AUTOMORPHISMS OF NORMAL PARTIAL TRANSFORMATION SEMIGROUPS

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1. Introduction. We let X be an arbitrary infinite set. A semigroup S of total or partial transformations of X is called $\mathscr{G}_X$-*normal* if $hSh^{-1}=S$, for all h in $\mathscr{G}_X$, the symmetric group on X. For example, the full transformation semigroup $\mathscr{T}_X$, the semigroup of all partial transformations $\mathscr{P}_X$, the semigroup of all 1–1 partial transformations $\mathscr{I}_X$ and all ideals of $\mathscr{T}_X$, $\mathscr{P}_X$ and $\mathscr{I}_X$ are $\mathscr{G}_X$-normal.

If S is a $\mathscr{G}_X$-normal semigroup then for each $h\in\mathscr{G}_X$ the map

$$\phi: f\mapsto hfh^{-1}\quad (f\in S)$$

is an *inner* automorphism of S. The set $\operatorname{Inn} S$ of all inner automorphisms of S is a subgroup of the group $\operatorname{Aut} S$ of all automorphisms of S. In **[3]** we showed that if S is a $\mathscr{G}_X$-normal subsemigroup of $\mathscr{T}_X$ then inner automorphisms exhaust all automorphisms of S, that is

$$\operatorname{Aut} S=\operatorname{Inn} S.$$

The purpose of this paper is to extend the above result to an arbitrary $\mathscr{G}_X$-normal subsemigroup S of $\mathscr{P}_X$ and therefore to give a complete description of all automorphisms of any $\mathscr{G}_X$-normal semigroup.

Schreier **[10]** in 1937 was the first to show that $\operatorname{Aut}\mathscr{T}_X=\operatorname{Inn}\mathscr{T}_X$. Since then many authors have described the automorphisms of various $\mathscr{G}_X$-normal semigroups: Mal'cev **[5]** (all ideals of $\mathscr{T}_X$); Liber **[4]** ($\mathscr{I}_X$ and all its ideals); Gluskin **[1]** ($\mathscr{P}_X$); Shutov **[8]** (the semigroup of all partial transformations shifting at most a finite number of elements); Shutov **[9]** (all ideals of $\mathscr{P}_X$); Schein **[6, 7]** (all $\mathscr{G}_X$-normal subsemigroups of $\mathscr{I}_X$, but see **[2]** for a special case). In **[11]** Sullivan showed that if S is a subsemigroup of $\mathscr{P}_X$ containing a constant idempotent with the range $\{x\}$, for each $x\in X$, then $\operatorname{Aut} S=\operatorname{Inn} S$. In particular if S is a $\mathscr{G}_X$-normal subsemigroup of $\mathscr{P}_X$ containing a constant map then $\operatorname{Aut} S=\operatorname{Inn} S$. Our result completes the task of characterization of all automorphisms of a $\mathscr{G}_X$-normal semigroup, subsuming previously stated results for $\mathscr{G}_X$-normal semigroups.

In this paper we continue the development of a technique involving the production of certain maximal one-sided ideals, first introduced in **[3]**. Here the assumption (made due to **[3]**) that S contains a proper partial transformation allows us to restrict ourselves to the study of only left ideals. Hence, unlike in **[3]**, a uniform proof is given for the case when $S\subseteq\mathscr{I}_X$ as well as when S contains transformations which are not 1–1.

2. Transitivity. We say that a semigroup S is *trivial* if $S\subseteq\{\Phi,\iota\}$, where Φ is the empty and ι is the identity transformation. In what follows S is non-trivial. The composition of transformations f and g in S defined by the formula

$$fg(x)=f(g(x)),\quad\text{where}\quad x\in X.$$

In this section we show that each non-trivial $\mathscr{G}_X$-normal semigroup S is transitive. If S also is a constant-free semigroup then it is 2-transitive (Definition 2.3).

