ON THE RESIDUAL FINITENESS OF PERMUTATIONAL PRODUCTS OF GROUPS

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1. Introduction

In a recent paper Gregorac [1] has posed the following question: if P is a permutational product of the amalgam $\mathscr{A} = A \cup B|H$ and if P is residually finite are all the permutational products of $\mathscr{A}$ residually finite?

The present note gives an example answering the question in the negative.

The notation will be that introduced by B. H. Neumann in [3]. In particular for elements $a \in A$, $b \in B$, $\rho(a)$, $\rho(b)$ will denote the corresponding permutations in the permutational product under consideration. We use $\{ \}$ to denote groups, $\langle \rangle$ for sets, $( )$ for the permuted symbols of the permutational products and also for permutations themselves.

I should like to thank the referee for his comments.

2. Preliminaries

In our example below we shall need the following information which is easily deduced from § 4 of B. H. Neumann [3].

Let

$$X = \{x, k; x^3 = k^2 = (xk)^2 = 1\},$$
$$Y = \{y, k; y^3 = k^2 = (yk)^2 = 1\},$$
$$K = \{k; k^2 = 1\}$$

and let $\mathscr{X}$ be the amalgam $X \cup Y|K$. Take $X_1 = \langle k, x, x^2 \rangle$ as a set of left coset representatives of X modulo K and Y_1, Y_2 as the sets $\langle 1, y, y^2 \rangle$ and $\langle k, y, y^2 \rangle$ respectively of left coset representatives of Y modulo K. If P_1 is the permutational product of $\mathscr{X}$ using X_1 and Y_1 then in P_1 $\rho(x)$ and $\rho(y)$ generate a subgroup S_1 of permutations which is soluble of length 2. If P_2 is the permutational product of $\mathscr{X}$ using X_1 and Y_2 then $P_2 \cong S_2 \times Z_2$ where $S_2 = \{\rho(x), \rho(y)\} \cong A_9$, the alternating group on nine symbols, and Z_2 is a cycle of order 2.

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3. The example

We construct $\mathscr{A}$ as follows. Take

$$A = \{a, h; a^3 = h^2 = (ah)^2 = 1\},$$
$$B = \{h, b_1, b_2, \cdots; h^2 = b_\iota^3 = (hb_\iota)^2 = [b_\iota, b_j] = 1\}$$

and $$H = \{h; h^2 = 1\}.$$

To construct a permutational product Q_1 of $\mathscr{A}$ which is residually finite choose, as (left) coset representatives of A modulo H, the elements 1, a, a^2 and, as (left) coset representatives of B modulo H, the elements of $C = \{b_1, b_2, \cdots\} < B$. It is easy to check that in Q_1 the equation $[\rho(a), \rho(b_\iota)] = 1$ holds for all b_ι. Hence $W_1 = \{\rho(a), \rho(b_1), \rho(b_2), \cdots\}$ is the direct product of a countable infinity of 3-cycles and is consequently residually finite (see Lemma 1.1 [2]). Further W_1 has index 2 in Q_1 and so, as an extension of a residually finite group by a finite group, Q_1 is residually finite (Lemma 1.5 [2]).

To obtain a permutational product Q_2 of $\mathscr{A}$ which is not residually finite we choose, as coset representatives of A and B modulo H, the same systems of representatives as above except that we choose the representative of H in each system to be 'h' rather than '1'. To prove Q_2 is not residually finite we show that Q_2 contains a subgroup isomorphic to the alternating group on a countable infinity of symbols. Clearly no residually finite group can contain an infinite simple group.

Let C be as above. First we select an arbitrary generator b_ι in C and split the set of coset representatives of B modulo H into disjoint subsets $\langle h, b_\iota, b_\iota^2\rangle$, $\langle t, tb_i, tb_\iota^2\rangle$ where t ranges over all non-identity elements of $C_\iota = \{b_1, b_2, \cdots, b_{\iota-1}, b_{\iota+1}, \cdots\} < C$. We now fix on such an element t and consider the effect of the elements $\rho(a)$, $\rho(b_\iota)$ firstly on the set of triplets $(\bar{a}, \bar{b}, \bar{h})$ where $\bar{a} \in \langle h, a, a^2\rangle$, $\bar{b} \in \langle t, tb_\iota, tb_\iota^2\rangle$, $\bar{h} \in \langle 1, h\rangle$. It is not difficult to see that the subgroup $T_\iota = \{\rho(a), \rho(b_\iota)\}$ permutes this set of 18 triplets amongst themselves in precisely the same way that $S_1 = \{\rho(x), \rho(y)\}$ permutes the corresponding triplets[1] in P_1. Since S_1 is soluble ot length 2 the elements of the second derived group T_ι'' of T_ι all map the 18 triplets under consideration identically. This argument clearly holds for any element $t(\neq 1)$ of C_ι.

