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Weak Approximation for Points with Coordinates in Rank-one Subgroups of Global Function Fields

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Abstract. For every affine variety over a global function field, we show that the set of its points with coordinates in an arbitrary rank-one multiplicative subgroup of this function field satisfies the required property of weak approximation for finite sets of places of this function field avoiding arbitrarily given finitely many places.

1 Introduction

Let K be a global function field over a finite field k of positive characteristic p. We denote by k^{alg} the algebraic closure of k inside a fixed algebraic closure K^{alg} of K. Let Σ_K be the set of all places of K. For each $v \in \Sigma_K$, denote by K_v the completion of K at v; by O_v, $\mathfrak{m}_v$, and $\mathbb{F}_v$ respectively the valuation ring, the maximal ideal, and the residue field associated with v. For each finite subset $S \subset \Sigma_K$, we denote by O_S the ring of S-integers in K. We fix a cofinite subset $\widetilde{S} \subset \Sigma_K$, and endow $\prod_{v\in\widetilde{S}} K_v$ with the natural product topology. For any semi-abelian variety A over K and any subset H of $A(K)$, we denote by $H_{\widetilde{S}}$ the image of H in $A(\prod_{v\in\widetilde{S}} K_v)$ under the diagonal embedding, and denote by $\overline{H_{\widetilde{S}}}$ its topological closure; an element $(\alpha_v)_{v\in\widetilde{S}} \in A(\prod_{v\in\widetilde{S}} K_v)$ lies in $\overline{H_{\widetilde{S}}}$ if and only if there is a sequence $(h_n)_{n\geq1}$ in H, such that for each $v \in \widetilde{S}$ the sequence $(h_n)_{n\geq1}$ in the complete topological space $A(K_v)$ converges to α_v. In view of a conjecture proposed by Poonen and Voloch, we propose the following question.

Main Question Let X be a closed K-subvariety of a semi-abelian variety A over K, and let H be a finitely generated subgroup of $A(K)$. Does the equality

$$X\Big(\prod_{v\in\widetilde{S}} K_v\Big) \cap \overline{H_{\widetilde{S}}} = \overline{\big(X(K)\cap H\big)_{\widetilde{S}}} \tag{1.1}$$

hold?

Remark 1.1 Poonen and Voloch [PV10, Conjecture C] conjecture that the Main Question has a positive answer in the case where A is an abelian variety and $H = A(K)$. They also prove that this conjecture holds in the case where A has no nonzero isotrivial

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quotient, X does not contain a translate of a positive-dimensional subvariety of A, and $A(K^{\text{sep}})[p^\infty]$ is finite [PV10, Theorem B].

Remark 1.2 Without requiring the subgroup $H \subset A(K)$ to be finitely generated, the Main Question would have a negative answer [Sun13, Example 1].

Remark 1.3 The author shows that the Main Question has a positive answer in the following two cases.

(i) There is an isogeny f from A to an semi-abelian variety A_0 defined over k^{alg} satisfying the condition that each translate $P + f(X)$ of $f(X)$, where $P \in A_0(K^{\text{alg}})$, contains no positive-dimensional closed K^{alg}-subvariety Y of the base change of A_0 to K^{alg} such that Y is the base change of a k^{alg}-subvariety of A_0. ([Sun13])

(ii) When X is a (finite) union of linear subvarieties of $A = \mathbb{G}_{\text{m}}^M$, *i.e.*, those subvarieties defined by linear forms in M variables and $H = \mathbb{A}^M(\Gamma)$ for some finitely generated subgroup $\Gamma \subset K^* = \mathbb{G}_{\text{m}}(K)$ satisfying the condition that X contains no linear K-subvariety Y with dimension greater than one such that the translate of Y by some element in $\mathbb{A}^M(\rho(\Gamma))$, where $\rho(\Gamma) = \bigcap_{m \ge 0}(K^{p^m})^*\Gamma$, is defined over k [Sun14].

The set $X(\prod_{v\in\widetilde{S}} K_v) \cap \overline{H_{\widetilde{S}}}$ is shown to be contained in a zero-dimensional variety under the condition on X in Remark 1.1 and that in Remark 1.3(i), respectively, and contained in a (finite) union of lines under the condition that in Remark 1.3(ii). Thus we have in the former situation that $X(\prod_{v\in\widetilde{S}} K_v) \cap \overline{H_{\widetilde{S}}} = (X(K) \cap H)_{\widetilde{S}}$ is a finite set, and in the latter situation that $X(\prod_{v\in\widetilde{S}} K_v) \cap \overline{\mathbb{A}^M(\Gamma)_{\widetilde{S}}}$ is the set of $\overline{\Gamma_{\widetilde{S}}}$-multiples of points in some fixed finite subset $Z \subset \mathbb{A}^M(\Gamma)$ such that $X(K) \cap \mathbb{A}^M(\Gamma)$ is the set of Γ-multiples of points in Z, *i.e.*,

$$X(K) \cap \mathbb{A}^M(\Gamma) = \left\{ (\gamma c_1, \dots, \gamma c_M) : \gamma \in \Gamma, (c_1, \dots, c_M) \in Z \right\},$$

and $X(\prod_{v\in\widetilde{S}} K_v) \cap \overline{\mathbb{A}^M(\Gamma)_{\widetilde{S}}}$ is the set of limits of sequences $((\gamma_n c_1, \dots, \gamma_n c_M))_{n\ge1}$, where $(c_1, \dots, c_M) \in Z$ and $(\gamma_n)_{n\ge1}$ is a sequence in Γ converging in $\mathbb{A}^M(\prod_{v\in\widetilde{S}} K_v^*)$. It is interesting to find cases in which the Main Question has an affirmative answer, while the set $X(\prod_{v\in\widetilde{S}} K_v) \cap \overline{H_{\widetilde{S}}}$ cannot be described as easily as above. On the other hand, in the known results of the Main Question, for a given setting $(K, \widetilde{S}, A, X, H)$ with $\dim X \ge 3$, it is almost impossible to verify whether the hypotheses are satisfied; thus, in this sense, these results are not practical. For these two reasons, we desire to have another approach to show that the Main Question has a positive answer.

Toward a positive answer to the Main Question, *i.e.*, to prove that (1.1) holds, the following idea is very straightforward: given an arbitrary $\alpha \in X(\prod_{v\in\widetilde{S}} K_v) \cap \overline{H_{\widetilde{S}}}$, *i.e.*, given a sequence $(h_n)_{n\ge1}$ in H converging to α, we aim to construct a sequence $(h'_n)_{n\ge1}$ in $X(K) \cap H$ such that the sequence $(h_n - h'_n)_{n\ge1}$ in H converges to the neutral element of $A(\prod_{v\in\widetilde{S}} K_v)$. This suffices for our purpose since the desired property will show that α is the limit of the sequence $(h'_n)_{n\ge1}$ in $X(K) \cap H$ in $A(\prod_{v\in\widetilde{S}} K_v)$. To avoid the complicated geometry that A may have, in this paper we implement the above idea on the same setting in Remark 1.3(ii), where $A = \mathbb{G}_{\text{m}}^M$ is embedded in the affine space $\mathbb{A}^M$ in the usual way.