For an f in $\mathscr{P}_X$ we denote the *range* of f by $R(f)$, the *domain* of f by $D(f)$ and the *partition* of f by $\pi(f)$ $(=\{f^{-1}(x):x\in R(f)\})$. If S is a subsemigroup of $\mathscr{P}_X$, let

$$D(S)=\{D(f):f\in S\} \quad\text{and}\quad \pi(S)=\{\pi(f):f\in S\}.$$

We say that $D(S)$ $(\pi(S))$ is *normal* if, for each $h\in\mathscr{G}_X$,

$$h(D(S))=D(S) \quad (h(\pi(S))=\pi(S)),$$

where $h(D(S))=\{h(A):A\in D(S)\}$, $h(\pi(S))=\{h(\mathscr{A}):\mathscr{A}\in\pi(S)\}$.

The following lemma is straightforward.

Lemma 2.1. *If S is a $\mathscr{G}_X$-normal semigroup, then $D(S)$ and $\pi(S)$ are normal.*

The proof of our next proposition coincides with the proof of result 1.3 of [**3**].

Proposition 2.2. *Every $\mathscr{G}_X$-normal semigroup is transitive.*

Definition 2.3. A semigroup S is 2-*transitive* if for any two ordered subsets $\{x, u\}$ and $\{y, v\}$ of X $(x\neq u,\ y\neq v)$ there exists an f in S with $f(x)=y$, $f(u)=v$.

Lemma 2.4. *If S is a $\mathscr{G}_X$-normal constant-free semigroup then each f in S has an infinite range.*

Proof. Suppose $R(f)$ is finite. Then either $D(f)$ is finite and $\exists g\in S$ with $|D(g)\cap R(f)|=1$ (by 2.1), or $\pi(f)$ contains an infinite subset A and $\exists q\in S$ with $R(f)\subseteq B\in\pi(q)$ (by 2.1). In either case S contains a constant map (gf or qf).

Proposition 2.5. *Every $\mathscr{G}_X$-normal constant-free semigroup S is 2-transitive.*

Proof. Take arbitrary ordered subsets $\{x, u\}$ and $\{y, v\}$ of X, $x\neq u$, $y\neq v$. We construct an f in S such that $f(x)=y$ and $f(u)=v$.

Firstly let x, y, u and v be distinct. Choose t in S with $t(x)=y$ (by 2.2) and let $z\in D(t)\setminus\{x,\ y,\ t^{-1}(x),\ t^{-1}(y)\}$ (if such z does not exist then $R(t)\subseteq\{x,\ y,\ t(y)\}$, a contradiction to 2.4). Let $g=(z, u)t(z, u)$ and $g(u)=(z, u)t(z)=w$ (here (z, u) denotes the permutation of X interchanging z and u and leaving all other elements of X fixed). Clearly $g(x)=y$, and if $w=v$, then $f=g$. If $w\neq v$, u then let $f=(v, w)g(v, w)$ (since $z\notin\{t^{-1}(x),\ t^{-1}(y)\}$, $w\neq x, y$, and this ensures $f(x)=y$).

Thus starting with $t\in S$, $t(x)=y$, we construct either the required f or a map g with $g(x)=y$, $g(u)=u$. Similarly, starting with $s\in S$, $s(u)=v$, we can construct either the required f or a map q with $q(u)=v$, $q(x)=x$. In the latter case we let $f=(u, v)g(u, v)q$.

Now assume that x, y, u and v are not all distinct. Choose a and b in $X\setminus\{x, y, u, v\}$, $a\neq b$, and with the aid of the first part of the proof construct r, $s\in S$ with $r(x)=a$, $r(u)=b$ and $s(a)=y$, $s(b)=v$. Then $f=sr$ is the required map.

3. Left ideals and automorphisms. Let S be a non-trivial $\mathscr{G}_X$-normal constant-free semigroup. If $S\subseteq\mathscr{T}_X$, then $\operatorname{Aut} S=\operatorname{Inn} S$ [**3**]. Hence we assume that S contains a proper partial transformation and show that all automorphisms of S are inner.

