ON GENERALIZED MORSE-TRANSUE FUNCTION SPACES

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1. Introduction. Marston Morse and William Transue **(6, 8)** have introduced and studied function spaces, called MT-spaces, for which the elements of the topological dual are of integral type. Their theory does not admit certain classical Banach function spaces including spaces of bounded functions and $\mathfrak{L}_C^\infty$ spaces. The theory of function spaces determined by a length function (λ-spaces) **(4, 5)**, which depends on a fixed measure, admits many of the maximal MT-spaces, the spaces $\mathfrak{L}_C^\infty$ and spaces of locally integrable functions but does not admit certain maximal MT-spaces including the space $\mathfrak{K}_C$ of complex continuous functions with compact supports.

In **(4)** the definition of MT-spaces was weakened by dropping the requirement that $\mathfrak{K}_C$ be dense in the space and making no hypothesis concerning the dual. The resulting spaces were called MT^*-spaces and the elements of intregal type in the dual then constituted the MT-conjugate of the space. A λ-space **(4)** is an MT^*-space if it contains $\mathfrak{K}_C$. The MT-spaces are just those MT^*-spaces for which the dual and MT-conjugate coincide. The space of bounded functions on a suitable space E is an MT^*-space that is neither an MT- nor a λ-space.

In the development of the theory of MT-spaces an important role was played by the fact that the semi-norm $\mathfrak{N}^A$ could be defined in A and extended to all of $\mathbf{C}^E$ by (3.2) below. Since there are MT^*-spaces for which the MT-conjugate reduces to the zero element of the dual (§ 3), (3.2) is not valid for every MT^*-space. For an $\mathfrak{N}^A$-extensible MT^*-space (Definition 3.2) (3.2) holds. Since $\mathfrak{N}^A$ is then a reflexive semi-norm, the MT-conjugate is then dense in the dual of A in the $\sigma(A', A)$ topology (Theorem 3.1). The $\mathfrak{N}^A$-extensible MT^*-spaces have many of the properties of general MT-spaces.

The last part of this paper is mainly concerned with the role played in the general theory of MT^*-spaces by the λ-spaces. When E is countable at infinity this can be simply stated as follows. If A is a λ-space containing $\mathfrak{K}_C$, A is an $\mathfrak{N}^A$-extensible MT^*-space for which every measure in $\mathfrak{A}^*$ is of base μ (Theorem 3.3.). Conversely if A is an $\mathfrak{N}^A$-extensible MT^*-space for which every measure in $\mathfrak{A}^*$ is of base μ, $\mathfrak{N}^A$ extended by (3.2) determines a length function λ (Theorem 4.1) and $\mathfrak{L}_C^\lambda$, the λ-space determined by λ, and Ω^A (§ 3), coincide on some μ-measurable set B with $E - B$ $\mathfrak{A}^*$-negligible (Theorem 4.3). If then A is an MT^*-space of Cauchy type, $A = \mathfrak{L}_C^\lambda = \Omega^A$

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on B. Thus an MT-space of Cauchy type on a locally compact space E that is countable at infinity coincides with a λ-space for μ on the restriction of E to some μ-measurable set B with $E - B$ $\mathfrak{A}^*$-negligible if and only if every element of A' is of base μ.

2. The MT-conjugates as vector spaces.

Let E be a locally compact space, $\mathbf{C}^E$ the vector space of functions on E valued in $\mathbf{C}$ the field of complex numbers. A semi-norm on a vector subspace A of $\mathbf{C}^E$ will be called monotone if $\mathfrak{N}^A(x) \leqslant \mathfrak{N}^A(y)$ when $|x(t)| \leqslant |y(t)|$, $x, y \in A$; non-trivial if $\mathfrak{N}^A(x) \not\equiv 0$ over A **(6)**.

Definition. A vector subspace A of $\mathbf{C}^E$ will be called an MT^*-space if it contains $\mathfrak{K}_C$, if with x it contains $|x|$ and $\bar{x}$ and if it has a non-trivial, monotone semi-norm $\mathfrak{N}^A$.

If A' is the dual of A topologized by $\mathfrak{N}^A$ as a semi-norm and if $y \in A'$, then the restriction of y to $\mathfrak{K}_C$ determines a C-measure $\hat{y}$ and

$$y(x) = \int x\, d\hat{y}, \tag{2.1}$$

for every $x \in \mathfrak{K}_C$ **(6)**. We denote by A^* the subspace of elements y of A' for which every $x \in A$ is $\hat{y}$-integrable with (2.1) holding and call such a y an element of integral type. We call A^* the MT-conjugate of A. As in **(6)** the mapping $y \to \hat{y}$ of A^* into $\mathfrak{M}_C$, the space of measures on E, is an isomorphism. We denote by $\mathfrak{A}'$ and $\mathfrak{A}^*$ the images of A', A^* in $\mathfrak{M}_C$ and call $\mathfrak{A}^*$ the MT-measure conjugate of A. We define for each $y \in A^*$, $\hat{y} \in \mathfrak{A}^*$,

$$|\hat{y}|_{\mathfrak{A}^*} = \sup_{\substack{x \neq 0 \\ x \in A}} |\int x\, d\hat{y}|/\mathfrak{N}^A(x) = \sup_{\substack{x \neq 0 \\ x \in A}} |y(x)|/\mathfrak{N}^A(x) = |y|_{A^*} = |y|_{A'},$$

where $|y|_{A'}$ is the usual norm on A'. There are corresponding definitions for real MT^*-spaces.

