by the equality $x = \frac{1}{2}(y+z)$ it follows that $y(t) = z(t)$ for μ- a.e. $t \in T$.

NECESSITY. Assume first that $\|x\|_\Phi = 1$, $I_\Phi(x) < 1$ and $\mu\big(\{t \in T : |x(t)| = a(\Phi)\}\big) < \mu(T)$. We shall prove that x is not an extreme point. We have $|x(t)| < a(\Phi)$ for $t \in A$, where $\mu(A) > 0$. By the assumption it must be $|x(t)| \le a(\Phi)$ for μ- a.e. $t \in T$ and

$$\forall \lambda \in (0,1) : I_\Phi(x/\lambda) = +\infty. \tag{1}$$

Indeed, in the opposite case it would be $I_\Phi(x/\lambda) < +\infty$ for some $\lambda \in (0,1)$. Choosing $n_0 \in \mathbb{N}$ such that $1 + 1/n_0 \le \lambda^{-1}$, we get $(1+1/n)|x(t)| \le \lambda^{-1}|x(t)|$ for every $t \in T$ and $n \ge n_0$. In virtue of the Lebesgue dominated convergence theorem, it follows that

$$\lim_{n\to\infty} I_\Phi\big((1+1/n)x\big) = I_\Phi(x).$$

Hence, $I_\Phi\big((1+1/k)x\big) \le 1$ for some $k \in \mathbb{N}$, which yields $\|x\|_\Phi \le (1+1/k)^{-1}$, a contradiction. Therefore condition (1) is proved.

Let $m \in \mathbb{N}$ be such that the set

$$A_m = \big\{t \in T : 1/m < |x(t)| < \min\{(1-1/m)a(\Phi), m\}\big\}$$

has positive measure. Obviously, this minimum is equal to m, when $a(\Phi)$ is infinite. Let $A \subset A_m$ be a set of positive and finite measure and $a > 1$ be such that $a(1-1/m) < 1$. Then

$$I_\Phi\big((x/\lambda)\chi_A\big) < +\infty \tag{2}$$

for every $\lambda \in (\frac{1}{a}, 1]$. Conditions (1) and (2) and the orthogonal additivity of the functional I_Φ yield

$$\forall \lambda \in (0,1) : I_\Phi\big((x/\lambda)\chi_{T\setminus A}\big) = +\infty. \tag{3}$$

Let $\varepsilon > 0$ be such that $I_\Phi(x) + 2\varepsilon < 1$ and define

$$B_n = \Big\{t \in A : \Phi\Big(x(t) + \frac{1}{n}\operatorname{sgn}\big(x(t)\big)\Big) \le \Phi\big(cx(t)\big)\Big\}$$

where $c \in (1,a)$. We have $B_n \uparrow$ and $\bigcup_{n=1}^\infty B_n = A$. By the Lebesgue dominated convergence theorem we get

$$\begin{aligned}\lim_{n\to\infty} I_\Phi\Big(\Big(x + \frac{1}{n}\operatorname{sgn}(x)\Big)\chi_{B_n}\Big) &= \lim_{n\to\infty} I_\Phi\Big(\Big(x - \frac{1}{n}\operatorname{sgn}(x)\Big)\chi_{B_n}\Big) \\ &= \lim_{n\to\infty} I_\Phi(x\chi_{B_n}) = I_\Phi(x\chi_A).\end{aligned} \tag{4}$$

Therefore, by the orthogonal additivity of I_Φ, we have

$$\lim_{n\to\infty} I_\Phi(x\chi_{T\setminus B_n}) = I_\Phi(x\chi_{T\setminus A}). \tag{5}$$

Let $k \in \mathbb{N}$ be such that $I_\Phi\Big(\big(x + \frac{1}{k}\operatorname{sgn}(x)\big)\chi_{B_k}\Big) \leq I_\Phi(x\chi_A) + \varepsilon$ and $I_\Phi(x\chi_{T\setminus B_k}) \leq I_\Phi(x\chi_{T\setminus A}) + \varepsilon$. Denote $B = B_k$ and define

$$y = x\chi_{T\setminus B} + \Big(x + \frac{1}{k}\operatorname{sgn}(x)\Big)\chi_B,$$

$$z = x\chi_{T\setminus B} + \Big(x - \frac{1}{k}\operatorname{sgn}(x)\Big)\chi_B.$$

We have $I_\Phi(y) \leq I_\Phi(x) + 2\varepsilon < 1$ and $I_\Phi(z) \leq I_\Phi(x) + 2\varepsilon < 1$. Moreover, in virtue of property (3), we have $I_\Phi(y/\lambda) = I_\Phi(z/\lambda) = +\infty$ for every $\lambda \in (0, 1)$. Therefore, $\|y\|_\Phi = \|z\|_\Phi = 1$. Since $y \neq z$ and $x = (y + z)/2$, it means that x is not an extreme point.