Denote the coordinates in $\mathbb{A}^M$ by $\mathbf{X} = (X_1, \ldots, X_M)$. For each polynomial $f \in K[X_1, \ldots, X_M]$, we denote by $H_f \subset \mathbb{A}^M$ the hypersurface defined by f. If the total degree of f is one, we say that H_f is a hyperplane. By a linear K-variety in $\mathbb{A}^M$, we mean an intersection of K-hyperplanes. In our setting, any subvariety X of A comes from some affine variety W in $\mathbb{A}^M$; thus, it is natural to work on the case where $H = \mathbb{A}^M(\Gamma)$ for a finitely generated subgroup $\Gamma \subset K^*$, where for any commutative ring R with unity, denote by R^* the group of its units. In this particular case, we simply have $X(\prod_{v \in \widetilde{S}} K_v) \cap \overline{H_{\widetilde{S}}} = W(\overline{\Gamma_{\widetilde{S}}})$ and $X(K) \cap H = W(\Gamma)$, where $\Gamma_{\widetilde{S}}$ and $\overline{\Gamma_{\widetilde{S}}}$ are defined above by viewing $K^* = \mathbb{G}_{\mathrm{m}}(K)$, and $W(\overline{\Gamma_{\widetilde{S}}})$ (resp., $W(\Gamma)$) is the set of points on W with coordinate components in $\overline{\Gamma_{\widetilde{S}}}$ (resp., in Γ). For each $v \in \Sigma_K$, we simply write $\overline{\Gamma_v}$ for $\overline{\Gamma_{\{v\}}}$.

Our main result is as follows.

Theorem 1.4 *Let $\Gamma \subset K^*$ be a finitely generated subgroup such that $\Gamma \cap O_S^*$ has rank at most one, where $S = \Sigma_K \smallsetminus \widetilde{S}$. Then for every closed K-variety W in an affine space, we have that $W(\overline{\Gamma_{\widetilde{S}}}) = \overline{W(\Gamma)_{\widetilde{S}}}$; equivalently, the Main Question always has a positive answer in the case where A is a direct product of copies of the multiplicative group, H is the subgroup of $A(K)$ consisting of elements with each component in Γ.*

Remark 1.5 Note that in the case where $\widetilde{S} = \Sigma_K$, the product formula implies that the subset $A(K)_{\widetilde{S}} = A(K)_{\Sigma_K}$ is discrete in $A(\prod_{v \in \Sigma_K} K_v)$, hence $\overline{H_{\Sigma_K}} = H_{\Sigma_K}$, and thus (1.1) holds trivially. On the other hand, given a fixed finitely generated subgroup $\Gamma \subset K^*$, the condition in Theorem 1.4 that $\Gamma \cap O_S^*$ has rank at most one imposes a restriction on S, and we can always find a finite subset $S \subset \Sigma_K$ with arbitrarily large cardinality such that this condition is satisfied. For example, in the case where $K = k(T)$ is a rational function field, Γ is generated by irreducible polynomials in $k[T]$, and S is a finite subset containing the place corresponding to $\frac{1}{T}$ and exactly one place corresponding to some of these irreducible polynomials generating Γ, we have that the rank of $\Gamma \cap O_S^*$ is exactly one.

Proved at the end of Section 2 as an immediate consequence of Theorem 1.4, the next corollary provides a local-global criterion of simultaneous solvability of finitely many multi-variate polynomials over K in the subgroup Γ. Note that the torsion subgroup $\mathrm{Tor}(\Gamma)$ of Γ is cyclic.

Corollary 1.6 *Let $\Gamma \subset K^*$ be the rank-one subgroup generated by γ and τ with $\tau \in \mathrm{Tor}(\Gamma)$. Let $f_j(X_1, \ldots, X_M) \in K[X_1, \ldots, X_M]$ for each $j \in \{1, \ldots, J\}$, where J is a natural number. Let $S_0 \subset \Sigma_K$ be a finite subset such that $S_0 \cup \widetilde{S} = \Sigma_K$, $\Gamma \subset O_{S_0}^*$, and $f_j(X_1, \ldots, X_M) \in O_{S_0}[X_1, \ldots, X_M]$ for each $j \in \{1, \ldots, J\}$. Consider the following statements:*

(L) *For every non-zero ideal $\mathfrak{a} \subseteq O_{S_0}$, there exists a tuple*

$$\left(e_{\mathfrak{a},m,i} : m \in \{1, \ldots, M\}, i \in \{0, 1\}\right)$$

of rational integers with length $2M$ such that for each $j \in \{1, \ldots, J\}$,

$$f_j(\tau^{e_{\mathfrak{a},1,0}} \gamma^{e_{\mathfrak{a},1,1}}, \ldots, \tau^{e_{\mathfrak{a},M,0}} \gamma^{e_{\mathfrak{a},M,1}}) \in \mathfrak{a}.$$

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(G) *There exists a tuple* $(e_{\mathfrak{a},m,i} : m \in \{1,\dots,M\}, i \in \{0,1\})$ *of rational integers with length* $2M$ *such that for each* $j \in \{1,\dots,J\}$,

$$f_j(\tau^{e_{0,1,0}}\gamma^{e_{0,1,1}},\dots,\tau^{e_{0,M,0}}\gamma^{e_{0,M,1}}) = 0.$$

Then (L) implies (G).

Remark 1.7 Since $\mathrm{Tor}(\Gamma)$ is finite, it is easy to see that Condition (L) stays equivalent when additionally requiring that $e_{\mathfrak{a},m,0}$ does not depend on the ideal $\mathfrak{a}$. Therefore, roughly speaking, Corollary 1.6 is a local-global criterion for the existence of M independent integer parameters. In the case where K is a number field, and $J = 1$ with $f_1(X_1,\dots,X_M) = \lambda_1 X_1 + \cdots + \lambda_M X_M$, Bartolome, Bilu, and Luca [BBL13] prove the analogous statement of Corollary 1.6 with additionally requiring in Condition (L) that there is a tuple $(n_{1,0},\dots,n_{M,0},n_{1,1},\dots,n_{M,1})$ of rational integers (independent on the ideal $\mathfrak{a}$) satisfying $n_{m',r}e_{\mathfrak{a},m,i} = n_{m,r}e_{\mathfrak{a},m',i}$ for every $m, m' \in \{1,\dots,M\}$, every $r \in \{0,1\}$, and every non-zero ideal $\mathfrak{a} \subseteq O_{S_0}$. This additional requirement makes their number-field result a local-global criterion for the existence of a single integer parameter.

In the rest of this paper, which is devoted to our proof of Theorem 1.4, we fix a natural number M, a closed K-variety W in $\mathbb{A}^M$, and a finitely generated subgroup $\Gamma \subset K^*$. We also drop the subscript $\widetilde{S}$ in the notation of topological closure so that $\overline{\Gamma_{\widetilde{S}}}$ and $\overline{W(\Gamma)_{\widetilde{S}}}$ are simply written as $\overline{\Gamma}$ and $\overline{W(\Gamma)}$ respectively.

2 The Proof of Theorem 1.4

Recall that we have fixed a closed K-variety W in $\mathbb{A}^M$ and a finitely generated subgroup $\Gamma \subset K^*$. We say that W is a homogeneous linear K-variety if its vanishing ideal is generated by linear forms over K. For any subgroup $\Delta \subset K^*$, we say that W is Δ-isotrivial if there is some $(\delta_1,\dots,\delta_M) \in \mathbb{A}^M(\Delta)$ such that the multiplicative translate $(\delta_1,\dots,\delta_M)\cdot W$ is a k-variety; we denote by $k(\Delta)$ the smallest subfield of K containing k and Δ, by $\rho(\Delta)$ the subgroup $\bigcap_{m\geq 0}(K^{p^m})^*\Delta$ of K^*, and by $\sqrt[K]{\Delta}$ the subgroup $\{x \in K^* : x^n \in \Delta \text{ for some } n \in \mathbb{N}\}$ of K^*. We need the following earlier result of the author.