A^* is a vector subspace of A'. Let $y_1, y_2 \in A^*$, $a, b \in C$. Then $z = ay_1 + by_2 \in A'$ and determines a C-measure $\hat{z}$. From (2.1) for $\mathfrak{K}_C$ it follows that $\hat{z} = a\hat{y}_1 + b\hat{y}_2$. By **(6**, Corollary 9.1**)** every $x \in A$ is $a\hat{y}_1 + b\hat{y}_2 = \hat{z}$-integrable and

$$\int x\, d\hat{z} = \int x\, d(a\hat{y}_1 + b\hat{y}_2) = ay_1(x) + by_2(x) = z(x).$$

The spaces A^* and $\mathfrak{A}^*$ are thus normed vector spaces, equivalent by definition.

Morse and Transue **(6**, p. 153**)** associate with each C-measure η on E a unique positive measure $|\eta|$ such that for $x \in K$, $x \geqslant 0$,

$$|\eta|(x) = \sup_{\substack{|u| \leqslant x \\ u \in \mathfrak{K}_C}} |\int u\, d\eta|. \tag{2.2}$$

The absolute measure $|\eta|$ defined by η then has a unique extension $|\eta|_e$ as a real C-measure on E **(6**, p. 151**)**.

Condition 2.1. *If* $\eta \in \mathfrak{A}^*$, $|\eta|_e \in \mathfrak{A}^*$ *and* $|\eta|_{\mathfrak{A}^*} = ||\eta|_e|_{\mathfrak{A}^*}$.

Condition (2.1) is the analogue for the MT-conjugate spaces of the condition for A that $|x| \in A$ if $x \in A$ (noting that the monotone property of $\mathfrak{N}^A$ implies that $\mathfrak{N}^A(x) = \mathfrak{N}^A(|x|)$). If, for a positive measure μ, the C-measure η is of base μ (that is, can be written in the form $g(t) \, . \, \mu$ with $g(t)$ locally μ-integrable (**3**, p. 42; **7**, § 3).

$$|g(t) \, . \, \mu| = |g(t)| \, . \, \mu. \tag{2.3}$$

When all the elements of $\mathfrak{A}^*$ are of base μ, A^* can be identified with the collection of functions $\{g(t)\}$. If then A^* is an MT^*-space Condition 2.1 is necessarily satisfied. We note also that it is trivially satisfied when $A^* = 0$, that it is satisfied by the measure dual of every MT-space (**6**, Lemma 11.2) and by the measure dual of every MT^*-space that is a λ-space with the MT- and λ-conjugates coinciding (**4**).

Suppose that α_i, $i = 1, 2, \ldots$, are positive measures with $\alpha_{i,e} \in \mathfrak{A}^*$ and that $\Sigma|\alpha_{i,e}|_{\mathfrak{A}^*} < \infty$. Then for every $x \in \mathfrak{K}$, $x \geqslant 0$,

$$\sum_1^\infty |\alpha_i(x)| = \sum_1^\infty \alpha_i(|x|) \leqslant \mathfrak{N}^A(x) \sum_1^\infty |\alpha_{i,e}|_{\mathfrak{A}^*} < \infty,$$

so that the α_i form a *summable family* of positive measures on E and determine a positive measure $\alpha_0 = \Sigma_1^\infty \alpha_i$ (**3**, § 3, no. 5).

THEOREM 2.1. *Let A be an MT^*-space for which Condition* 2.1 *holds. If every real x in A is α_0-integrable for every α_0 defined as in the preceding paragraph, then $\mathfrak{A}^*$ is complete.*

Proof. The theorem is trivial when $\mathfrak{A}^* = 0$. In the general case let $\{\eta_n\}$ denote a Cauchy sequence in $\mathfrak{A}^*$ and choose a subsequence $\{\eta_{n_i}\}$ with

$$|\eta_{n_1}|_{\mathfrak{A}^*} + \sum_1^\infty |\eta_{n_{i+1}} - \eta_{n_i}|_{\mathfrak{A}^*} = L < \infty.$$

Define

$$\alpha_1 = |\eta_{n_1}|, \alpha_i = |\eta_{n_{i+1}} - \eta_{n_i}|, i = 2, 3, \ldots, \alpha_0 = \sum_1^\infty \alpha_i.$$

Condition 2.1 implies that each $\alpha_{i,e}$ is in $\mathfrak{A}^*$ with

$$|\alpha_{1,e}|_{\mathfrak{A}^*} = |\eta_{n_1}|_{\mathfrak{A}^*},$$
$$|\alpha_{i,e}|_{\mathfrak{A}^*} = |\eta_{n_{i+1}} - \eta_{n_i}|_{\mathfrak{A}^*}, i = 1, 2, \ldots.$$