Assume now that $\|x\|_\Phi = 1$ and there exists a set $C \in \Sigma$, $\mu(C) > 0$, such that $x(t) \notin \operatorname{Ext}(\Phi)$ (equivalently $|x(t)| \notin \operatorname{Ext}(\Phi)$) for every $t \in C$. Obviously, the equality $|x(t)| = a(\Phi)$ for t in a set of positive measure is possible only in the case when $\Phi\big(a(\Phi)\big) < +\infty$. Since $a(\Phi) \in \operatorname{Ext}(\Phi)$ when $a(\Phi) < +\infty$ and $\Phi\big(a(\Phi)\big) < +\infty$, we have $|x(t)| < a(\Phi)$ for t in a set of positive measure whenever $a(\Phi) < +\infty$. According to the previous part of the proof, x is not extreme if $I_\Phi(x) < 1$. Therefore, we can assume that $I_\Phi(x) = 1$. It follows that there are Σ-measurable functions w and v, with values different from zero on C, such that $\Phi\big(w(t)\big) < +\infty$ and $\Phi\big(v(t)\big) < +\infty$, and

$$
\begin{aligned}
0 &< w(t) < x(t) < v(t),\\
x(t) &= \frac{1}{2}\big(w(t) + v(t)\big),\\
\Phi\big(x(t)\big) &= \frac{1}{2}\Big(\Phi\big(w(t)\big) + \Phi\big(v(t)\big)\Big)
\end{aligned}
\tag{6}
$$

for every $t \in C$. Let $D \subset C$, $D \in \Sigma$, be such that

$$0 \leq \int_D \Phi\big(w(t)\big)\, d\mu < +\infty \text{ and } 0 \leq \int_D \Phi\big(v(t)\big)\, d\mu < +\infty.$$

Condition (6) implies

$$\int_D \Phi\big(x(t)\big)\, d\mu - \int_D \Phi\big(w(t)\big)\, d\mu = \int_D \Phi\big(v(t)\big)\, d\mu - \int_D \Phi\big(x(t)\big)\, d\mu. \tag{7}$$

Define a Σ-measurable function f by $f(t) = \Phi\big(v(t)\big) - \Phi\big(w(t)\big)$. This function generates a nonatomic measure on $D \cap \Sigma$ by the formula

$$\upsilon_f(A) = \int_A f(t)\, d\mu.$$

So, there is a (measurable) subset E in $D \cap \Sigma$ such that

$$\int_E f(t)\, d\mu = \int_{D\setminus E} f(t)\, d\mu.$$

Hence in view of equality (7) we get

$$
\begin{aligned}
\int_E \Phi\big(v(t)\big)\, d\mu + \int_{D\setminus E} \Phi\big(w(t)\big)\, d\mu &= \int_E \Phi\big(w(t)\big)\, d\mu + \int_{D\setminus E} \Phi\big(v(t)\big)\, d\mu\\
&= \int_D \Phi\big(x(t)\big)\, d\mu
\end{aligned}
$$

Define

$$y(t) = x(t)\chi_{T\setminus D}(t) + w(t)\chi_E(t) + v(t)\chi_{D\setminus E}(t),$$
$$z(t) = x(t)\chi_{T\setminus D}(t) + v(t)\chi_E(t) + w(t)\chi_{D\setminus E}(t),$$

We have $x = (y+z)/2$ and $I_\Phi\big((y+z)/2\big) = I_\Phi(x) = I_\Phi(y) = I_\Phi(z) = 1$. Thus, $\|x\|_\Phi = \|y\|_\Phi = \|(y+z)/2\|_\Phi = 1$. Since $y \neq z$, x is not extreme. The proof of the theorem is finished.

Since for every $x \in E^\Phi(\mu)$, where Φ is a finite-valued Orlicz function, we have $\|x\|_\Phi = 1$ if and only if $I_\Phi(x) = 1$, in view of Theorem 1, we obtain the following:

THEOREM 2. *Let Φ be a finite-valued Orlicz function and μ be a positive nonatomic measure. A point $x \in S_E\Phi$ is extreme if and only if for μ- a.e. $t \in T$ there holds $x(t) \in$* Ext(Φ).