Proposition 2.1 ([Sun14, Proposition 6]) *Let d be the dimension of W. Suppose that W is a union of homogeneous linear K-varieties, and that each d-dimensional irreducible component of W is not $\rho(\Gamma)$-isotrivial. Then there exists a finite union V of homogeneous linear K-subvarieties of W with dimension smaller than d such that $W(\overline{\Gamma_v}) = V(\overline{\Gamma_v})$ for every $v \in \Sigma_K$; in particular, we have $W(\overline{\Gamma}) = V(\overline{\Gamma})$.*

Remark 2.2 Proposition 2.1 is motivated by an earlier result of Derksen and Masser [DM12], which roughly says that the weaker conclusion $W(\Gamma) = V(\Gamma)$ holds under the slightly stronger hypothesis that each d-dimensional irreducible component of W is not $\sqrt[K]{\Gamma}$-isotrivial. For a homogeneous linear K-variety W that is $\sqrt[K]{\Gamma}$-isotrivial, they also express $W(\Gamma)$ in terms of Frobenius orbits of the Γ-valued points of its

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proper homogeneous linear K-subvarieties; however, this expression has no "adelic analog". For precise details, see Appendix A .

One of the key ingredients in the proof of the main result in this paper is the following unexpected application of Proposition 2.1.

Proposition 2.3 *For any closed K-variety $W \subset \mathbb{A}^M$, there exists some closed $k(\rho(\Gamma))$-subvariety V of W such that $W(\overline{\Gamma_v}) = V(\overline{\Gamma_v})$ for every $v \in \Sigma_K$; in particular, we have $W(\overline{\Gamma}) = V(\overline{\Gamma})$.*

Proof Let $\{f_j : 1 \le j \le J\} \subset K[X_1, \dots, X_M]$ be a set of polynomials defining W. Choose $D \in \mathbb{N}$ such that for each $j \in \{1, \dots, J\}$, we can write

$$f_j(X_1, \dots, X_M) = \sum_{(d_1,\dots,d_M)\in\{0,1,\dots,D\}^M} c_{(j,d_1,\dots,d_M)} X_1^{d_1} \cdots X_M^{d_M}$$

with each $c_{(j,d_1,\dots,d_M)} \in K$. Consider the tuple $\mathbf{Y} = (Y_{(d_1,\dots,d_M)})_{(d_1,\dots,d_M)\in\{0,1,\dots,D\}^M}$ of new variables, in which we define linear forms

$$\ell_j(\mathbf{Y}) = \sum_{(d_1,\dots,d_M)\in\{0,1,\dots,D\}^M} c_{(j,d_1,\dots,d_M)} Y_{(d_1,\dots,d_M)}$$

for each $j \in \{1, \dots, J\}$. Let $N = (D+1)^M$ and $W' \subset \mathbb{A}^N$ be the homogeneous linear variety defined by $\{\ell_j : 1 \le j \le J\}$. By Proposition 2.1, there exists a finite union V' of homogeneous linear K-subvarieties of W' such that each irreducible component of V' is $\rho(\Gamma)$-isotrivial and that $W'(\overline{\Gamma_v}) = V'(\overline{\Gamma_v})$ for every $v \in \Sigma_K$. In particular, each irreducible component of V' is defined over $k(\rho(\Gamma))$; thus, so is V'. Let $\{g'_j : 1 \le j \le J'\} \subset k(\rho(\Gamma))[\mathbf{Y}]$ be a set of polynomials defining V'. For each $j \in \{1, \dots, J'\}$, we construct $f'_j(X_1, \dots, X_M)$ by substituting each variable $Y_{(d_1,\dots,d_M)}$ in $g'_j(\mathbf{Y})$ by the monomial $X_1^{d_1} \cdots X_M^{d_M}$; thus, we have that $f'_j(X_1, \dots, X_M) \in k(\rho(\Gamma))[X_1, \dots, X_M]$. Let $V \subset \mathbb{A}^M$ be the $k(\rho(\Gamma))$-variety whose vanishing ideal is generated by $\{f'_j : 1 \le j \le J'\}$. For every $j \in \{1, \dots, J'\}$ and every $(x_1, \dots, x_M) \in V(K^{\mathrm{alg}})$, we have $f'_j(x_1, \dots, x_M) = 0$, thus the point $(x_1^{d_1} \cdots x_M^{d_M})_{(d_1,\dots,d_M)\in\{0,1,\dots,D\}^M} \in \mathbb{A}^N(K^{\mathrm{alg}})$ is a zero of $g'_j(\mathbf{Y})$ by construction; this shows that $(x_1^{d_1} \cdots x_M^{d_M})_{(d_1,\dots,d_M)\in\{0,1,\dots,D\}^M} \in V'(K^{\mathrm{alg}}) \subset W'(K^{\mathrm{alg}})$, and thus the construction yields $(x_1, \dots, x_M) \in W(K^{\mathrm{alg}})$. Hence, we see that $V \subset W$. Similar reasoning gives the other desired conclusion that $W(\overline{\Gamma_v}) = V(\overline{\Gamma_v})$ for every $v \in \Sigma_K$. ∎

For any finitely generated subgroup $\Delta \subset K^*$, [Vol98, Lemma 3] shows that $\rho(\Delta) \subset \sqrt[K]{\Delta}$, and thus that Δ and $\rho(\Delta)$ have the same rank; we also note that $\Delta \subset \rho(\Delta) = \rho(\rho(\Delta))$, by definition.

Proposition 2.4 *Letting $S = \Sigma_K \setminus \Sigma$, there exists a free subgroup $\Phi \subset O_S^*$ that has the same rank as $\Gamma \cap O_S^*$ and satisfies the following property: if $V(\overline{\Phi}) = \overline{V(\Phi)}$ for every closed $k(\Phi)$-variety $V \subset \mathbb{A}^M$, then $W(\overline{\Gamma}) = \overline{W(\Gamma)}$ for every closed K-variety $W \subset \mathbb{A}^M$.*

Proof Let Φ be a maximal free subgroup of the finitely generated abelian group $\rho(\Gamma \cap O_S^*)$. Since $\Phi \subset \rho(\Phi) \subset \rho(\rho(\Gamma \cap O_S^*)) = \rho(\Gamma \cap O_S^*)$, it follows that Φ is a

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maximal free subgroup of $\rho(\Phi)$, and this implies that $\rho(\Phi) = \mathrm{Tor}(\rho(\Phi))\Phi = k^*\Phi$. Letting $S_0 \subset \Sigma_K$ be a finite subset such that $\Gamma \subset O^*_{S_0}$, we see that the image of Γ in $\prod_{v\in\Sigma} K^*_v$ is contained in $(\prod_{v\in\Sigma\cap S_0} K^*_v) \times (\prod_{v\in\Sigma\smallsetminus S_0} O^*_v)$; since $S = \Sigma_K \smallsetminus \Sigma$, this shows that the image of $\Gamma \cap O^*_S$ in $\prod_{v\in\Sigma} K^*_v$ is exactly the intersection of the image of Γ in $\prod_{v\in\Sigma} K^*_v$ with the open subgroup $(\prod_{v\in\Sigma\cap S_0} O^*_v) \times (\prod_{v\in\Sigma\smallsetminus S_0} K^*_v)$ of $\prod_{v\in\Sigma} K^*_v$. It follows that $\Gamma \cap O^*_S$ is open in Γ. Since the index of $\Phi \cap \Gamma \cap O^*_S$ in $\Gamma \cap O^*_S$ is finite, [Sun14, Corollary 2] shows that $\Phi \cap \Gamma \cap O^*_S$ is open in $\Gamma \cap O^*_S$, and thus is open in Γ. We note that $\Phi \cap \Gamma \cap O^*_S = \Phi \cap \Gamma$, since $\Phi \subset O^*_S$.