DEFINITION 3.1. Given distinct $f, g \in S$ let

$$\mathscr{L}(f, g) = \{l \in S : lf = lg\}.$$

Then $\mathscr{L}(f, g)$ is a left ideal of S, which we call a *function left ideal.*

We will show in 3.12 that there always exist $f, g \in S$ with $\mathscr{L}(f, g) \neq \{\Phi\}$. However, $\mathscr{L}(f, g)$ may consist of the empty map. Let S, for example, be the semigroup of all 1–1, onto transformations f with $|X\backslash D(f)| = |X|$. Choose an f in S. Clearly $X\backslash D(f) \in D(S)$, and so we can choose a g in S with $D(g) = X\backslash D(f)$. Then $\mathscr{L}(f, g) = \{\Phi\}$, because for any $l \in S$, $lf = lg$ implies

$$D(f) \supseteq D(lf) = D(lg) \subseteq D(g) = X\backslash D(f),$$

so $lg = \Phi$. But then $D(l) \cap X = D(l) \cap R(g) = \Phi$, the empty set. Thus $l = \Phi$.

If $\phi \in \operatorname{Aut} S$, then for any $f, g \in S$

$$\phi(\mathscr{L}(f, g)) = \phi(\{l \in S : lf = lg\}) = \{l' \in S : l'\phi(f) = l'\phi(g)\} = \mathscr{L}(\phi(f), \phi(g)).$$

Similar equality holds for $\phi^{-1} \in \operatorname{Aut} S$ and we deduce the following result.

LEMMA 3.2. *Any $\phi \in \operatorname{Aut} S$ permutes function left ideals and $\phi(\mathscr{L}(f, g)) = \mathscr{L}(\phi(f), \phi(g))$.*

Our aim is to translate the definition of $\mathscr{L}(f, g)$ from the language of transformations to the language of subsets of X (Proposition 3.11), and to obtain a bijection of X associated with ϕ, specifically, with the permutation of function left ideals by ϕ.

DEFINITION 3.3. Let $x \in X$ and

$$\mathscr{L}(x) = \{l \in S : x \in X\backslash D(l)\}.$$

Then $\mathscr{L}(x)$ is a left ideal of S, which we call a *point left ideal.*

Notice that since S contains a proper partial transformation, 2.1 ensures that $\mathscr{L}(x) \neq \Phi$, for any $x \in X$.

LEMMA 3.4. *Given $x, y \in X$ the following three statements are equivalent*:

$$\text{(i) } \mathscr{L}(x) \subseteq \mathscr{L}(y); \qquad \text{(ii) } x = y; \qquad \text{(iii) } \mathscr{L}(x) = \mathscr{L}(y).$$

Proof. Implications (ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(i) are trivial. To show (i)$\Rightarrow$(ii) assume $x \neq y$, and choose, by 2.1, an $A \in D(S)$ with $x \in A'$ $(= X\backslash A)$, $y \in A$. If $f \in S$ with $D(f) = A$, then $f \in \mathscr{L}(x)\backslash\mathscr{L}(y)$, proving (i)$\Rightarrow$(ii).

Define a map $\theta : X \to \{\mathscr{L}(x) : x \in X\}$ via $\theta(x) = \mathscr{L}(x)$, for each $x \in X$. Clearly θ is onto and 3.4 ensures θ is 1–1. Hence the next lemma.

LEMMA 3.5. *θ is a bijection.*

Let $\mathscr{P}_2$ be the set of all doubletons $\{a, b\}$ in X, $a \neq b$.

DEFINITION 3.6. Given $A \in \mathscr{P}_2$, $A = \{a, b\}$, let

$$L(A) = \{l \in S : l(a) = l(b)\},$$
$$\mathscr{L}(A) = L(A) \dot{\cup} (\mathscr{L}(a) \cap \mathscr{L}(b)).$$

Then $\mathscr{L}(A)$ is a left ideal of S which we call a *set left ideal.*

REMARK. It is convenient to extend Definitions 3.3 and 3.6 by letting

$$\mathscr{L}(\Phi) = S.$$

Recall that $\pi(S)$ is normal for $\mathscr{G}_X$-normal S (Lemma 2.1). Thus $L(A) = \Phi$ for some $A \in \mathscr{P}_2$ if and only if $L(A) = \Phi$ for all $A \in \mathscr{P}_2$, i.e. if and only if $S \subseteq \mathscr{I}_X$. If $S \subseteq \mathscr{I}_X$ then $\mathscr{L}(A) = \mathscr{L}(a) \cap \mathscr{L}(b)$ $(a, b \in A)$ is a *degenerate set left ideal.* The next lemma reveals that for any $A = \{a, b\} \in \mathscr{P}_2$, $\mathscr{L}(a) \cap \mathscr{L}(b) \neq \Phi$, ensuring that $\mathscr{L}(A) \neq \Phi$.