COROLLARY 1. *In the case of a nonatomic measure μ, every extreme point of $S\big(L^1(\mu) \cap L^\infty(\mu)\big)$, for the space $L^1(\mu) \cap L^\infty(\mu)$ equipped with the* norm $\|x\| = \max\{\|x\|_{L^1}, \|x\|_{L^\infty}\}$, *is of the form: $x(t) = \varkappa(t)\chi_A(t)$, where $A \in \Sigma$, $\mu(A) =$* $\min\{1, \mu(T)\}$, *and $\varkappa(t)$ is a Σ-measurable function such that $|\varkappa(t)| = 1$ for μ- a.e. $t \in A$.*

This follows by Theorem 1 and by the fact that this space is exactly the Orlicz space $L^\Phi(\mu)$ for $\Phi(u) = |u|$, if $|u| \leq 1$, and $\Phi(u) = +\infty$, if $|u| > 1$, and by the equality $\|x\|_\Phi = \max\{\|x\|_{L^1}, \|x\|_{L^\infty}\}$ for $x \in L^\Phi(\mu)$.

COROLLARY 2. *The Orlicz space $L^\Phi(\mu) = L^1(\mu) + L^\infty(\mu)$ is generated by the Orlicz function Φ defined by*

$$\Phi(u) = \begin{cases} 0, & \text{for } |u| \leq 1, \\ |u| - 1, & \text{for } |u| > 1. \end{cases}$$

It follows by Theorem 1 that the unit sphere of this space equipped with the norm $\|\cdot\|_\Phi$ contains no extreme point whenever μ is nonatomic.

COROLLARY 3. *In the case of a nonatomic measure μ the only extreme points of S_{L^∞} are functions x such that $|x(t)| = 1$ for μ- a.e. $t \in T$.*

This follows by the facts that $L^\Phi(\mu)$ is isometrically isomorphic to the Orlicz space $L^\Phi(\mu)$ generated by the Orlicz function Φ such that $\Phi(u) = 0$ if $|u| \leq 1$ and $\Phi(u) = +\infty$ if $|u| > 1$, and Ext(Φ) $= \{\pm 1\}$.

In order to characterize exposed points of S_{L^Φ}, some additional notation must be introduced. For a given Orlicz function Φ define the following (countable) sets.

F_1 denotes a set of all points $u > 0$ such that Φ is affine in some right-hand side neighbourhood of u and in no left-hand side neighbourhood of u.

F_2 denotes a set of all points $u > 0$ such that Φ is affine in some left-hand side neighbourhood of u and in no right-hand side neighbourhood of u.

$$A_1 = \{u \in F_1 : \Phi'_+(u) = \Phi'_-(u)\}, \quad B_1 = F_1 \setminus A_1,$$
$$A_2 = \{u \in F_2 : \Phi'_+(u) = \Phi'_-(u)\}, \quad B_2 = F_2 \setminus A_2.$$

From the definition of the subdifferential of Φ at u it follows that

$$\partial\Phi(u) = \begin{cases} [\Phi'_-(u), \Phi'_+(u)], & \text{if } u \geq 0, \\ [\Phi'_+(u), \Phi'_-(u)], & \text{if } u < 0 \end{cases}$$

for every finite-valued Orlicz function Φ. Moreover $uv = \Phi(u) + \Phi^*(v)$ if and only if $v \in \partial\Phi(u)$ or $u \in \partial\Phi^*(v)$, for every $u, v \in \mathbb{R}$, where Φ^* denotes the Young's conjugate to the function Φ.

THEOREM 3. *Let Φ be a finite-valued Orlicz function vanishing only at zero and μ be a positive nonatomic measure. A point $x_0 \in S_{L^\Phi}$ is exposed if and only if:*

(i) $I_\Phi(x_0) = 1$,

(ii) $|x_0(t)| \in \operatorname{Ext}(\Phi)$ *for μ- a.e. $t \in T$,*

(iii) there is a (measurable) function $\eta\colon T \longrightarrow \mathbb{R}$ such that $\eta(t) \in \partial\Phi\big(|x_0(t)|\big)$ for μ- a.e. $t \in T$ and $\int_T |x_0(t)|\,\eta(t)\,d\mu < +\infty$ (equivalently $\eta \in L^{\Phi^*}$),

(iv) $\mu\big(|x_0|^{-1}(A_1)\big)\,\mu\big(|x_0|^{-1}(A_2)\big) = 0$, *where* $|x_0|^{-1}(A_i) = \big\{t \in T : |x_0(t)| \in A_i\big\}$.