We now consider the effect of $\rho(a)$, $\rho(b_\iota)$ on the 18 triplets $(\bar{a}, b^*, \bar{h})$ where $\bar{a}, \bar{h}$ are as above and $b^* \in \langle h, b_\iota, b_\iota^2\rangle$. Clearly T_ι permutes this set of triplets amongst themselves in precisely the same way that S_2 permutes the corresponding set of triplets in P_2. Since $S_2 \cong A_9$, $S_2 = S_2''$. Thus any element of S_2 can be written as a product of commutators of the form $[[\alpha, \beta], [\gamma, \delta]]$ where each of $\alpha, \beta, \gamma, \delta$ is a product of $\rho(x)$, $\rho(y)$ and their inverses.

If we now consider the corresponding elements in T_ι'' (obtained by replacing

[1] That is the set of 18 triplets $(\bar{x}, \bar{y}, \bar{k})$ where $\bar{x} \in X_1$, $\bar{y} \in Y_1$ and $\bar{k} \in \langle 1, k\rangle$.

$\rho(x)$ by $\rho(a)$ and $\rho(y)$ by $\rho(b_\iota)$) the above considerations show that T_ι'' generates a copy of A_9 on the $(\bar{a}, b^*, \bar{h})$ and maps the $(\bar{a}, \bar{b}, \bar{h})$ identically.

Finally we select a subset of the set of all triplets on which Q_2 acts and label them as follows.

$(1) \cdots (h, h, 1)$	$(1') \cdots (h, h, h)$
$(2) \cdots (a^2, h, h)$	$(2') \cdots (a, h, 1)$
$(3) \cdots (a, h, h)$	$(3') \cdots (a^2, h, 1)$

and, for each integer $n = 1, 2, 3, \cdots$ label

$(6n-2) \cdots (h, b_n^2, h)$	$(6n-2') \cdots (h, b_n, 1)$
$(6n-1) \cdots (a^2, b_n^2, 1)$	$(6n-1') \cdots (a, b_n, h)$
$(6n) \cdots (a, b_n^2, 1)$	$(6n') \cdots (a^2, b_n, h)$
$(6n+1) \cdots (h, b_n, h)$	$(6n+1') \cdots (h, b_n^2, 1)$
$(6n+2) \cdots (a^2, b_n, 1)$	$(6n+2') \cdots (a, b_n^2, h)$
$(6n+3) \cdots (a, b_n, 1)$	$(6n+3') \cdots (a^2, b_n^2, h)$.

Now given any integer $m > 3$ write $m = 6n+r$ where $-2 \leqq r \leqq 3$. The above considerations showed that $T_n'' = \{\rho(a), \rho(b_n)\}$ generates a copy of A_9 on (1), (2), (3), $(6n-2), \cdots (6n+3)$; $(1')$, $(2')$, $(3')$, $(6n-2') \cdots, (6n+3')$ and leaves all other triplets invariant. In fact it is not difficult to check that T_n'' (and hence T_n) contains each of the permutations of the form $(12k)(1'2'k')$ where $k \in \langle 6n-2, \cdots, 6n+3 \rangle$ (that is, a permutation moving $1, 2, k, 1', 2', k'$ as shown and fixing all the other triplets on which Q_2 acts), and also the permutation $(1\,2\,3)$ $(1'2'3')$. Hence in Q_2 the subgroup $W_2 = \{\rho(a), \rho(b_1), \rho(b_2), \cdots\}$ contains all permutations of the form $(1\,2\,m)(1'2'm')$ with $m \geqq 3$. Hence W contains a subgroup isomorphic to the alternating group on a countable infinity of symbols thus proving that Q_2 is not residually finite.

References

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