LEMMA 3.7. *There exists an A in $D(S)$ with $|A'| \geqslant 2$.*

Proof. Choose a proper partial transformation f in S and let $x \in X \backslash D(f)$, $y \in D(f)$, $f(y) = z$. Take g in S with $z \in X \backslash D(g)$ (by 2.1) and let $t = gf$. Then $x, y \in X \backslash D(t)$ and we let $A = D(t)$.

REMARK 3.8. By applying the arguments of the proof of Lemma 3.7 to the map t instead of f it is easy to produce an $A \in D(S)$ with $|A'| \geqslant 3$.

LEMMA 3.9. *Given A and B in $\mathscr{P}_2$, the following three statements are equivalent*:

(i) $\mathscr{L}(A) \subseteq \mathscr{L}(B)$; (ii) $A = B$; (iii) $\mathscr{L}(A) = \mathscr{L}(B)$.

Proof. Implications (ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(i) are trivial. We show (i)$\Rightarrow$(ii). Assume $x \in B \backslash A$ and let $C = (A \cup B) \backslash \{x\}$. Clearly, $|C| \leqslant 3$. Using Remark 3.8 and the normality of $D(S)$ (see 2.1) choose an f in S with $x \in D(f)$ and $C \subseteq X \backslash D(f)$. Then $f \in \mathscr{L}(A) \backslash \mathscr{L}(B)$, so $\mathscr{L}(A) \nsubseteq \mathscr{L}(B)$, proving (i)$\Rightarrow$(ii).

NOTATION 3.10. Given f and g in S, let

$$\Delta(f, g) = f(D(f) \backslash D(g)) \cup g(D(g) \backslash D(f)),$$
$$\mathscr{D}(f, g) = \{\{f(x), g(x)\} : x \in D(f) \cap D(g), f(x) \neq g(x)\}.$$

PROPOSITION 3.11. *Let $f, g \in S$ with $f \neq g$ and $\mathscr{L}(f, g) \neq \{\Phi\}$. Then*

$$\mathscr{L}(f, g) = \Big(\bigcap_{x \in \Delta(f,g)} \mathscr{L}(x)\Big) \cap \Big(\bigcap_{A \in \mathscr{D}(f,g)} \mathscr{L}(A)\Big).$$

Proof. Let $l \in \mathscr{L}(f, g)$, $x \in \Delta(f, g)$ and without loss of generality let $f(y) = x$ for some $y \in D(f) \backslash D(g)$ (Notation 3.10). If $x \in D(l)$, then $lf = lg$ implies that $lf(y) = lg(y)$, and so $y \in D(g)$, a contradiction. Thus $x \notin D(l)$ and

$$l \in \mathscr{L}(x). \tag{1}$$

Now let $A \in \mathscr{D}(f, g)$, $A = \{f(z), g(z)\}$. Then either $l \in \mathscr{L}(f(z)) \cap \mathscr{L}(g(z))$, or $A \cap D(l) \neq \Phi$, and $lf = lg$ implies $lf(z) = lg(z)$, whence $l \in L(A)$. We conclude that

$$l \in \mathscr{L}(A). \tag{2}$$

Since (1) and (2) hold for all $x \in \Delta(f, g)$ and $A \in \mathscr{D}(f, g)$, we deduce that

$$\mathscr{L}(f, g) \subseteq \Big(\bigcap_{x \in \Delta(f,g)} \mathscr{L}(x)\Big) \cap \Big(\bigcap_{A \in \mathscr{D}(f,g)} \mathscr{L}(A)\Big).$$