By hypothesis each real $x \in A$ is α_0-integrable so that (**3**, Proposition 5, 3°)

$$\int x \, d\alpha_0 = \sum_1^\infty \int x \, d\alpha_i.$$

If $x \in A$, $x = x_1 + ix_2$, with x_1 and x_2 real and in A, x is $\alpha_{0,e}$-integrable (**6**, Lemma 4.3) and

$$\int x\,d\alpha_{0e} = \int x_1\,d\alpha_0 + i\int x_2\,d\alpha_0 = \sum_1^\infty \int x_1\,d\alpha_i + i\sum_1^\infty \int x_2\,d\alpha_i$$
$$= \sum_1^\infty \int x\,d\alpha_{i,e};$$
$$\left|\int x\,d\alpha_{0,e}\right| \leqslant \sum_1^\infty \int |x|\,d\alpha_{i,e} \leqslant L\,\mathfrak{N}^A(x).$$

It follows that $\int x\,d\alpha_{0,e}$ determines a continuous linear functional y of integral type with $\hat{y} = \alpha_{0,e}$ and therefore $\alpha_{0,e} \in \mathfrak{A}^*$.

For each $x \in A$,

$$[\eta_{n_i+1}(x) - \eta_{n_i}(x)]$$

is a Cauchy sequence in $\mathbf{C}$ since

$$\left|\sum_p^q [\eta_{n_i+1}(x) - \eta_{n_i}(x)]\right| \leqslant \sum_p^q \alpha_i(|x|) \to 0$$

as $p, q \to \infty$. Thus

$$\eta(x) = \eta_{n_1}(x) + \sum_1^\infty [\eta_{n_1+i}(x) - \eta_{n_i}(x)] = \lim_{i\to\infty} \eta_{n_i}(x) \tag{2.4}$$

is defined in $\mathbf{C}$ for every $x \in A$. Now η is linear on A and continuous since

$$|\eta(x)| \leqslant \alpha_0(|x|) \leqslant L\mathfrak{N}^A(x) \tag{2.5}$$

for all $x \in A$. Thus η determines an element of A'.

It follows from (2.2) and (2.5) that $|\eta|(x) \leqslant \alpha_0(x)$ for every $x \geqslant 0$, $x \in \mathfrak{R}$. This implies that $|\eta|^*(x) \leqslant \alpha^*{}_0(x)$ for every $x \geqslant 0$. Thus every α_0-negligible set is $|\eta|$-negligible and every α_0-measurable function is $|\eta|$-measurable **(2**, p. 180**)**. Thus if $x \in A$, $|x|$ is $|\eta|$-measurable and x is η-measurable **(6**, p. 168**)**. Since

$$\int |x| d\eta \leqslant \int |x| d\alpha_0 \leqslant L\,\mathfrak{N}^A(x) < \infty,$$

every x in A is η-integrable **(6**, Theorem 9.4**)**. This with (2.5) shows that $\int x\,d\eta$ determines an element $y \in A^*$ with $\hat{y} = \eta$ so that $\eta \in \mathfrak{A}^*$.

Then

$$|\eta - \eta_{n_i}|_{\mathfrak{A}^*} = \sup_{0\neq x\in A} |\textstyle\int x\,d(\eta - \eta_{n_i})|/\mathfrak{N}^A(x)$$
$$\leqslant \sup_{0\neq x\in A} \sum_{i+1}^\infty \int |x|\,d\alpha_j/\mathfrak{N}^A(x)$$
$$\leqslant \sum_{i+1}^\infty |\alpha_{j,e}|_{\mathfrak{A}^*}$$

which approaches zero as $i \to \infty$. The full sequence $\{\eta_n\}$ then converges to η in $\mathfrak{A}^*$ so that $\mathfrak{A}^*$ is complete.

COROLLARY. *If E is countable at infinity and A is an MT^*-space for which Condition* 2.1 *holds, A^* and $\mathfrak{A}^*$ are Banach spaces.*

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Proof. By **(3**, Corollaire 2, p. 28**)** every $x \in A$ is α_0-integrable.

Length functions for a positive measure μ are defined in **(4, 5)**. We denote by $\mathfrak{L}^\lambda$, $\mathfrak{L}_C{}^\lambda$ the subspaces of $\mathbf{R}^E$ and $\mathbf{C}^E$ respectively consisting of μ-measurable functions $x(t)$ with $\lambda(x) = \lambda(|x|) < \infty$ **(**cf. **5**, p. 577**)**. (If $x(t) \in \mathbf{C}^E$, it is μ-measurable for $\mu > 0$ if its Riesz components are μ-measurable **(6**. p. 168**)**.)

We show that if $A = \bar{\mathfrak{L}}_C{}^1(E, \mu)$ **(4**, § 2**)** with E and μ defined as in **(2**, Exercise 4, pp. 116**)** A^* is not complete. We define $g_i(P) = 1/\ln n$, $P = (1/n, k/n^2)$, $n = 2, 3, \ldots i$; $g_i(P) = 0$ elsewhere; $g(P) = 1/\ln n$, $P = (1/n, k/n^2)$, $n = 2, 3, \ldots$; $g(P) = 0$ elsewhere. The g_i form a Cauchy sequence in A' and converge to g. Each $g_i \,.\, \mu$ is in $\mathfrak{A}^*$ but $g \,.\, \mu$ is not.

The λ-conjugate of every λ-space is complete since it is also a λ-space **(4)**. Thus the MT-conjugate of an arbitrary λ-space containing $\mathfrak{K}_C$ is complete when it coincides with the λ-conjugate.