Fix a closed K-variety $W \subset \mathbb{A}^M$. Consider an arbitrary $\mathbf{x} \in W(\overline{\Gamma})$, which is the limit of a sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ in $\mathbb{A}^M(\Gamma)$. Since $\Phi \cap \Gamma$ is open in Γ, we can assume that $\mathbf{x}_n = \mathbf{r}\mathbf{y}_n$ with some $\mathbf{r} \in \mathbb{A}^M(\Gamma)$ and a sequence $(\mathbf{y}_n)_{n\in\mathbb{N}}$ in $\mathbb{A}^M(\Phi \cap \Gamma)$. Note that the sequence $(\mathbf{y}_n)_{n\in\mathbb{N}}$ converges to $\mathbf{r}^{-1}\mathbf{x} \in (\mathbf{r}^{-1}W)(\overline{\Phi})$. Recalling that $\rho(\Phi) = k^*\Phi$, Proposition 2.3 says that there exists some closed $k(\Phi)$-subvariety V of $\mathbf{r}^{-1}W$ such that $(\mathbf{r}^{-1}W)(\overline{\Phi}) = V(\overline{\Phi})$. Assuming $V(\overline{\Phi}) = \overline{V(\Phi)}$, we see that

$$\mathbf{r}^{-1}\mathbf{x} \in (\mathbf{r}^{-1}W)(\overline{\Phi}) = V(\overline{\Phi}) = \overline{V(\Phi)} \subset \overline{(\mathbf{r}^{-1}W)(\Phi)} \subset \mathbf{r}^{-1}(\overline{W(\Phi)}),$$

i.e., $\mathbf{x} \in \overline{W(\Phi)}$ is the limit of some sequence $(\mathbf{x}'_n)_{n\in\mathbb{N}}$ in $W(\Phi)$. Letting $(\mathbf{x}''_n)_{n\in\mathbb{N}} \subset \mathbb{A}^M(\Phi\Gamma)$ be the sequence defined by $\mathbf{x}''_{2n-1} = \mathbf{x}_n$ and $\mathbf{x}''_{2n} = \mathbf{x}'_n$, we see that the sequence $(\mathbf{x}''_n)_{n\in\mathbb{N}} \subset \mathbb{A}^M(\Phi\Gamma)$ is Cauchy. As abelian groups, $\Phi\Gamma/\Gamma$ is isomorphic to $\Phi/(\Gamma\cap\Phi)$, which is finite by the construction of Φ; thus, [Sun14, Corollary 2] shows that Γ is open in $\Phi\Gamma$. It follows that $\Phi\cap\Gamma$ is open in $\Phi\Gamma$. Hence, for sufficiently large $n \in \mathbb{N}$, we have that $(\mathbf{r}^{-1}\mathbf{x}_n)^{-1}(\mathbf{r}^{-1}\mathbf{x}'_n) = (\mathbf{x}''_{2n-1})^{-1}\mathbf{x}''_{2n} \in \mathbb{A}^M(\Phi\cap\Gamma)$; since $\mathbf{r}^{-1}\mathbf{x}_n = \mathbf{y}_n \in \mathbb{A}^M(\Phi\cap\Gamma)$, we conclude that $\mathbf{r}^{-1}\mathbf{x}'_n \in \mathbb{A}^M(\Phi\cap\Gamma)$, and thus $\mathbf{x}'_n \in \mathbb{A}^M(\Gamma) \cap W(\Phi) \subset W(\Gamma)$, *i.e.*, $\mathbf{x} \in \overline{W(\Gamma)}$. This finishes the proof. ∎

We make the following convention. For a polynomial $Q(T) \in k[T]$ and a rational function $P(T) \in k(T)$, we say that $Q(T)$ divides $P(T)$ if any zero of $Q(T)$ in k^{alg} is not a pole of $\frac{P(T)}{Q(T)}$. The long proof of the following proposition, which is the core in the proof of Theorem 1.4, is postponed to Section 3.

Proposition 2.5 *Let $\mathcal{S}$ be a finite set of irreducible polynomials in $k[T]$. Let J be a natural number. For each $j \in \{1,\ldots,J\}$, let $f_j(X_1,\ldots,X_M) \in k[T][X_1,\ldots,X_M]$.*

Assume that there exists a sequence $\{(e_{1,n},\ldots,e_{M,n})\}_{n\geq 1}$ in $\mathbb{A}^M(\mathbb{Z})$ satisfying the following condition.

> *For every $Q(T) \in k[T]$ not divisible by any element in $\mathcal{S}$, there is an $N_Q \in \mathbb{N}$ such that for any $n \geq N_Q$ we have that $Q(T)$ divides $f_j(T^{e_{1,n}},\ldots,T^{e_{M,n}})$ for all $j \in \{1,\ldots,J\}$.*

Then there exists a sequence $\{(e'_{1,n},\ldots,e'_{M,n})\}_{n\in\mathcal{N}}$ in $\mathbb{A}^M(\mathbb{Z})$ indexed by an infinite subset $\mathcal{N} \subset \mathbb{N}$ with the following properties:

(i) *For each $n \in \mathcal{N}$ we have $f_j(T^{e'_{1,n}},\ldots,T^{e'_{M,n}}) = 0$ for all $j \in \{1,\ldots,J\}$.*

(ii) *For every $\widetilde{Q}(T) \in k[T]$ not divisible by T, there is an $\widetilde{N}_{\widetilde{Q}} \in \mathbb{N}$ such that for any $n \in \mathcal{N}$ with $n \geq \widetilde{N}_{\widetilde{Q}}$ we have that $\widetilde{Q}(T)$ divides $T^{e_{i,n}} - T^{e'_{i,n}}$ for all i.*

The next theorem follows formally from Proposition 2.5.

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Theorem 2.6 *Let W be a closed $k(\Gamma)$-variety in $\mathbb{A}^M$. Suppose that Γ is free with rank one, is contained in O_S^*, where $S = \Sigma_K \smallsetminus \Sigma$. Then we have that $W(\overline{\Gamma}) = \overline{W(\Gamma)}$.*

Proof Let γ be a generator of Γ. Let $\Sigma|_{k(\gamma)} \subset \Sigma$ be the subset satisfying the following property: for each $v \in \Sigma$, there exists a unique $w \in \Sigma|_{k(\gamma)}$ such that both v and w restrict to the same place of $k(\gamma)$. Consider the k-isomorphism between fields

$$k(T) \longrightarrow k(\gamma), \quad T \longmapsto \gamma. \tag{2.1}$$

Through the isomorphism (2.1), the set $\Sigma|_{k(\gamma)}$ is injectively mapped onto a subset of the set of places of $k(T)$. For each $v \in \Sigma|_{k(\gamma)}$, we have that $\gamma \in O_v^*$; let $P_v(T) \in k[T]$ be the irreducible polynomial corresponding to the image of v under this map. Let $\mathcal{S}$ be the complement of the subset $\{P_v(T) : v \in \Sigma|_{k(\gamma)}\}$ of the set of all irreducible polynomials in $k[T]$. Note that $\mathcal{S}$ is a finite set containing the polynomial T, and that $k[\Gamma] \subset \prod_{v\in\Sigma} O_v$, where $k[\Gamma]$ is the smallest subring of K containing both k and Γ.