Conversely, let

$$l \in \Big(\bigcap_{x \in \Delta(f,g)} \mathscr{L}(x)\Big) \cap \Big(\bigcap_{A \in \mathscr{D}(f,g)} \mathscr{L}(A)\Big).$$

Firstly observe that

$$D(lf) = D(lg). \tag{3}$$

Indeed, assume that $z \in D(lf) \backslash D(lg)$. Then $z \in D(g)$ (otherwise $f(z) \in \Delta(f, g)$ and so $l \in \mathscr{L}(f(z))$, implying $z \notin D(lf)$). Now $f(z) \neq g(z)$ means that $\{f(z), g(z)\} = A \in \mathscr{D}(f, g)$, and so $l \in \mathscr{L}(A)$. Since $g(z) \notin D(l)$, we must also have that $f(z) \notin D(l)$, or $z \notin D(lf)$, a contradiction which proves (3).

Now take $z \in D(lf) = D(lg)$. If $f(z) = g(z)$, then certainly $lf(z) = lg(z)$. If $f(z) \neq g(z)$, then $\{f(z), g(z)\} = A \in \mathscr{D}(f, g)$. Since $l \in \mathscr{L}(A)$ and $A \subseteq D(l)$ we conclude that $l \in L(A)$, or $lf(z) = lg(z)$ again. Thus $lf = lg$, or $l \in \mathscr{L}(f, g)$.

Proposition 3.12. *Given an A in $\mathscr{P}_2$ and an x in X there exist f, g, p and q in S such that*

$$\mathscr{L}(A) = \mathscr{L}(f, g), \qquad \mathscr{L}(x) = \mathscr{L}(p, q)$$

and there is a k in S such that $p = kf$, $q = kg$.

Proof. Take an A in $\mathscr{P}_2$. On account of Proposition 3.11 it is sufficient to construct f and g such that $D(f) = D(g)$ (and hence $\Delta(f, g) = \Phi$) and $\mathscr{D}(f, g) = \{A\}$. Choose $t \in S$ with $A \subseteq X \backslash D(t)$ (by 3.7) and let $c, d \in R(t)$, where $c \neq d$ (note that S is constant-free). Let $A = \{a, b\}$ and $s \in S$ take c to a and d to b (see 2.5). Then $f = st$ and $g = (a, b)f(a, b) = (a, b)f$ are the required transformations with $\mathscr{L}(f, g) = \mathscr{L}(A)$.

Now let $x \in X$ and choose $k \in S$ such that $k(a) = x$ and $b \in X \backslash D(k)$. (To construct such k choose by 2.1 a map q in S with $a \in D(q)$ and $b \in X \backslash D(q)$, by 2.2 a map p in S which takes $q(a)$ to x, and let $k = pq$.) It is easy to check that $\mathscr{D}(kf, kg) = \Phi$ and $\Delta(kf, kg) = \{x\}$, whence 3.11 ensures that $\mathscr{L}(kf, kg) = \mathscr{L}(x)$. We let $p = kf$, $q = kg$.

We will show (Proposition 3.14) that each maximal function left ideal of S is either a point left ideal or a non-degenerate set left ideal, and these exhaust all maximal function left ideals.

LEMMA 3.13. *For all A in $\mathscr{P}_2$ and x in X:*

(i) $\mathscr{L}(x) \nsubseteq \mathscr{L}(A)$,
(ii) $\mathscr{L}(A) \subseteq \mathscr{L}(x)$ *implies* $\mathscr{L}(A)$ *is degenerate.*

Proof. (i) Let $A = \{a, b\}$ and assume that $a \neq x$. With the aid of Lemmas 2.1 and 3.7 choose a $B \in D(S)$ with $a \in B$ and $b, x \in B'$, together with $f \in S$ such that $D(f) = B$. Then $f \in \mathscr{L}(x) \backslash \mathscr{L}(A)$.

(ii) If $\mathscr{L}(A) = L(A) \cup (\mathscr{L}(a) \cap \mathscr{L}(b)) \subseteq \mathscr{L}(x)$, then $L(A) \subseteq \mathscr{L}(x)$. Assume $\mathscr{L}(A) \neq \Phi$, then $x \notin A$ and each g such that $A \cup \{x\} \subseteq D(g)$ and $g(a) = g(b)$ (chosen by Lemma 2.1) is in $L(A) \backslash \mathscr{L}(x)$. Thus $L(A) = \Phi$, and so $\mathscr{L}(A)$ is degenerate.