3. $\mathfrak{N}^A$-extensible MT*-spaces.

For a normed or semi-normed space X we let X_u denote the subunit elements of X, that is, the elements with norm or semi-norm not exceeding unity **(**cf. **6**, p. 171**)**.

Definition 3.1. A semi-norm $\mathfrak{N}^A$ on an MT^*-space A will be called *reflexive* if for every $x \in A$,

$$\mathfrak{N}^A(x) = \sup_{\eta \in \mathfrak{A}^*_u} |\textstyle\int x\, d\eta|. \tag{3.1}$$

THEOREM 3.1. *In order that $\mathfrak{N}^A$ be a reflexive semi-norm on the MT^*-space A it is necessary and sufficient that A^*_u be dense in A_u' for the $\sigma(A', A)$ topology.*

Proof. Since A_u' and A_u^* are *équilibré* parts of A', the polars of A_u' and A_u are respectively $A_u'^0 = (x \in A : |y(x)| \leqslant 1 \text{ for all } y \in A_u')$ and $A_u^{*0} = (x \in A : |y(x)| \leqslant 1 \text{ for all } y \in A^*_u)$ **(1**, p. 52**)**. We first show that $A^{*0}_u = A_u'^0$. Since $A^*_u \subset A_u'$, $A_u'^0 \supset A^{*0}_u$ and it is sufficient to prove the opposite inequality. If $x \in A^{*0}_u$, the hypothesis that $\mathfrak{N}^A$ is reflexive implies that

$$\mathfrak{N}^A(x) = \sup_{y \in \mathfrak{A}^*_u} |\textstyle\int x\, d\hat{y}| \leqslant 1.$$

Thus $|y(x)| \leqslant \mathfrak{N}^A(x)|y|_{A'} \leqslant 1$ if $y \in A_u'$ so that $x \in A_u'^0$.

Thus $A^{*0}_u = A_u'^0$ and it follows that $A^{*00}_u = A_u'^{00} = A_u'$. Since A^*_u is convex and contains 0, the argument of **(1**, Proposition 3, p. 52**)** shows that $A_u' = A^{*00}_u$ is the closure of A^*_u for $\sigma(A', A)$.

We next prove that the condition is sufficient. Since the definition of $|y|_A{}^*$ implies that $\geqslant$ holds in (3.1) we need only show that, given $\epsilon > 0$, there exists $y \in A^*_u$ with $\mathfrak{N}^A(x) \leqslant |\int x\, dy| + \epsilon$.

By an extension of the Hahn-Banach Theorem there exsits $y_0 \in A_u'$ with $y_0(x) = \mathfrak{N}^A(x)$, $|y_0|_{A'} = 1$. The set $[y \in A';\ |(y - y_0)(x)| < \epsilon]$ is a neighbourhood of y_0 for the $\sigma(A', A)$ topology and by hypothesis contains $y_1 \in A^*_u$. Then

$$0 \leqslant \mathfrak{N}^A(x) - |\textstyle\int x\, d\hat{y}_1| \leqslant |y_0(x) - y_1(x)| = |(y_0 - y_1)(x)| < \epsilon.$$

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We note the analogy with the relation between E and E'' for Banach spaces (**1**, Proposition 5, p. 114).

Definition 3.2. A semi-norm on an MT^*-space will be called *extensible* if A satisfies Condition 2.1 and $\mathfrak{N}^A$ is reflexive. An MT^*-space will be called $\mathfrak{N}^A$-extensible if it has an extensible semi-norm.

For an extensible semi-norm

$$(3.2) \qquad \mathfrak{N}^A(x) = \sup_{\eta \in \mathfrak{A}^*_u} \int^* |x| d|\eta|$$

holds with outer integrals replaced by integrals for every $x \in A$. Formula (3.2) then extends the definition of $\mathfrak{N}^A$ to all of $\mathbf{C}^E$ and all of $\bar{\mathbf{R}}^E$.

Given a collection of C-measures $\mathfrak{M}$ a function $x \in \mathbf{C}^E$ or $\bar{\mathbf{R}}^E$ will be called $\mathfrak{M}$-negligible if $|x(t)|$ is $|\eta|$-negligible for every $\eta \in \mathfrak{M}$. $\mathfrak{M}$-negligible sets, $\mathfrak{M}$-equivalence and almost everywhere ($\mathfrak{M}$) are then defined by analogy with the case where $\mathfrak{M}$ reduces to a single C-measure η. When A is an $\mathfrak{N}^A$-extensible MT^*-space, $\mathfrak{N}^A(x) = 0$ if x is $\mathfrak{A}^*$-negligible. If then $x(t)$ is defined and valued in $\mathbf{C}$ or $\bar{\mathbf{R}}$ almost everywhere ($\mathfrak{A}^*$), x is $\mathfrak{A}^*$-equivalent to some $\dot{x}$ in $\mathbf{C}^E$ or $\bar{\mathbf{R}}^E$ and we define $\mathfrak{N}^A(x) = \mathfrak{N}^A(\dot{x})$. When $\mathfrak{A}^* \equiv 0$ every function is $\mathfrak{A}^*$-negligible but $\mathfrak{N}^A(x) > 0$ holds for some $x \in A$.