Write $W = \bigcap_{j=1}^J H_{f_j}$, where $f_j(X_1,\dots,X_M) \in k[\gamma][X_1,\dots,X_M]$ for each j. Let $\{(\gamma^{e_{1,n}},\dots,\gamma^{e_{M,n}})\}_{n\geq 1}$ be a sequence in $\mathbb{A}^M(\Gamma)$ that converges to a point

$$(x_1,\dots,x_M) \in W(\overline{\Gamma}) \subset \mathbb{A}^M\Big(\prod_{v\in\Sigma} K_v^*\Big),$$

where $e_{i,n} \in \mathbb{Z}$. In fact, this sequence lies in the image of $\mathbb{A}^M(\prod_{v\in\Sigma|_{k(\gamma)}} k(\gamma)_v^*)$ in $\mathbb{A}^M(\prod_{v\in\Sigma} K_v^*)$ under the natural map, where $k(\gamma)_v$ denotes the topological closure of the subfield $k(\gamma)$ in K_v. Note that this image is a closed subset. The topology on $\overline{\Gamma}$ is induced from the usual product topology on $\prod_{v\in\Sigma} k(\gamma)_v^*$, and the latter topology is the same as the subspace topology restricted from the usual product topology on $\prod_{v\in\Sigma} k(\gamma)_v$. Thus, for each $i \in \{1,\dots,M\}$, the sequence $(\gamma^{e_{i,n}})_{n\geq 1}$ converges to x_i in $\prod_{v\in\Sigma} k(\gamma)_v$. Therefore, from the continuity of each f_j at $(x_1,\dots,x_M) \in \mathbb{A}^M(\prod_{v\in\Sigma} k(\gamma)_v)$, we see that each sequence $(f_j(\gamma^{e_{1,n}},\dots,\gamma^{e_{M,n}}))_{n\geq 1}$ converges to $f_j(x_1,\dots,x_M) = 0$ in $\prod_{v\in\Sigma} k(\gamma)_v$. Consider the sequence $\{(e_{1,n},\dots,e_{M,n})\}_{n\geq 1}$ in $\mathbb{A}^M(\mathbb{Z})$. Fix an arbitrary $Q(T) \in k[T]$ not divisible by any element in $\mathcal{S}$. Thus we have the prime decomposition $Q(T) = \prod_{v\in\Sigma|_{k(\gamma)}} P_v(T)^{n_v}$ in $k[T]$, where there are only finitely many $v \in \Sigma|_{k(\gamma)}$ with $n_v > 0$. In particular,

$$U_Q = \prod_{\substack{v\in\Sigma|_{k(\gamma)}\\ n_v=0}} k(\gamma)_v \times \prod_{\substack{v\in\Sigma|_{k(\gamma)}\\ n_v>0}} \big(\mathfrak{m}_v \cap k(\gamma)_v\big)^{n_v}$$

is an open subset in $\prod_{v\in\Sigma|_{k(\gamma)}} k(\gamma)_v$ endowed with the product topology. Note that $f_j(\gamma^{e_{1,n}},\dots,\gamma^{e_{M,n}}) \in k[\gamma,\gamma^{-1}]$ for each $j \in \{1,\dots,J\}$ and $n \in \mathbb{N}$. The intersection of U_Q with the image of $k[\gamma,\gamma^{-1}]$ in $\prod_{v\in\Sigma|_{k(\gamma)}} k(\gamma)_v$ is the image of $Q[\gamma]k[\gamma,\gamma^{-1}]$, which is thus an open subset of $k[\gamma,\gamma^{-1}]$ containing zero with respect to the subspace topology restricted from $\prod_{v\in\Sigma|_{k(\gamma)}} k(\gamma)_v$. Therefore, from the fact that each sequence $(f_j(\gamma^{e_{1,n}},\dots,\gamma^{e_{M,n}}))_{n\geq 1}$ converges to zero in $\prod_{v\in\Sigma} k(\gamma)_v$, it follows that there is an $N_Q \in \mathbb{N}$ such that for any $n \geq N_Q$ we have that $f_j(\gamma^{e_{1,n}},\dots,\gamma^{e_{M,n}}) \in Q[\gamma]k[\gamma,\gamma^{-1}]$ for each $j \in \{1,\dots,J\}$; thus by the isomorphism (2.1), we have that $Q(T)$ divides $f_j(T^{e_{1,n}},\dots,T^{e_{M,n}})$, because 0 is not a zero of $Q(T)$. Therefore, the assumption of

Proposition 2.5 is verified. Applying the isomorphism (2.1) to the conclusion of Proposition 2.5, we see that there exists a sequence $\{(e'_{1,n},\dots,e'_{M,n})\}_{n\in\mathcal{N}}$ in $\mathbb{A}^M(\mathbb{Z})$ indexed by an infinite subset $\mathcal{N}\subset\mathbb{N}$ satisfying the following properties:

- For each $n\in\mathcal{N}$ we have $f_j(\gamma^{e'_{1,n}},\dots,\gamma^{e'_{M,n}})=0$ for all $j\in\{1,\dots,J\}$.
- For every $\widetilde{Q}(T)\in k[T]$ not divisible by T, there is an $\widetilde{N}_{\widetilde{Q}}\in\mathbb{N}$ such that for any $n\in\mathcal{N}$ with $n\geq\widetilde{N}_{\widetilde{Q}}$ we have that $\gamma^{e_{i,n}}-\gamma^{e'_{i,n}}\in\widetilde{Q}(\gamma)k[\gamma,\gamma^{-1}]$ for all $i\in\{1,\dots,M\}$.

The first property says that $(\gamma^{e'_{1,n}},\dots,\gamma^{e'_{M,n}})\in W(\Gamma)$ for each $n\in\mathcal{N}$. On the other hand, because the image of $k[\gamma,\gamma^{-1}]$ in $\prod_{v\in\Sigma|_{k(\gamma)}}k(\gamma)_v$ lies in $\prod_{v\in\Sigma|_{k(\gamma)}}(O_v\cap k(\gamma)_v)$, one can argue similarly as above that the topology on $k[\gamma,\gamma^{-1}]$, which is induced from the usual product topology on $\prod_{v\in\Sigma}k(\gamma)_v$, is generated by those subsets $\widetilde{Q}(\gamma)k[\gamma,\gamma^{-1}]$ with $\widetilde{Q}(T)\in k[T]$ not divisible by any element in the set $\mathcal{S}$. Since $\mathcal{S}$ contains the polynomial T, the second property implies that for each $i\in\{1,\dots,M\}$ the sequence $(\gamma^{e_{i,n}}-\gamma^{e'_{i,n}})_{n\in\mathcal{N}}$ converges to zero in $\prod_{v\in\Sigma}k(\gamma)_v$; this shows that the two sequences $(\gamma^{e_{i,n}})_{n\in\mathcal{N}}$ and $(\gamma^{e'_{i,n}})_{n\in\mathcal{N}}$ converge to the same element in $\prod_{v\in\Sigma}k(\gamma)_v$. Hence, for each $i\in\{1,\dots,M\}$, the sequence $(\gamma^{e'_{i,n}})_{n\in\mathcal{N}}$ converges to x_i in $\prod_{v\in\Sigma}k(\gamma)_v$; since $x_i\in\prod_{v\in\Sigma}k(\gamma)_v^*$, it follows from what is explained above that the same convergence also happens in $\prod_{v\in\Sigma}k(\gamma)_v^*$. This shows that

$$(x_1,\dots,x_M)\in\overline{\{(\gamma^{e'_{1,n}},\dots,\gamma^{e'_{M,n}})\}_{n\in\mathcal{N}}}\subset\overline{W(\Gamma)},$$

which completes the proof. ∎

Proof of Theorem 1.4 Combine Proposition 2.4 and Theorem 2.6. ∎

Proof of Corollary 1.6 Condition (L) is equivalent to $W(\overline{\Gamma_{\Sigma_K\smallsetminus S_0}})\neq\varnothing$ by a compactness argument using the assumption $\Gamma\subset O^*_{S_0}$, while Condition (G) is clearly equivalent to $W(\Gamma)\neq\varnothing$; these two conditions are equivalent, since Theorem 1.4 implies that $W(\Gamma)_{\Sigma_K\smallsetminus S_0}$ is dense in the subspace $W(\overline{\Gamma_{\Sigma_K\smallsetminus S_0}})$ of the topological space $\mathbb{A}^M(\prod_{v\in\Sigma_K\smallsetminus S_0}K_v^*)$, where $W\subset\mathbb{A}^M$ is the variety defined by $f_j(X_1,\dots,X_M)=0$ for each $j\in\{1,\dots,J\}$. ∎