PROPOSITION 3.14. *Let $f, g \in S$. Then $\mathscr{L}(f, g)$ is a maximal function left ideal if and only if either $\mathscr{L}(f, g) = \mathscr{L}(x)$, $x \in X$, or $\mathscr{L}(f, g) = \mathscr{L}(A)$, where $\mathscr{L}(A)$ is non-degenerate, $A \in \mathscr{P}_2$.*

Proof. Firstly, assume that $\mathscr{L}(f, g)$ is a maximal function left ideal. Let $x \in \Delta(f, g)$. By 3.12 there exist $p, q \in S$ such that $\mathscr{L}(p, q) = \mathscr{L}(x)$. Hence $\mathscr{L}(f, g) \subseteq \mathscr{L}(x) = \mathscr{L}(p, q)$ (by 3.11). The maximality of $\mathscr{L}(f, g)$ implies

$$\mathscr{L}(f, g) = \mathscr{L}(x) = \mathscr{L}(p, q).$$

Similarly, if $A \in \mathscr{D}(f, g)$ then there are also $t, s \in S$ with $\mathscr{L}(t, s) = \mathscr{L}(A)$ (by 3.12) and $\mathscr{L}(f, g) \subseteq \mathscr{L}(A) = \mathscr{L}(t, s)$ (by 3.11), implying that

$$\mathscr{L}(f, g) = \mathscr{L}(A) = \mathscr{L}(t, s),$$

because of the maximality of $\mathscr{L}(f, g)$. Suppose $\mathscr{L}(A)$ is degenerate, then for $a \in A$, by 3.4,

$$\mathscr{L}(f, g) = \mathscr{L}(A) \subsetneq \mathscr{L}(a) = \mathscr{L}(l, r),$$

for some $l, r \in S$ (by 3.12), a contradiction to the maximality of $\mathscr{L}(f, g)$.

For the converse, assume that $\mathscr{L}(f, g) = \mathscr{L}(x)$, for some $x \in X$. To show that $\mathscr{L}(f, g)$ is maximal suppose that there are $p, q \in S$ with $\mathscr{L}(p, q) \supseteq \mathscr{L}(f, g)$, that is, by 3.11,

$$\mathscr{L}(x) = \mathscr{L}(f, g) \subseteq \mathscr{L}(p, q) = \Big(\bigcap_{y \in \Delta(p,q)} \mathscr{L}(y)\Big) \cap \Big(\bigcap_{B \in \mathscr{D}(p,q)} \mathscr{L}(B)\Big). \tag{4}$$

If $\mathscr{D}(p, q) \neq \Phi$, then $\mathscr{L}(x) \subseteq \mathscr{L}(B)$, for every $B \in \mathscr{D}(p, q)$, contradicting 3.13(i). Thus $\mathscr{D}(p, q)$ is empty and, for every $y \in \Delta(p, q)$, $\mathscr{L}(x) \subseteq \mathscr{L}(y)$. Lemma 3.4 ensures that $\Delta(p, q) = \{x\}$ and we deduce from (4) that $\mathscr{L}(f, g) = \mathscr{L}(p, q)$.

Finally assume that $\mathscr{L}(f, g) = \mathscr{L}(A)$, $A \in \mathscr{P}_2$, and $\mathscr{L}(A)$ is non-degenerate. If $\mathscr{L}(f, g) \subseteq \mathscr{L}(t, s)$ for $t, s \in S$, then 3.11 implies

$$\mathscr{L}(A) = \mathscr{L}(f, g) \subseteq \mathscr{L}(t, s) = \Big(\bigcap_{z \in \Delta(t,s)} \mathscr{L}(z)\Big) \cap \Big(\bigcap_{C \in \mathscr{D}(p,q)} \mathscr{L}(C)\Big). \tag{5}$$

If $\Delta(t, s) \neq \Phi$, then $\mathscr{L}(A) \subseteq \mathscr{L}(z)$, for each $z \in \Delta(t, s)$, contradicting 3.13(ii). Hence

$\Delta(t, s) = \Phi$ and, for each $C \in \mathscr{D}(p, q)$, $\mathscr{L}(A) \subseteq \mathscr{L}(C)$. Thus $\mathscr{D}(p, q) = \{A\}$ (3.9) and we deduce from (5) that $\mathscr{L}(f, g) = \mathscr{L}(t, s)$.