THEOREM 3.2. *For* $1 \leqslant p < \infty$, $A = \bar{\mathfrak{L}}_C{}^p(E, \mu)$ *is an* $\mathfrak{N}^A$*-extensible* MT^**-space.*

LEMMA 3.1. *If* $A = \bar{\mathfrak{L}}_C{}^\lambda(E, \mu)$ *is an* MT^**-space for which the* λ*-conjugate contains the* MT*-conjugate, then Condition* 2.1 *is satisfied and every element of* $\mathfrak{A}^*$ *is of base* μ.

Proof of Lemma 3.1. Every g in the λ-conjugate is locally μ-integrable and therefore determines a measure $g \,.\, \mu$ (that is, a measure of base μ) (**4**, § 3). If $g \in A^*$, $\hat{g} = g \,.\, \mu$ and thus the elements of $\mathfrak{A}^*$ are of base μ.

The definition of the λ-conjugate then implies that $|g(t)| \in \mathfrak{L}^{\lambda *}$. By (**7**, § 3) $|g| \,.\, \mu = |g \,.\, \mu|$. Now $\int |x||g| d\mu \leqslant \lambda(x)\lambda^*(g) < \infty$ and the $g \,.\, \mu$-integrability of x implies that $\int |x| d|g \,.\, \mu| < \infty$ (**6**, Theorem 9.4). Thus by (**7**, Theorem 1.1), for every $x \in A$,

$$|g|(x) = \int x|g| d\mu = \int x \, d(|g| \,.\, \mu)_e$$

so that $(|g| \,.\, \mu)_e \in \mathfrak{A}^*$. It then follows from the definitions that

$$|(|g| \,.\, \mu)_e|_{\mathfrak{A}^*} = \lambda^*(|g|) = \lambda^*(g) = |g \,.\, \mu|_{\mathfrak{A}^*}.$$

Proof of Theorem 3.2. It remains to be shown that $\bar{\mathfrak{N}}^p$ is reflexive as a semi-norm on $A = \bar{\mathfrak{L}}_C{}^p$. Since $\bar{\mathfrak{N}}^p$ is reflexive as a length function,

$$\bar{\mathfrak{N}}^p(x) = \sup_{g \in (\mathfrak{L}_C^q)_u} \left| \int xg \, d\mu \right| \geqslant \sup_{g \in \mathfrak{A}^*_u} \left| \int xg \, d\mu \right|,$$

and it is sufficient to determine $g \in A^*{}_u$ with $|\int xg \, d\mu|$ arbitrarily near to $\bar{\mathfrak{N}}^p(x)$.

If $\mathfrak{N}^p(x) < \infty$ there exists $E_0 = \bigcup_1^\infty K_i$, where $\{K_i\}$ is an increasing sequence of compact sets for which, writing f_B for the product of the function $f(t)$ and the characteristic function of the set B,

$$\overline{\mathfrak{N}}^p(x) = \overline{\mathfrak{N}}^p(x_{E_0}) = \mathfrak{N}^p(x_{E_0})$$

(4, § 2**)**. Now

$$x_{E_0} \in \mathfrak{L}_C{}^p$$

and $\mathfrak{L}_C{}^p$ is $\mathfrak{N}^A$-extensible as an MT-space. Thus

$$\overline{\mathfrak{N}}^p(x) = \mathfrak{N}^p(x_{E_0}) = \sup_{g \in (\mathfrak{L}_C^q)_u} |\int x_{E_0}\, g\, d\,\mu|.$$

Since E_0 is μ-measurable and

$$|g_{E_0}(t)| \leqslant |g(t)|,\; g_{E_0} \in \mathfrak{L}^q{}_u$$

if $g \in L_u{}^q$. Thus

$$\overline{\mathfrak{N}}^p(x) = \mathfrak{N}^p(x_{E_0}) = \sup_{g_{E_0} \in (\mathfrak{L}_C^q)_u} |\int x\, g_{E_0}\, d\,\mu|.$$

For $g \in (\mathfrak{L}_C{}^q)_u$ fixed,

$$|\int x\, g_{K_i}\, d\,\mu| \rightarrow |\int x\, g_{K_0}\, d\,\mu|$$

as $i \rightarrow \infty$ and

$$g_{K_i} \in (\mathfrak{L}_C^q)_u.$$

Thus for i sufficiently large and a suitable

$$g \in (\mathfrak{L}_C^q)_u,\, g_{K_i} \in (\mathfrak{L}_C^q)_u$$

with $|\int x\, g_{K_i} d\mu|$ arbitrarily near $\overline{\mathfrak{N}}^p(x)$. The C-measure $g_{K_i}\,.\,\mu$ has compact support so that CK_i is $g_{K_i}\,.\,\mu$-negligible **(2**, Proposition 5, p. 119**)**. Thus if $f \in A$ and fg_{K_i} vanishes in E,

$$\int^*|f|d|g_{K_i}\,.\,\mu| \leqslant \int^*|f|_{CK_i}\, d(|g_{K_i}|\,.\,\mu) + \int^*|f|_{K_i} d(|g_{K_i}|\,.\,\mu) = \int|fg_{K_i}|d\mu = 0$$

and the complex analogue of **(4**, Theorem 3.1**)** implies that

$$g_{K_i}\,.\,\mu \in \mathfrak{A}_u{}^*.$$

THEOREM 3.3. *If λ is a reflexive length function for the positive measure μ, if E is countable at infinity or if E is arbitrary and $A = \mathfrak{L}_C{}^\lambda$ is an MT^*-space for which the MT- and λ-conjugates coincide, then A is $\mathfrak{N}^A$-extensible and every measure in $\mathfrak{A}^*$ is of base μ.*

Proof. Theorem 3.3 is a consequence of Lemma 3.1 and the fact that the reflexivity of $\mathfrak{N}^A = \lambda$ as a length function implies that it is reflexive as a semi-norm on A.