PROOF. SUFFICIENCY. Assume that conditions (i)–(iv) are satisfied. Condition (iii) is equivalent to the fact that there exists a regular (order continuous) support functional at x_0. Every function η satisfying condition (iii) generates a regular support functional ξ_g at x_0 by the formula

$$\xi_g(x) = \int_T x(t)g(t)\,d\mu \quad \big(\forall x \in L^\Phi(\mu)\big), \tag{8}$$

where

$$g(t) = \frac{\eta(t)\operatorname{sgn}\big(x_0(t)\big)}{\int_T \eta(t)|x_0(t)|\,d\mu} \tag{9}$$

and $\eta(t) \in \partial\Phi\big(|x_0(t)|\big)$ μ- a.e. In fact it is evident that $\xi_g(x_0) = 1$. Moreover, for every $x \in L^\Phi(\mu)$ with $\|x\|_\Phi \leq 1$, we have

$$\begin{aligned} |\xi_g(x)| &= \frac{\big|\int_T \eta(t)\operatorname{sgn}\big(x_0(t)\big)x(t)\,d\mu\big|}{\int_T \eta(t)|x_0(t)|\,d\mu} \\ &= \frac{\big|\int_T \eta(t)\operatorname{sgn}\big(x_0(t)\big)x(t)\,d\mu\big|}{I_{\Phi^*}(\eta) + I_\Phi(x_0)} \\ &\leq \frac{\int_T \eta(t)|x(t)|\,d\mu}{I_{\Phi^*}(\eta) + 1} \\ &\leq \frac{I_{\Phi^*}(\eta) + I_\Phi(x)}{I_{\Phi^*}(\eta) + 1} \leq 1. \end{aligned}$$

Therefore for the dual norm $\|\cdot\|$ we have $\|\xi_g\| = 1$, which means that ξ_g is a support functional at x_0.

Note, that in view of our assumptions we can find a function η satisfying condition (iii) and such that

$$\eta(t) \in \operatorname{Int}\Big(\partial\Phi\big(|x_0(t)|\big)\Big) \tag{10}$$

for every $t \in T$ such that $x_0(t) \notin \mathrm{Smooth}(\Phi)$. Therefore in the remaining part of the proof of the sufficiency, we assume that η satisfies condition (10).

Let $y \in L^\Phi(\mu)$, $y \neq x_0$, be such that $\xi_g(y) = 1$. In order to prove that x_0 is exposed it suffices to show that $\|y\|_\Phi > 1$, *i.e.* $I_\Phi(y) > 1$. Define $h = y - x_0$. We have $y = x_0 + h$, where $\xi_g(h) = 0$. Since $\eta(t)$ belongs to the subdifferential of Φ at $|x_0(t)|$, we have

$$\Phi\big(y(t)\big) = \Phi\big(x_0(t)\big) + \eta(t)\operatorname{sgn}\big(x_0(t)\big)h(t) + \omega\big(h(t)\big), \tag{11}$$

where $\omega\big(h(t)\big) \geq 0$ by convexity of Φ. Since $\xi_g(h) = 0$, we have

$$\int_T \eta(t)\operatorname{sgn}\big(x_0(t)\big)h(t)\,d\mu = 0. \tag{12}$$

Integrating both sides of equality (11) and taking into account the last equality, we get

$$I_\Phi(y) = I_\Phi(x_0) + \int_T \omega\big(h(t)\big)\,d\mu = 1 + \int_T \omega\big(h(t)\big)\,d\mu. \tag{13}$$

Therefore, it suffices to prove that $\int_T \omega\big(h(t)\big)\,d\mu > 0$. Since $\eta(t) \geq 0$, $\eta \not\equiv 0$ and $h \not\equiv 0$, condition (12) yields that both sets

$$A_+ = \big\{t \in T : \operatorname{sgn}\big(x_0(t)\big)h(t) > 0\big\},$$
$$A_- = \big\{t \in T : \operatorname{sgn}\big(x_0(t)\big)h(t) < 0\big\}$$

have positive measure. It follows by assumption (iii) that either $\mu\Big(|x_0|^{-1}(A_1)\Big) = 0$ or $\mu\Big(|x_0|^{-1}(A_2)\Big) = 0$. Moreover, we have

$$\omega\big(h(t)\big) > 0 \text{ for } \mu\text{- a.a. } t \in A_+ \text{ whenever } \mu\Big(|x_0|^{-1}(A_1)\Big) = 0, \tag{14}$$

$$\omega\big(h(t)\big) > 0 \text{ for } \mu\text{- a.a. } t \in A_- \text{ whenever } \mu\Big(|x_0|^{-1}(A_2)\Big) = 0. \tag{15}$$