It is clear from 3.2 that each automorphism ϕ of S permutes maximal function left ideals. Our aim is to show that ϕ also permutes point left ideals. If all the set left ideals are degenerate, that is $S \subseteq \mathscr{I}_X$, then, as the above proposition reveals, the point left ideals are the only maximal function left ideals. In the next proposition we formulate a property which distinguishes the non-degenerate set left ideals and is preserved under ϕ.

PROPOSITION 3.15. *Let $S \nsubseteq \mathscr{I}_X$ and $\mathscr{L}(f, g)$ be a maximal function left ideal. Then $\mathscr{L}(f, g)$ is a set left ideal if and only if*

$$\forall \text{ maximal function left ideal } L\ \exists k \in S \text{ such that } \mathscr{L}(kf, kg) = L. \tag{6}$$

Proof. Assume firstly that $\mathscr{L}(f, g) = \mathscr{L}(A)$ (non-degenerate), $A = \{a, b\} \in \mathscr{P}_2$. We show that (6) holds. If $L = \mathscr{L}(x)$, for some $x \in X$, then we appeal to Lemma 3.12. Hence assume $L = \mathscr{L}(B)$, for some $B \in \mathscr{P}_2$. Choose k in S mapping A onto B (by 2.5). Then $D(kf) = D(kg)$ and so $\Delta(kf, kg) = \Phi$. (Indeed, assume, for example, that $u \in D(kf)\backslash D(kg)$. Then $u \in D(f) = D(g)$, since $\Delta(f, g) = \Phi$, by 3.11 and 3.13(ii), $f(u) \in D(k)$ and $g(u) \notin D(k)$. Thus $f(u) \neq g(u)$, so that by Lemma 3.9 $\{f(u), g(u)\} = A \subseteq D(k)$, a contradiction.) Also, $\mathscr{D}(kf, kg) = \{B\}$, since $kf(u) \neq kg(u)$, for some $u \in D(kf)$, implies that $f(u) \neq g(u)$, or $\{f(u), g(u)\} = A$, again by 3.9, and so by the choice of k, $\{kf(u), kg(u)\} = B$. Proposition 3.11 ensures that $\mathscr{L}(kf, kg) = \mathscr{L}(B)$, proving (6).

For the converse, assume that $\mathscr{L}(f, g)$ satisfies (6) and is a point left ideal $\mathscr{L}(x)$ (Proposition 3.14). Let $L = \mathscr{L}(A)$, $A \in \mathscr{P}_2$, be a non-degenerate set left ideal (recall, $S \nsubseteq \mathscr{I}_x$), and $k \in S$ be such that $\mathscr{L}(kf, kg) = \mathscr{L}(A)$. Then by 3.11 and 3.13(ii), $\Delta(kf, kg) = \Phi$, that is $D(kf) = D(kg)$. Since $\mathscr{L}(fg) = \mathscr{L}(x)$, it follows from 3.11 and 3.13(i) that $\Delta(f, g) \neq \Phi$. Assume without loss of generality that $x = f(y)$, where $y \in D(f)\backslash D(g)$. If $x \in D(k)$, then $y \in D(kf) = D(kg) \subseteq D(g)$, a contradiction. Hence $x \notin D(k)$ and so $k \in \mathscr{L}(x)$, which means that $kf = kg$, a contradiction to the assumption that $\mathscr{L}(kf, kg) = \mathscr{L}(A)$.