When A is an $\mathfrak{N}^A$-extensible MT^*-space we denote by $\mathfrak{F}^A$ the vector subspace of $\mathbf{C}^E$ of mappings x with $\mathfrak{N}^A(x) < \infty$. Then $\mathfrak{N}^A$ is a non-trivial, monotone semi-norm on $\mathfrak{F}^A$ and $\mathfrak{F}^A$ is an MT^*-space for which Condition 2.1 holds. If for each $\eta \neq 0$ in $\mathfrak{A}^*$ there exists a relatively compact set $e(\eta)$ that is not η-measurable the MT-conjugate of $\mathfrak{F}^A$ reduces to the zero element of A'. Such non-measurable sets exist, for example, if $A = \mathfrak{L}_C{}^p\ (E, \mu)$ with $E = (0, 1)$ and μ Lebesgue measure on E, $1 \leqslant p < \infty$. In contrast, if E is arbitrary, if $A = \mathfrak{K}_C$ and $\mathfrak{N}^A$ is the uniform semi-norm, $\mathfrak{N}^A$ extends to $\mathbf{C}^E$ in the form **(6**, Theorem 15.3**)**,

$$\mathfrak{N}^A(x) = \sup_{t \in E} |x(t)|$$

and $\mathfrak{F}^A$ is the space of all bounded functions on E which is an $\mathfrak{N}^A$-extensible MT^*-space.

We note that if $B = \mathfrak{F}^A$, where A is an arbitrary $\mathfrak{N}^A$-extensible MT^*-space, $\mathfrak{N}^A(x) > 0$ is possible for a $\mathfrak{B}^*$-negligible function in B but $\mathfrak{N}^A(x) = 0$ for every $\mathfrak{A}^*$-negligible function in B.

The properties of the extended semi-norm $\mathfrak{N}^A$ and of $\mathfrak{F}^A$ for MT-spaces **(6**, § 12**)** extend to $\mathfrak{N}^A$-extensible MT^*-spaces with A'-negligibility replaced by $\mathfrak{A}^*$-negligibility. In particular $\mathfrak{F}^A$ is complete.

Generalizing **(6)** we define

$$\Omega^A = \bigcap_{\eta \in \mathfrak{A}^*} \mathfrak{L}_C^1(E, \eta)$$

for every MT^*-space A. We define $\Omega_0{}^A = \Omega^A \cap \mathfrak{F}^A$. Then $\Omega_0{}^A$ is an MT^*-space with $\mathfrak{N}^A$ (extended) as a semi-norm.

THEOREM 3.4. *If A is an $\mathfrak{N}^A$-extensible MT^*-space and if A^* is complete or, more generally, tonnelé* **(1**, § 1**)**, *then* $\Omega_0{}^A = \Omega^A$.

Proof. The argument of **(8**, Theorem 5.1**)** applies. We note in particular that $\Omega_0{}^A = \Omega^A$ for every $\mathfrak{N}^A$-extensible MT^*-space A if E is countable at infinity (Theorem 2.1, Corollary).

4. λ-spaces generated by $\mathfrak{N}^A$-extensible MT*-spaces.

THEOREM 4.1. *Let A be an $\mathfrak{N}^A$-extensible MT^*-space, μ a positive measure on E. Then $\mathfrak{N}^A$, extended by* (3.2), *defines a length function for μ if and only if every μ-negligible set is $\mathfrak{A}^*$-negligible.*

Proof. By (3.1) and the subsequent remarks $\mathfrak{N}^A(x)$ is defined for every $x(t)$ that is defined almost everywhere ($\mathfrak{A}^*$) and valued in $\bar{\mathbf{R}}^E$ and therefore for every $x(t)$, μ-measurable and defined, non-negative and valued in $\bar{\mathbf{R}}$ almost everywhere ($\mathfrak{A}^*$). That $\mathfrak{N}^A$ then satisfies Conditions (L2)–(L5) for length functions **(5)** is then easily verified. We verify (L5). If $x_n(t) \in \bar{\mathbf{R}}^E$ is non-negative and μ-measurable, $n = 1, 2, \ldots$, and if $x_n(t)$ increases to $x(t)$ as $n \to \infty$, then for each $\eta \in \mathfrak{A}^*$,

$$\int^* x(t)\, d|\eta| = \sup_n \int^* x_n(t)\, d|\eta|,$$

by **(2**, Theorem 3, p. 110**)**. Thus

$$\mathfrak{N}^A(x) = \sup_{\eta\in\mathfrak{A}_u^*} \int^* x(t)\, d|\eta| = \sup_{\eta\in\mathfrak{A}_u^*} \sup_n \int^* x_n(t)\, d|\eta|$$
$$= \sup_n \mathfrak{N}^A(x_n).$$