3 The Proof of Proposition 2.5

For any $a\in\mathbb{N}$ and $b\in\mathbb{N}\smallsetminus p\mathbb{N}$, consider the polynomial

$$g_{a,b}(T)=\frac{T^{ab}-1}{T^a-1}\in k[T].$$

The following result is proved in the author's recent work [Sunar].

Lemma 3.1 *Let $f(T)=\sum_{i\in I}c_iT^{e_i}\in k(T)$ with each $c_i\in k$ and $e_i\in\mathbb{Z}$, where I is a finite index set. Let $a\in\mathbb{N}$, $b\in\mathbb{N}\smallsetminus p\mathbb{N}$ with b greater than the cardinality of I. Denote by $\mathscr{C}$ the collection of those partitions $\mathscr{P}$ of the set I such that for each set $\Omega\in\mathscr{P}$ we have $\sum_{i\in\Omega}c_i=0$ and for each nonempty proper subset $\Omega'\subset\Omega$ we have $\sum_{i\in\Omega'}c_i\neq0$. Suppose that $g_{a,b}(T)$ divides $f(T)$. Then there is some $\mathscr{P}\in\mathscr{C}$ such that for each set $\Omega\in\mathscr{P}$ and each $i_1,i_2\in\Omega$, we have that ab divides $e_{i_1}-e_{i_2}$.*

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Proved by an elementary linear-algebra argument, the following result plays a crucial role so that Proposition 24 in the author's recent work [Sunar] can be generalized to Proposition 2.5, which is the core in the proof of Theorem 1.4.

Lemma 3.2 *Let $\mathcal{N} \subset \mathbb{N}$ be a subset such that for each $m \in \mathbb{N}$ there is some $n \in \mathcal{N}$ divisible by m. Let $a_{j,i} \in \mathbb{Z}$ and $b_j \in \mathbb{Z}$, $(j,i) \in \{1,\dots,J\} \times \{1,\dots,M\}$ be fixed integers. Suppose that for each $n \in \mathcal{N}$, there are some $e_{i,n}$, $i \in \{1,\dots,M\}$ such that n divides $b_j - \sum_{i=1}^{M} a_{j,i} e_{i,n}$ for each j. Then there is some $n_0 \in \mathcal{N}$ with the following property: for each $n \in \mathcal{N}$ divisible by n_0, there are some $e'_{i,n}$, $i \in \{1,\dots,M\}$, such that $\frac{n}{n_0}$ divides $e_{i,n} - e'_{i,n}$ and that $b_j = \sum_{i=1}^{M} a_{j,i} e'_{i,n}$ for each j.*

Proof Consider the J-by-$(M+1)$ matrix $(a_{j,i} \mid b_j)$, where j indexes rows and i indexes the first M columns. Applying a sequence of the following operations: interchanging any two rows or any two of the first M columns, multiplying some row by an integer, adding some row to another one, we can transform this matrix to $(a'_{j,i} \mid b'_j)$ such that for some $R \le \min\{J,M\}$, we have that $a'_{j,i} = 0$ for any

$$(j,i) \in \big(\{1,\dots,J\} \times \{1,\dots,R\}\big) \cup \big(\{R+1,\dots,J\} \times \{1,\dots,M\}\big)$$

with $i \neq j$, and that $a'_{i,i} \neq 0$ if and only if $i \in \{1,\dots,R\}$. Then there is some permutation σ on $\{1,\dots,M\}$ such that n divides $b'_j - \sum_{i=1}^{M} a'_{j,i} e_{\sigma(i),n}$ for each $n \in \mathcal{N}$ and each $j \in \{1,\dots,J\}$. By the properties of $\mathcal{N}$, there is some $n_0 \in \mathcal{N}$ divisible by $\prod_{i=1}^{R} a'_{i,i}$. For any $j \in \{1,\dots,R\}$ and any $n \in \mathcal{N}$ divisible by n_0, from the fact that

$$b'_j - \sum_{i=1}^{M} a'_{j,i} e_{\sigma(i),n} = b'_j - a'_{j,j} e_{\sigma(j),n} - \sum_{i=R+1}^{M} a'_{j,i} e_{\sigma(i),n}$$

is divisible by $n \in a'_{j,j}\mathbb{Z}$, we see that $a'_{j,j}$ divides $b'_j - \sum_{i=R+1}^{M} a'_{j,i} e_{\sigma(i),n}$, and thus there exists a unique $e'_{\sigma(j),n} \in \mathbb{Z}$ satisfying $b'_j - a'_{j,j} e'_{\sigma(j),n} - \sum_{i=R+1}^{M} a'_{j,i} e_{\sigma(i),n} = 0$; hence, n divides $a'_{j,j}(e'_{\sigma(j),n} - e_{\sigma(j),n})$. For any $j \in \{1,\dots,R\}$, since is n_0 divisible by $a'_{j,j}$, we conclude that $e_{\sigma(j),n} - e'_{\sigma(j),n}$ is divisible by $n/a'_{j,j}$ and thus by n/n_0 as desired. For any $j \in \{R+1,\dots,J\}$ and any $n \in \mathcal{N}$ divisible by n_0, we simply define $e'_{\sigma(j),n} = e_{\sigma(j),n}$; thus n/n_0 divides $e_{\sigma(j),n} - e'_{\sigma(j),n}$ trivially. For every pair $(j,i) \in \{R+1,\dots,J\} \times \{1,\dots,M\}$, we have $a'_{j,i} = 0$, hence b'_j is divisible by every integer, and therefore $b'_j = 0$. Combined with the construction of $e'_{\sigma(j),n}$ for any $j \in \{1,\dots,R\}$, we see that for any $j \in \{1,\dots,J\}$ and any $n \in \mathcal{N}$ divisible by n_0, we always have $b'_j - \sum_{i=1}^{M} a'_{j,i} e'_{\sigma(i),n} = 0$. Transforming the matrix $(a'_{j,i} \mid b'_j)$ back to $(a_{j,i} \mid b_j)$, we obtain that $b_j - \sum_{i=1}^{M} a_{j,i} e'_{i,n} = 0$, as desired. ■