PROPOSITION 3.16. *Let $\phi \in$ Aut S. Given $x \in X$ there exists $y \in X$ such that $\phi(\mathscr{L}(x)) = \mathscr{L}(y)$.*

Proof. Let $x \in X$ and choose $f, g \in S$ with $\mathscr{L}(f, g) = \mathscr{L}(x)$ (by 3.12). Proposition 3.14 ensures that $\mathscr{L}(f, g)$ is a maximal function left ideal. Whence

$$\phi(\mathscr{L}(x)) = \phi(\mathscr{L}(f, g)) = \mathscr{L}(\phi(f), \phi(g)) \quad \text{(by 3.2)}$$

is a maximal function left ideal. If S contains only degenerate set left ideals then $\mathscr{L}(\phi(f), \phi(g)) = \mathscr{L}(y)$ as required. Hence assume that there are non-degenerate set left ideals. Since $\mathscr{L}(f, g) = \mathscr{L}(x)$, by 3.15 there exists a maximal function left ideal L such that for any $k \in S$, $\mathscr{L}(kf, kg) \neq L$, or for any $k' \in S$, $\mathscr{L}(k'\phi(f), k'\phi(g)) \neq \phi(L)$. With the aid of 3.2 we deduce that $\phi(L)$ is a maximal function left ideal. Then 3.15 ensures that $\mathscr{L}(\phi(f), \phi(g)) = \mathscr{L}(y)$, for some $y \in X$.

Using the above proposition define a map

$$\eta:\{\mathscr{L}(x):x\in X\}\to\{\mathscr{L}(x):x\in X\}\quad\text{via}\quad \eta(\mathscr{L}(x))=\phi(\mathscr{L}(x)),$$

for each $\mathscr{L}(x)$. Similarly, by considering the automorphism ϕ^{-1}, define a map

$$\xi:\{\mathscr{L}(x):x\in X\}\to\{\mathscr{L}(x):x\in X\}\quad\text{via}\quad \xi(\mathscr{L}(x))=\phi^{-1}(\mathscr{L}(x)).$$

Certainly ξ is the inverse of η and so we have proved the following.

LEMMA 3.17. *η is a bijection.*

By Lemma 3.4, $\mathscr{L}(x)=\mathscr{L}(y)$ if and only if $x=y$ $(x, y\in X)$. We can therefore now define a map $h:X\to X$ by $h(x)=y$, where y is given by $\eta(\mathscr{L}(x))=\mathscr{L}(y)$, for $x\in X$. Thus, with the notation of 3.5,

$$h=\theta^{-1}\eta\theta.$$

By 3.17, h is a bijection; that is, $h\in\mathscr{G}_X$. We call h the *bijection associated with ϕ.*

Now we will prove the main result of this paper.

THEOREM 3.18. *If S is a $\mathscr{G}_X$-normal subsemigroup of $\mathscr{P}_X$, then* $\operatorname{Aut} S=\operatorname{Inn} S$.

Proof. If S consists of total transformations we appeal to [**3**, Theorem 1.1]. If S contains a constant map, the result is given in [**11**, Theorem 2]. Thus we assume that S is a constant-free semigroup containing a proper partial transformation, and so $\mathscr{L}(x)\neq\Phi$ for every $x\in X$.

Take $f\in S$, $x\in D(f)$ and let $f(x)=y$. Since $f\notin\mathscr{L}(x)$, also $\phi(f)\notin\eta(\mathscr{L}(x))=\mathscr{L}(h(x))$, where h is the bijection associated with ϕ. Hence $h(x)\in D(\phi(f))$.

Now observe that for any k in $\mathscr{L}(y)$, $kf\in\mathscr{L}(x)$, hence for any k' in $\mathscr{L}(h(y))$, $k'\phi(f)\in\mathscr{L}(h(x))$. Let $\phi(f)h(x)=z$. If $z\neq h(y)$, we can always choose k' in $\mathscr{L}(h(y))$ with $z\in D(k')$ (Lemma 2.1). But then $k'\phi(f)\notin\mathscr{L}(h(x))$, a contradiction which shows that $z=h(y)$. Thus

$$\phi(f)h(x)=h(y)=hf(x).$$

Since this is true for all x in $D(f)$, we conclude that

$$\phi(f)=hfh^{-1},$$

and, since f is an arbitrary element of S, the result follows.

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