If (L1) **(5)** holds every μ-negligible set is $\mathfrak{A}^*$-negligible. Conversely if every μ-negligible set is $\mathfrak{A}^*$-negligible, $\mathfrak{N}^A$ is defined and non-negative for every $x(t)$ that is non-negative a.e. (μ) (and therefore a.e. ($\mathfrak{A}^*$)) and if $x(t)$ is μ-negligible and $e = [t : x(t) \neq 0]$, e is μ-negligible **(2**, Theorem 1, p. 119**)** and therefore $\mathfrak{A}^*$-negligible. This implies that $x(t)$ is η-negligible for every $\eta \in \mathfrak{A}^*$ and (3.2) then shows that $\mathfrak{N}^A(x) = 0$ giving (L1).

We note that there exist $\mathfrak{N}^A$-extensible MT^*-spaces, in fact MT-spaces on a compact set E, for which $\mathfrak{N}^A$ cannot define a length function for any measure μ. Consider the MT-space $A = \mathfrak{C}_C(E)$ of complex valued functions continuous in $E = [0, 1]$ with semi-norm $\mathfrak{N}^A(x) = \sup_{t\in E}|x(t)|$ and suppose that $\mathfrak{N}^A$ defines a length function for some positive measure μ. Then, since $\mathfrak{A}^*$ contains all the point measures, the empty set is the only $\mathfrak{A}^*$-negligible set and therefore, by the preceding theorem, the only μ-negligible set. For each t, $0 \leqslant t \leqslant 1$, the set $\{t\}$ consisting of the point t is closed and therefore μ-measurable and $\mu(\{t\}) > 0$. For some $a > 0$ there is a collection of points t_i of E with $\mu(\{t_i\}) > a$, $i = 1, 2, \ldots$. Thus for the characteristic function of E, χ_E,

$$\mu(\chi_E) = \mu(E) \geqslant \lim_n \mu(\cup_1^n t_i) \geqslant \lim_n na = \infty,$$

contradicting the assumption that μ is a measure since $\chi_E \in \mathfrak{C}_C$.

The following theorem is a partial converse of Theorem 3.3.

THEOREM 4.2. *Let A be an $\mathfrak{N}^A$-extensible MT^*-space, μ a positive measure on E and suppose that all of the elements of $\mathfrak{A}^*$ are of base μ. Suppose that every μ-negligible set is $\mathfrak{A}^*$-negligible and that every $\mathfrak{A}^*$-negligible set is locally μ-negligible. Then $A \subset \bar{A} \subset \mathfrak{L}_C^\lambda = \Omega_0^A \subset \mathfrak{F}^A$.*

Proof. By Theorem 4.1 $\mathfrak{N}^A$ determines a length function λ for μ. We denote by $\mathfrak{L}_C^\lambda$ the λ-space determined by λ. By hypothesis every $\eta \in \mathfrak{A}^*$ can be written $\eta = g \,.\, \mu$ where $g(t)$ is locally μ-integrable. We identify the functions $g(t)$ with A^*, the measures $g \,.\, \mu$ with $\mathfrak{A}^*$. If $E(g) = (t : g(t) \neq 0)$, $E(g)$ is μ-measurable and, for every $x \in \Omega^A$, $x_{E(g)}(t)$ is μ-measurable **(3**, Proposition 3, p. 43**)**. Given a compact set K in E with $\mu(K) > 0$ consider, for all $g \in A^*$, the collection of subsets $E(g)$ of K with $\mu[E(g)] > 0$. From this collection form a maximal collection of disjoint sets and let B denote their union. Since this collection will be at most countable B will be μ-measurable. If $g \in A^*$, $g_{K-B} \in A^*$ and $\mu[\mathrm{E}(g_{K-B})] = 0$ for otherwise $B \cup E(g_{K-B})$ properly contains B contradicting the definition of B. Thus, for every $g \in A^*$, $g(t) = 0$ almost everywhere in $K - B$, $g \,.\, \mu(K - B) = 0$ and $K - B$ is A^*-negligible and therefore, by hypothesis, $K - B$ is μ-negligible. If $x \in \Omega^A$, x_B is μ-measurable and therefore x_K is μ-measurable. It follows from **(2**, Proposition 4, p. 182**)** that every $x \in \Omega^A$ is μ-measurable. If $x \in \Omega_0^A$, $\mathfrak{N}^A(x) < \infty$ and $x \in \mathfrak{L}_C^\lambda$. Thus $A \subset \Omega_0^A \subset \mathfrak{L}_C^\lambda$. Since $\mathfrak{L}_C^\lambda$ is complete it is closed in $\mathfrak{F}^A$ and contains $\bar{A}$, the closure of A.