Since $\omega\big(h(t)\big) \geq 0$ for μ- a.e. $t \in T$ inequalities (14) and (15) yield $\int_T \omega\big(h(t)\big)\,d\mu > 0$. In virtue of (13), we get $I_\Phi(y) > 1$ which implies the desired inequality $\|y\|_\Phi > 1$, and the proof of sufficiency is finished.

NECESSITY. Conditions (i) and (ii) are necessary by Theorem 1. Now, we shall prove the necessity of condition (iii). If condition (iii) is not satisfied then there exists no regular support functional at x_0 (see [4], Lemma 3). Now, we shall prove that in this case every support functional at x_0 must be singular. Every functional $x^* \in (L_\Phi)^*$ is uniquely represented in the following form (see [1], [7], [8] and [13])

$$x^* = \xi_g + \xi^s,$$

where $g \in L_{\Phi^*}(\mu)$ and

$$\xi_g(x) = \int_T g(t)x(t)\,d\mu \quad \big(\forall x \in L^\Phi(\mu)\big),$$

and ξ^s denotes a singular functional, *i.e.* $\xi^s(x) = 0$ for all $x \in E^\Phi(\mu)$. Assume that $\|x^*\| = 1$ and $x^*(x_0) = 1 = \|x_0\|_\Phi$. Since additionally $\|x^*\| = \|\xi_g\| + \|\xi^s\|$ (see [1]), we get

$$1 = x^*(x_0) = \xi_g(x) + \xi^s(x) \leq \|\xi_g\|\,\|x_0\| + \|\xi^s\|\,\|x_0\| = \|\xi_g\| + \|\xi^s\| = \|x^*\| = 1,$$

whence

$$\xi_g(x_0) = \|\xi_g\| \, \|x_0\|_\Phi = \|\xi_g\| \text{ and } \xi^s(x_0) = \|\xi^s\| \, \|x_0\|_\Phi = \|\xi^s\|.$$

Therefore it must be

$$g(t) = \eta(t)\varkappa(t) \Big/ \int_T \eta(t) \frac{|x_0(t)|}{\|\xi_g\|} \, d\mu, \tag{16}$$

where $\eta(t) \in \partial\Phi\big(|x_0(t)|\big)$ for μ- a.e. $t \in T$ and $\varkappa(t)$ is any Σ-measurable function satisfying $|\varkappa(t)| = 1$ for μ- a.e. $t \in T$, $\varkappa(t) = \operatorname{sgn}\big(x_0(t)\big)$ for μ- a.e. $t \in \operatorname{supp}(x_0) \cap \operatorname{supp}(\eta)$ (see [4], [7], [8], [9] and [10]). Note that for Φ smooth at zero $\operatorname{supp}(x_0) \supset \operatorname{supp}(\eta)$. Since $g \in L_{\Phi^*}(\mu)$, it follows from (16) that $\eta \in L_{\Phi^*}(\mu)$, and hence $\int_T |x_0(t)| \eta(t) \, d\mu < +\infty$. Therefore, in the case when condition (iii) does not hold, every support functional at x_0 must be singular. Moreover it must be

$$I_\Phi(\lambda x_0) = +\infty \quad (\forall \lambda > 1), \tag{17}$$

because in the opposite case every support functional at x_0 must be regular (see [4], Lemma 2 and [8], Theorem 3.2); that means in virtue of the Hahn-Banach theorem that condition (iii) holds.