Proof of Proposition 2.5 Choose $D \in \mathbb{N}$ such that for each $j \in \{1,\dots,J\}$, we can write

$$f_j(X_1,\dots,X_M) = \sum_{(d_0,d_1,\dots,d_M)\in\{0,1,\dots,D\}^{M+1}} c_{(j,d_0,d_1,\dots,d_M)} T^{d_0} X_1^{d_1} \cdots X_M^{d_M}$$

with each $c_{(j,d_0,d_1,\dots,d_M)} \in k$. For each $j \in \{1,\dots,J\}$, denote by $\mathscr{C}_j$ the collection of those partitions $\mathscr{P}$ of the set $\{0,1,\dots,D\}^{M+1}$ such that for each set $\Omega \in \mathscr{P}$, we have $\sum_{(d_0,d_1,\dots,d_M)\in\Omega} c_{(j,d_0,d_1,\dots,d_M)} = 0$ and for each nonempty proper subset $\Omega' \subset \Omega$ we

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have $\sum_{(d_0,d_1,\dots,d_M)\in\Omega'} c_{(d_0,d_1,\dots,d_M)} \neq 0$. By [Sunar, Remark 14], we can choose some $a_0 \in \mathbb{N} \setminus p\mathbb{N}$ and $b_0 \in \mathbb{N} \setminus p\mathbb{N}$ with $b_0 > (D+1)^{M+1}$ such that for any $a \in a_0\mathbb{N}$ the polynomial $g_{a,b_0}(T)$ is not divisible by any element in $\mathcal{S}$. By our assumption, there is a strictly increasing sequence $\{N_a\}_{a\in a_0\mathbb{N}}$ such that $g_{a,b_0}(T)$ divides

$$f_j(T^{e_{1,N_a}},\dots,T^{e_{M,N_a}}) = \sum_{(d_0,d_1,\dots,d_M)\in\{0,1,\dots,D\}^{M+1}} c_{(j,d_0,d_1,\dots,d_M)} T^{d_0+d_1e_{1,N_a}+\cdots+d_Me_{M,N_a}}$$

for any $j \in \{1,\dots,J\}$. Thus, by Lemma 3.1, for any $a \in a_0\mathbb{N}$ and any $j \in \{1,\dots,J\}$, there is some $\mathscr{P}_{j,a} \in \mathscr{C}_j$ such that for each set $\Omega \in \mathscr{P}_{j,a}$, each $(d_0,d_1,\dots,d_M)$ and $(d_0',d_1',\dots,d_M')$ in Ω, any $i \in \{1,\dots,M\}$ and any $n \geq N_a$, we have that ab_0 divides both $d_0 - d_0' + (d_1 - d_1')e_{1,N_a} + \cdots + (d_M - d_M')e_{M,N_a}$. Consider the subset $\{\prod_{i=1}^{a_0+n} i : n \in \mathbb{N}\} \subset a_0\mathbb{N}$. For each $j \in \{1,\dots,J\}$ the collection $\mathscr{C}_j$ is finite, while $\{\prod_{i=1}^{a_0+n} i : n \in \mathbb{N}\}$ is infinite; thus, there is an infinite subset $\mathcal{A}$ of $\{\prod_{i=1}^{a_0+n} i : n \in \mathbb{N}\}$ that is contained in $a_0\mathbb{N}$, such that for each $j \in \{1,\dots,J\}$ the collection $\{\mathscr{P}_{j,a} : a \in \mathcal{A}\}$ consists of only one partition, denoted by $\mathscr{P}_j$. Since $\mathcal{A} \subset \{\prod_{i=1}^{a_0+n} i : n \in \mathbb{N}\}$ is an infinite subset, it has the property that for each $m \in \mathbb{N}$, there is some $a \in \mathcal{A}$ divisible by m.

For any $a \in \mathcal{A}$, any $j \in \{1,\dots,J\}$, we observe that $(e_{1,N_a},\dots,e_{M,N_a})$ satisfies the condition that for each set $\Omega \in \mathscr{P}_j$, each $(d_0,d_1,\dots,d_M)$ and $(d_0',d_1',\dots,d_M')$ in Ω, we have that a divides $d_0 - d_0' + (d_1 - d_1')e_{1,N_a} + \cdots + (d_M - d_M')e_{M,N_a}$. Applying Lemma 3.2, we obtain some $n_0 \in \mathcal{A}$ with the following property: for each $a \in \mathcal{A}$ divisible by n_0, there are some e'_{i,N_a}, $i \in \{1,\dots,M\}$, such that $\frac{a}{n_0}$ divides $e_{i,N_a} - e'_{i,N_a}$ and that for each $j \in \{1,\dots,J\}$, each set $\Omega \in \mathscr{P}_j$, each $(d_0,d_1,\dots,d_M)$ and $(d_0',d_1',\dots,d_M')$ in Ω, we have

$$d_0 - d_0' + (d_1 - d_1')e'_{1,N_a} + \cdots + (d_M - d_M')e'_{M,N_a} = 0;$$

thus, we can let $m_{a,j,\Omega} = d_0 + d_1e'_{1,N_a} + \cdots + d_Me'_{M,N_a}$ for any $(d_0,d_1,\dots,d_M) \in \Omega$. Letting $\mathcal{N} = \{N_a : a \in \mathcal{A} \cap n_0\mathbb{N}\}$, which is an infinite subset of $\mathbb{N}$, since $\mathcal{A} \subset a_0\mathbb{N}$ and the sequence $\{N_a\}_{a\in a_0\mathbb{N}}$ is strictly increasing, we now show that the constructed sequence $\{(e'_{1,n},\dots,e'_{M,n})\}_{n\in\mathcal{N}}$ satisfies the desired properties. To verify property (i), we fix some $j \in \{1,\dots,J\}$ and $n = N_a \in \mathcal{N}$ with $a \in \mathcal{A} \cap n_0\mathbb{N}$. From construction, we have

$$\begin{aligned} f_j(T^{e'_{1,n}},\dots,T^{e'_{M,n}}) &= \sum_{(d_0,d_1,\dots,d_M)\in\{0,1,\dots,D\}^{M+1}} c_{(j,d_0,d_1,\dots,d_M)} T^{d_0+d_1e'_{1,N_a}+\cdots+d_Me'_{M,N_a}} \\ &= \sum_{\Omega\in\mathscr{P}_j} T^{m_{a,j,\Omega}} \sum_{(d_0,d_1,\dots,d_M)\in\Omega} c_{(j,d_0,d_1,\dots,d_M)} = 0, \end{aligned}$$

as desired. To verify property (ii), we fix some $\widetilde{Q}(T) \in k[T]$, not divisible by T. Since each zero of $\widetilde{Q}(T)$ is in $(k^{\mathrm{alg}})^*$ and thus has a finite order, we can use the property that for each $m \in \mathbb{N}$ there is some element of $\mathcal{A} \cap n_0\mathbb{N}$ divisible by m and get some $a \in \mathcal{A} \cap n_0\mathbb{N}$ such that $\widetilde{Q}(T)$ divides $T^a - 1$. Using this property again yields some $a_{\widetilde{Q}} \in \mathcal{A} \cap n_0\mathbb{N}$ divisible by an_0. Let $\widetilde{N}_{\widetilde{Q}} = N_{a_{\widetilde{Q}}}$ and fix some $n \in \mathcal{N}$ with $n \geq \widetilde{N}_{\widetilde{Q}} = N_{a_{\widetilde{Q}}}$. Then $n = N_{a'} \in \mathcal{N}$ with some $a' \in \mathcal{A} \cap n_0\mathbb{N}$; the latter condition implies, by construction, that a'/n_0 divides $e_{i,N_{a'}} - e'_{i,N_{a'}} = e_{i,n} - e'_{i,n}$ for each $i \in \{1,\dots,M\}$.