To prove that $\mathfrak{L}_C^\lambda \subset \Omega_0^A$ we must show that every μ-measurable function $x(t)$ with $\mathfrak{N}^A(x) < \infty$ is in $\mathfrak{L}_C^1(g \,.\, \mu)$ for every $g \in A^*$. Every $x(t) \in \mathfrak{L}_C^\lambda$ is μ-measurable by definition so that the Riesz components of $x(t)$ are μ-measurable (**6**, p. 168). The Riesz components are then measurable ($|g \,.\, \mu| = |g| \,.\, \mu$) for every $g \in A^*$ (**3**, Proposition 3, p. 43). Thus $x(t)$ is measurable $(g \,.\, \mu)$ for every $g \in A^*$. Since for each $g \in A^*$, $|g \,.\, \mu|_e \in \mathfrak{A}^*$, it follows from (3.2) and (**6**, Theorem 9.4) that $x(t) \in \mathfrak{L}_C^1(g \,.\, \mu)$.

We note that if to each compact set K corresponds $g(t) \in A^*$ with $g(t) \neq 0$ a.e. (μ) in K, every $\mathfrak{A}^*$-negligible set is locally μ-negligible. This is true in particular if $\mathfrak{A}^*$ contains $\mathfrak{K}_C$ or the characteristic function of every compact set.

THEOREM 4.3. *Suppose that E is countable at infinity or that $E = E_0 \cup_1^\infty K_i$, with each K_i compact and E_0 locally μ-negligible, μ a positive measure. Let A be an $\mathfrak{N}^A$-extensible MT*-space for which all of the elements of A^* are of base μ. Then, if E_0 is $\mathfrak{A}^*$-negligible, the normed spaces $\mathbf{L}_C^{\mathfrak{N}^A}$ and $\tilde{\Omega}_0^A$ associated with $\mathfrak{L}_C^{NA}$ and Ω_0^A are equivalent and contain $\tilde{A}$, the normed space associated with A.*

Proof. As in Theorem 4.2 each K_i is the union of a μ-measurable set B_i and an $\mathfrak{A}^*$-negligible set. If $B = \cup_1^\infty B_i$, x_B is μ-measurable for every $x \in \Omega^A$. Every $g \in A^*$ vanishes a.e. $(\mathfrak{A}^*)$ in $\cup_1^\infty K_i - B$. If not, for some g, i,

$$\mu[E(g_{K_i - B})] > 0,$$

contradicting the definition of B_i. It follows that $B' = E - B$ is $\mathfrak{A}^*$-negligible. Thus for each $x(t) \in \Omega^A$, $x_B(t)$ is μ-measurable and $\mathfrak{N}^A(x - x_B) = 0$. If then $x(t) \in \Omega_0^A$, $x_B(t) \in \mathfrak{L}_C^\lambda$, $\lambda = \mathfrak{N}^A$, with $\mathfrak{N}^A(x) = \mathfrak{N}^A(x_B)$ and $\tilde{\Omega}_0^A \subset \mathbf{L}_C^\lambda$. The proof that $\mathbf{L}_C^\lambda \subset \tilde{\Omega}_0^A$ is similar to the corresponding part of the proof of Theorem 4.2.

When E is countable at infinity $\Omega_0^A = \Omega^A$. The space E defined in (**2**, Exercise 4, p. 116) is of the form $D \cup_1^\infty K_i$ with D locally μ-negligible for the measure μ defined there. For the spaces $\bar{\mathfrak{L}}_C^p$, $1 \leqslant p \leqslant \infty$, D is $\mathfrak{A}^*$-negligible.

We note that if $\mathfrak{K}_C$ is dense in $\mathfrak{L}_C^\lambda$, $\bar{A} = \mathfrak{L}_C^\lambda$ in Theorem 4.2 and $\bar{\tilde{A}} = \mathbf{L}_C^\lambda$ in Theorem 4.3.

5. MT*-spaces of Cauchy type. If A is an MT^*-space, let B be the vector subspace of A over R of real mappings in A, $\mathbf{B}$ the associated real normed vector space. As in (**8**), with a natural definition of a partial order on $\mathbf{B}$, $\mathbf{B}$ becomes a "Riesz space."

Definition 5.1. A complete $\mathfrak{N}^A$-extensible MT^*-space will be called an MT^*-space of *Cauchy type* if each subset H of $\mathbf{B}$, bounded in norm and filtering for the relation $\leqslant$ defines a Cauchy filter.

For a maximal MT-space the definition reduces to that given in **(8**, § 1**)**. The theory of MT-spaces of Cauchy type given in **(8)** extends to MT^*-spaces of Cauchy type with A'-negligibility replaced by $\mathfrak{A}^*$-negligibility and with Ω^A replaced by Ω_0^A.

THEOREM 5.1. *If A is an MT^*-space of Cauchy type then $A = \Omega_0^A$. If the hypotheses of Theorem 4.2 are then satisfied, $A = \mathfrak{L}_C^\lambda = \Omega_0^A$ and, if the hypotheses of Theorem 4.3 are satisfied $\tilde{A} = \mathbf{L}_C^\lambda = \tilde{\Omega}_0^A$.*

We note that if $A = \mathfrak{L}_C^\lambda$ is an MT^*-space of Cauchy type, the analogue of **(8**, Corollary 6.1**)** implies that λ satisfies (L9) **(4**, ((L9) as modified on p. 592)**)**. Thus if $E = [0, 1]$, μ Lebesgue measure, the space $\mathfrak{L}_C^\infty$ (E, μ) is not of Cauchy type.

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