Define the sets

$$A_n = \{ t \in \operatorname{supp}(x_0) : |x_0(t)| \leq n \}.$$

We have $A_n \uparrow$ and $\mu\big(\bigcup_n A_n\big) = \mu(T)$. Therefore, there exists $m \in \mathbb{N}$ such that $\mu(A_m) > 0$. Define $y = x_0 \chi_{T \setminus A_m}$. Obviously $x_0 = y + x_0 \chi_{A_m}$, and $\|y\|_\Phi = 1$, because $I_\Phi(\lambda y) = +\infty$ for every $\lambda > 1$. Take an arbitrary support functional x^* at x_0. Since, by the previous considerations, x^* is singular and $x_0 \chi_{A_m} \in E^\Phi(\mu)$, we have $x^*(y) = x^*(x_0) - x^*(x_0 \chi_{A_m}) = x^*(x_0) = 1$. This means that x_0 is not exposed.

To finish the proof it suffices to show that x_0 is not exposed whenever condition (iv) is not satisfied. There are $a_1 \in A_1$ and $a_2 \in A_2$ such that $\mu\big(|x_0|^{-1}(a_1)\big) > 0$ and $\mu\big(|x_0|^{-1}(a_2)\big) > 0$. Let B_1 and B_2 be subsets of $|x_0|^{-1}(a_1)$ and $|x_0|^{-1}(a_2)$, respectively, of the same positive and finite measure. There are positive numbers ε and δ such that $a_1 + \varepsilon$ and $a_2 - \delta$ are in the same interval of the affinity of Φ as the points a_1 and a_2, respectively, and $\Phi(a_1) + \Phi(a_2) = \Phi(a_1 + \varepsilon) + \Phi(a_2 - \delta)$. Define

$$y(t) = x_0(t)\chi_{T \setminus (B_1 \cup B_2)}(t) + (a_1 + \varepsilon) \operatorname{sgn}\big(x_0(t)\big)\chi_{B_1}(t) + (a_2 - \delta) \operatorname{sgn}\big(x_0(t)\big)\chi_{B_2}(t).$$

We have

$$\begin{aligned} I_\Phi(y) &= I_\Phi(x_0 \chi_{T \setminus (B_1 \cup B_2)}) + \Phi(a_1 + \varepsilon)\mu(B_1) + \Phi(a_2 - \delta)\mu(B_2) \\ &= I_\Phi(x_0 \chi_{T \setminus (B_1 \cup B_2)}) + \big(\Phi(a_1 + \varepsilon) + \Phi(a_2 - \delta)\big)\mu(B_1) \\ &= I_\Phi(x_0 \chi_{T \setminus (B_1 \cup B_2)}) + \big(\Phi(a_1) + \Phi(a_2)\big)\mu(B_1) \\ &= I_\Phi(x_0 \chi_{T \setminus (B_1 \cup B_2)}) + \Phi(a_1)\mu(B_1) + \Phi(a_2)\mu(B_2) = I_\Phi(x_0) = 1. \end{aligned}$$

Therefore $\|y\|_\Phi = 1$. Since $\Phi'_+\big(|y(t)|\big) = \Phi'_-\big(|y(t)|\big) = \Phi'_+\big(|x_0(t)|\big) = \Phi'_-\big(|x_0(t)|\big)$ for $t \in B_1 \cup B_2$ and $\operatorname{sgn}\big(y(t)\big) = \operatorname{sgn}\big(x_0(t)\big)$ for every $t \in \operatorname{supp}(x_0)$, in view of the equality in Young's inequality, we get

$$(18)\qquad \begin{aligned} \xi_g(y) &= \frac{1}{c}\Big(\int_T \eta(t)\operatorname{sgn}\big(x_0(t)\big)y(t)\,d\mu\Big) = \frac{1}{c}\int_T \eta(t)|y(t)|\,d\mu \\ &= \frac{1}{c}\big(I_{\Phi^*}(\eta) + I_\Phi(y)\big) = \frac{1}{c}\big(I_{\Phi^*}(\eta) + 1\big), \end{aligned}$$

where $c = \int_T \eta(t)|x_0(t)| / \|\xi_g\|\,d\mu$ (see (16)), and

$$(19)\qquad \xi_g(x_0) = \frac{1}{c}\int_T \eta(t)|x_0(t)|\,d\mu = \frac{1}{c}\big(I_{\Phi^*}(\eta) + I_\Phi(x_0)\big) = \frac{1}{c}\big(I_{\Phi^*}(\eta) + 1\big),$$

i.e. $\xi_g(x_0) = \xi_g(y)$. Since $y - x_0 \in E^\Phi(\mu)$, we also have $\xi^s(x_0) = \xi^s(y)$. Therefore $x^*(y) = x^*(x_0) = 1$. Since $y \neq x_0$, this means that x_0 is not exposed, and the proof of Theorem 3 is finished.