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Since the sequence $\{N_a\}_{a\in a_0\mathbb{N}}$ is strictly increasing, this implies that $a' \geq a_{\widetilde{Q}}$; by construction, both a' and $a_{\widetilde{Q}}$ are in the set $\{\prod_{i=1}^{a_0+n} i : n \in \mathbb{N}\}$; thus, we get that a' is divisible by $a_{\widetilde{Q}}$. Because $a_{\widetilde{Q}}/n_0$ is divisible by a, we conclude that a divides a'/n_0 and thus divides $e_{i,n} - e'_{i,n}$ for each $i \in \{1, \dots, M\}$; equivalently, we have shown that $T^{e_{i,n}} - T^{e'_{i,n}}$ is divisible by $T^a - 1$ and thus by $\widetilde{Q}(T)$ as desired. This completes the proof. ∎

A Appendix

Recall that we say that an affine variety $W \subset \mathbb{A}^M$ is a homogeneous linear K-variety if its vanishing ideal is generated by linear forms over K. A homogeneous linear K-variety is called a *coset* if its vanishing ideal is generated by linear forms over K with at most two nonzero terms. The following result is proved in [DM12, Section 9].

Theorem A.1 (Derksen-Masser [DM12]) *Let $\Gamma \subset K^*$ be a finitely generated subgroup and let $W \subset \mathbb{A}^M$ be a homogeneous linear K-variety whose dimension is at least two. Suppose that W is not a coset.*

(i) *If W is not $\sqrt[K]{\Gamma}$-isotrivial, then there exists a finite union V of proper homogeneous linear K-subvarieties of W such that $W(\Gamma) = V(\Gamma)$.*

(ii) *If W is $\sqrt[K]{\Gamma}$-isotrivial,* i.e., *there is some $(x_1, \dots, x_M) \in \mathbb{A}^M(\sqrt[K]{\Gamma})$ such that the multiplicative translate $(x_1, \dots, x_M) \cdot W$ is a k-variety, then there exists a finite union V of proper homogeneous linear K-subvarieties of W such that*

$$\big((x_1, \dots, x_M) \cdot W\big)(\Gamma) = \bigcup_{e\in\mathbb{N}\cup\{0\}} \Big(\big((x_1, \dots, x_M) \cdot V\big)(\Gamma)\Big)^{|k|^e}.$$

Example A.2 Proposition 2.1 provides an "adelic analog" of Theorem A.1(i). However, there is no such analog of Theorem A.1(ii). To see this, consider the example where $W = H_f \subset \mathbb{A}^3$ with $f(X_1, X_2) = X_1 + X_2 - X_3 \in \mathbb{F}_p[X_1, X_2]$. Note that W is an irreducible homogeneous $\mathbb{F}_p$-variety of dimension two. By Theorem A.1(ii), there exists a finite union V of proper homogeneous linear K-subvarieties of W such that $W(\Gamma) = \bigcup_{e\in\mathbb{N}\cup\{0\}} (V(\Gamma))^{p^e}$. Nevertheless, in the case where $K = \mathbb{F}_p(T)$ and $\Gamma = \{cT^n(1-T)^m : c \in \mathbb{F}_p^*, (n,m) \in \mathbb{Z}^2\}$ and Σ is the maximal subset of Σ_K such that $\Gamma \subset O_v^*$ for each $v \in \Sigma$, we claim that

$$W(\overline{\Gamma}) \notin \bigcup_{e\in\mathbb{N}\cup\{0\}} \big(V(\overline{\Gamma})\big)^{p^e}$$

for any finite union V of proper homogeneous linear K-subvarieties V of W. To see this, assume for the contradiction that there exists such V satisfying

$$W(\overline{\Gamma}) \subset \bigcup_{e\in\mathbb{N}\cup\{0\}} (V(\overline{\Gamma}))^{p^e}.$$

Since the subvariety $V \subset W = H_f$ must have dimension one, it follows that the K-variety $\phi_3(V)$, where $\phi_3 \colon \mathbb{A}^3 \to \mathbb{A}^2$ given by $(x_1, x_2, x_3) \mapsto (\frac{x_1}{x_3}, \frac{x_2}{x_3})$, is a zero-dimensional subvariety of $\phi_3(W)$, which is the line in $\mathbb{A}^2$ defined by $X_1 + X_2 - 1 = 0$.

On the other hand, the sequence $(T^{p^{n!}})_{n\geq 0}$ converges to some element $\alpha = (\alpha_v)_{v\in\Sigma} \in \overline{\Gamma_T} \subset \overline{\Gamma}$, where $\Gamma_T \subset \Gamma$ is the cyclic subgroup generated by T [Sun13, Example 1]. Similarly, we see that $1-\alpha \in \overline{\Gamma_{1-T}} \subset \overline{\Gamma}$, where $\Gamma_{1-T} \subset \Gamma$ is the cyclic subgroup generated by $1-T$. Thus we have that $(\alpha, 1-\alpha) \in \phi_3(W)(\overline{\Gamma}) \subset \bigcup_{e\in\mathbb{N}\cup\{0\}}(\phi_3(V)(\overline{\Gamma}))^{p^e}$. Based on these facts, our claim can be proved by either of the following two arguments.

- Example 1 in [Sun13] also shows that $\alpha \notin K^*$, thus $(\alpha, 1-\alpha) \notin \mathbb{A}^2(K^*)$. However, since $\phi_3(V)$ has dimension zero, we have that $\phi_3(V)(\overline{\Gamma}) = \phi_3(V)(\Gamma)$ by [Sun14, Theorem 2]. This leads to

$$(\alpha, 1-\alpha) \in \bigcup_{e\in\mathbb{N}\cup\{0\}} \left(\phi_3(V)(\overline{\Gamma})\right)^{p^e} = \bigcup_{e\in\mathbb{N}\cup\{0\}} \left(\phi_3(V)(\Gamma)\right)^{p^e} \subset \mathbb{A}^2(K^*),$$

a contradiction that proves our claim.

- For each $v \in \Sigma$, note that $\alpha_v \in O_v^*$ and let $P_v(T) \in \mathbb{F}_p[T]$ be the unique irreducible polynomial such that $P_v(T) \in \mathfrak{m}_v$; thus we have that

$$P_v(T^{p^{n!}}) = P_v(T)^{p^{n!}} \in \mathfrak{m}_v^{p^{n!}}$$

for each $n \geq 0$, which implies that $P_v(\alpha_v) \in \cap_{n\geq 0}\mathfrak{m}_v^{p^{n!}} = \{0\}$; it follows that P_v is the minimal polynomial for α_v over $\mathbb{F}_p$. On the other hand, for any $\alpha \in k^{\mathrm{alg}}$ and any $e \in \mathbb{N}\cup\{0\}$, the element α^{p^e} is a zero of the minimal polynomial for α over $\mathbb{F}_p$. As $Z \subset \mathbb{A}^2$ is a zero-dimensional K-variety, it follows that the degrees of minimal polynomials of torsion points in $(\phi_3(V)(\overline{\Gamma_v}))^{p^e}$ are uniformly bounded over all $(v, e) \in \Sigma\times(\mathbb{N}\cup\{0\})$. Because the degree of P_v can be arbitrarily large as v ranges over Σ, there must be some $v_0 \in \Sigma$ such that $(\alpha_{v_0}, 1-\alpha_{v_0}) \notin \bigcup_{e\in\mathbb{N}\cup\{0\}}(\phi_3(V)(\overline{\Gamma_{v_0}}))^{p^e}$; since $\phi_3(V)(\overline{\Gamma}) \subset \prod_{v\in\Sigma}\phi_3(V)(\overline{\Gamma_v})$, this proves our claim.

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