NOTE. If x_0 is an exposed point of S_{L^Φ}, the points $\big(x_0(t), \Phi(x_0(t))\big)$ need not be exposed point of the epigraf of Φ for μ- a.e. $t \in T$.

Indeed, let $(\mathbb{R}_+, \Sigma, \mu)$ be the Lebesgue measure space and Φ be an Orlicz function defined by

$$\Phi(u) = \begin{cases} |u|, & \text{if } |u| \le 1, \\ u^2/2 + 1/2, & \text{if } 1 < |u| \le 2, \\ 2|u| - 3/2, & \text{if } 2 < |u| \le 3, \\ u^2/3 + 3/2, & \text{if } |u| > 3. \end{cases}$$

Let $A, B \in \Sigma$ be disjoint sets such that $\mu(A) = 1/2$, $\mu(B) = 1/9$. Define $x_0 = \chi_A + 3\chi_B$. Then $I_\Phi(x_0) = 1$, whence $\|x_0\|_\Phi = 1$. In virtue of Theorem 3 it follows that x_0 is an exposed point of S_{L^Φ}. However, it is evident that $\big(x_0(t), \Phi(x_0(t))\big)$ is not exposed point of the epigraph of Φ for any $t \in \operatorname{supp}(x_0)$.

It is well known that if Φ satisfies the suitable Δ_2-condition then Φ is finite-valued, $E^\Phi(\mu) = L^\Phi(\mu)$ and we have $\|x\|_\Phi = 1$ if and only if $I_\Phi(x) = 1$ (see [5] and [16]). For a finite-valued Orlicz function every functional $x^* \in S_{(E^\Phi)^*}$ which attains its norm at $x_0 \in S_{L^\Phi}$ is of the form (8), where g is given by formula (9) with η being a Σ-measurable function with values $\eta(t)$ in the interval $\big[\Phi'_-\big(|x_0(t)|\big), \Phi'_+\big(|x_0(t)|\big)\big]$ for μ- a.e. $t \in \operatorname{supp}(x_0) \cap \operatorname{supp}(\eta)$. Therefore condition (iii) in Theorem 2 is then satisfied and we can formulate the following characterization of an exposed point in the spaces $E^\Phi(\mu)$.

THEOREM 4. *Let μ be nonatomic and Φ be a finite-valued Orlicz function vanishing only at zero. A point $x_0 \in S_{E^\Phi}$ is exposed if and only if:*

(i) $|x_0(t)| \in \operatorname{Ext}(\Phi)$ for μ- a.a. $t \in T$, and

(ii) $\mu\big(|x_0|^{-1}(A_1)\big)\,\mu\big(|x_0|^{-1}(A_2)\big) = 0$.

THEOREM 5. *Let μ be a non-atomic positive and σ-finite measure. Φ be an Orlicz function vanishing only at zero and with $a(\Phi) < +\infty$ and x_0 be a function in S_{L^Φ} with $|x_0(t)| = a(\Phi)$ for μ- a.a. $t \in T$. Then x_0 is an exposed point.*

PROOF. Let g be a positive function on the whole T such that $\int_T g(t)\,d\mu = 1$. Such a function g exists by the σ-finiteness of μ. Let us define $f(t) = g(t)\operatorname{sgn}\big(x_0(t)\big)/\,a(\Phi)$ and

$$\xi_f(x) = \int_T f(t)x(t)\,d\mu \quad \big(\forall x \in L^\Phi(\mu)\big).$$

Obviously, $\xi_f \in \big(L^\Phi(\mu)\big)^*$ and $\xi_f(x_0) = 1 = \|x_0\|_\Phi$. Moreover, for any $x \in L^\Phi(\mu)$ with $\|x\|_\Phi \leq 1$, we have $I_\Phi(x) \leq 1$ and so $|x(t)| \leq a(\Phi)$ for μ- a.e. $t \in T$, whence

$$\begin{aligned} |\xi_f(x)| &= \frac{1}{a(\Phi)}\left|\int_T g(t)x(t)\operatorname{sgn}\big(x_0(t)\big)\,d\mu\right| \\ &\leq \frac{1}{a(\Phi)}\int_T g(t)|x(t)|\,d\mu \leq \int_T g(t)\,d\mu = 1, \end{aligned}$$

which yields $\|\xi_f\| = 1$. To finish the proof it suffices to show that if $y \in S_{L^\Phi}$ and $y \neq x_0$, then $\xi_f(y) < 1$. It follows by the assumptions that $I_\Phi(y) \leq 1$ and so $|y(t)| \leq a(\Phi) = |x_0(t)|$ and $\operatorname{sgn}\big(x_0(t)\big)y(t) < |x_0(t)|$ on a set of positive measure. Therefore $\xi_f(y) < \xi_f(x_0) = 1$, which finishes the proof.

COROLLARY 4. *For μ as in Theorem 5 every extreme point of $S_{L^\infty(\mu)}$ is an exposed point.*

This follows immediately by Corollary 3 and Theorem 5.

Now we shall give an example of Orlicz space $L^\Phi(\mu)$ such that S_{L^Φ} contains an extreme point that is not exposed.

COROLLARY 5. *Using Theorems 1 and 4, it is now easy to construct Orlicz spaces whose unit sphere contains extreme points that are not exposed. For example, if μ is nonatomic with $\mu(T) \geq 2$ and if we define*

$$\Phi(u) = \begin{cases} 2u^2/3, & \text{if } 0 < |u| \leq 1/2, \\ 2|u|/3 - 1/6, & \text{if } 1/2 < |u| \leq 3/2, \\ 2u^2/9 + 1/3, & \text{if } |u| > 3/2. \end{cases}$$

the function $x = \frac{1}{2}\chi_A + \frac{3}{2}\chi_B$, where A and B are disjoint sets of measure 1, is an extreme point of S_{L^Φ} that is not exposed.

NOTE. Convex sets in $\mathbb{R}^2$ that are unit balls of some Orlicz spaces were characterized in [3].

REMARK. Applying analogous methods to the ones used in this paper we can extend our results to Musielak-Orlicz spaces over nonatomic measure space (T, Σ, μ).

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References

1. T. Ando, *Linear functionals on Orlicz spaces*, Nieuw. Arch. Wisk. **8**(1980), 1–16.

2. P. Greim, *Strongly exposed points in Bochner L^p-spaces*, Proc. Amer. Math. Soc. **88**(1983), 81–84.

3. R. Grząślewicz, *Finite dimensional Orlicz spaces*, Bull. Acad. Polon. Sci. Math. (5–6) **33**(1985), 277–283.

4. R. Grząślewicz and H. Hudzik, *Smooth points of Orlicz spaces equipped with Luxemburg norm*, Math. Nachr. **155**(1992), 31–45.

5. H. Hudzik, *Strict convexity of Musielak-Orlicz spaces with Luxemburg* norm, Bull. Acad. Polon. Sci. Math. (5–6) **29**(1981), 235–247.

6. A. Kamińska, *Rotundity of Orlicz-Musielak sequence spaces*, Bull. Acad. Polon. Sci. Math. (3–4) **29**(1981), 137–144.

7. A. Kozek, *Orlicz spaces of functions with values in Banach spaces*, Comment. Math. (2) **19**(1977), 299–288.

8. ———, *Convex integral functionals on Orlicz spaces*, Comment. Math. (1) **21**(1979), 109–135.

9. M. A. Krasnoselskiĭ and Ya. B. Rutickiĭ, *Convex functions and Orlicz spaces*, Groningen, 1961, translation.

10. W. Kurc, *Strongly exposed points in Orlicz spaces of vector-valued functions, I*, Comment. Math. (1) **27**(1987), 121–134.

11. W. A. J. Luxemburg, *Banach function spaces*, Thesis, Delft, 1955.

12. J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Math. **1034**, Berlin-Heidelberg-New York, Springer, 1983.

13. M. Nowak, *Linear functionals in Orlicz sequence spaces without local convexity*, Internat. J. Math. & Math. Sci. (2) **15**(1992), 241–254.

14. W. Orlicz, *On integral representability of linear functionals over the space of ϕ-integrable functions*, Bull. Acad. Polon. Sci. Math. **8**(1960), 567–569.

15. K. Sundaresan, *On the strict convexity of certain Banach spaces*, Pacific J. Math. **15**(1965), 1083–1086.

16. B. Turett, *Rotundity of Orlicz spaces*, Proc. Kon. Ned. Akad. Wet. **79**(1976), 462–469.

17. C. Wu and S. Chen, *Extreme points and rotundity of Musielak-Orlicz spaces*, preprint in Chinese.

18. ———, *Extreme points and rotundity of Orlicz-Musielak spaces*, J. Math. Res and Exposition **2**(1988), 195–200.

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