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DECREASE OF BOUNDED HOLOMORPHIC FUNCTIONS ALONG DISCRETE SETS

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Abstract We provide uniqueness results for holomorphic functions in the Nevanlinna class which bridge those previously obtained by Hayman and by Lyubarskii and Seip. In particular, we propose certain classes of hyperbolically separated sequences in the disc, in terms of the rate of non-tangential accumulation to the boundary (the outer limits of this spectrum of classes being, respectively, the sequences with a non-tangential cluster set of positive measure, and the sequences satisfying the Blaschke condition). For each of those classes, we give a critical condition of radial decrease on the modulus which will force a Nevanlinna class function to vanish identically.

Keywords: uniqueness theorems; essential minorants; positive harmonic functions; maximal functions

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1. Definitions and previous results

Let $\mathbb{D}$ be the unit disc in the complex plane. We are interested in the allowable decrease of the modulus of a non-trivial bounded holomorphic function f along a discrete sequence $\{a_k\}$ in $\mathbb{D}$. Notice that these problems are actually the same if we replace the class of bounded holomorphic functions, H^∞, by the Nevanlinna class N, since on the one hand $H^\infty \subset N$ and on the other hand, for any f in N, we can write $f = f_1/f_2$, where $|f_2(z)| \leqslant 1$ whenever $z \in \mathbb{D}$, thus $|f_1(z)| \leqslant |f(z)|$, and $f_1 \in H^\infty$. This also means in particular that any Hardy space H^p could be substituted for H^∞. On the other hand, it seems clear that the situation in Bergman spaces has to be quite different (see [**2**] for related results). Previous results about these problems can be found in [**7**], [**3**], [**6**], [**8**] and recent work by Eiderman and Essén [**4**]. Results in a similar spirit about positive harmonic functions can be found in [**1**].

We now proceed to describe the results in [**6**] and [**8**] that served as an inspiration for the present work.

Let $\Gamma_\alpha(\mathrm{e}^{\mathrm{i}\theta})$ stand for the Stolz angle of aperture $\alpha > 0$ with its vertex at the point $\mathrm{e}^{\mathrm{i}\theta}$ on the circle:

$$\Gamma_\alpha(\mathrm{e}^{\mathrm{i}\theta}) := \{\zeta \in \mathbb{D} : |1 - \zeta \mathrm{e}^{-\mathrm{i}\theta}| < (1+\alpha)(1 - |\zeta|)\}.$$

To simplify notation, we sometimes use the single lowercase letters a, b to stand for the sequences $\{a_k\}, \{b_k\}, \dots$. We denote by $\mathrm{NT}(a)$ the set of points on the circle which are non-tangential limit points of the sequence $\{a_k\}$:

$$\mathrm{NT}(a) := \{\mathrm{e}^{\mathrm{i}\theta} \in \partial\mathbb{D} : \exists \alpha > 0 : \mathrm{e}^{\mathrm{i}\theta} \in \overline{\Gamma_\alpha(\mathrm{e}^{\mathrm{i}\theta}) \cap \{a_k\}}\}. \tag{1.1}$$

A first result concerning decrease of bounded holomorphic functions is given, along with more elaborate theorems, in [**6**].

Theorem A. *Any function f in H^∞ such that $\lim_{k\to\infty} f(a_k) = 0$ must vanish identically if and only if $|\mathrm{NT}(a)| > 0$.*

When $\{a_k\}$ is a Blaschke sequence, that is, when it satisfies the Blaschke condition

$$\sum_{k=1}^{\infty} (1 - |a_k|) < \infty,$$

there exists a non-trivial bounded holomorphic function vanishing exactly on the sequence. It is easy to prove directly that a Blaschke sequence a verifies $|\mathrm{NT}(a)| = 0$ (see below). Thus questions of decrease only make sense for non-Blaschke sequences, and we are especially interested in the intermediate case of non-Blaschke sequence with $|\mathrm{NT}(a)| = 0$.

It is often useful to restrict attention to sequences which are hyperbolically *separated*.

Definition 1.1. The Gleason (or pseudo-hyperbolic) distance is given by

$$d_{\mathrm{G}}(z, w) := \left| \frac{z - w}{1 - \bar{w} z} \right|.$$

A sequence $\{a_k\}$ is called *separated* if and only if $\inf_{j \neq k} d_{\mathrm{G}}(a_k, a_j) > 0$, i.e.

$$\delta := \inf_{j \neq k} \left| \frac{a_k - a_j}{1 - \bar{a}_k a_j} \right| > 0.$$

The number δ is sometimes called the *separation constant*.

Following the idea of [**8**], given a class of sequences $\mathcal{S}$ in the disc, we give the following definition.

Definition 1.2. A non-increasing function g from $[0, 1)$ to $(0, 1]$, tending to 0 as x tends to 1, is an $(H^\infty$-$)$*essential minorant* for $\mathcal{S}$ if and only if given any sequence $a = \{a_k\}$ in $\mathcal{S}$, any bounded holomorphic function f verifying $|f(a_k)| \leqslant g(|a_k|)$ for all k must vanish identically.

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For instance, a consequence of the 'if' part of Theorem A is that any function g such that $\lim_{x\to 1} g(x) = 0$ is an essential minorant for the class of sequences a such that $|\mathrm{NT}(a)| > 0$.

The following theorem is is the main result of [**8**].

Theorem B. *A non-increasing function g is an H^∞-essential minorant for the separated non-Blaschke sequences if and only if*

$$\int_0^1 \frac{\mathrm{d}r}{(1-r)\log(1/g(r))} < \infty. \tag{1.2}$$

This theorem reveals the following remarkable fact: given a function f in H^∞, when one measures the decrease of $|f|$ outside of a fixed hyperbolic neighbourhood of the zero set of f, it cannot go below a certain critical velocity given by condition (1.2).

We introduce a transformation of the function g measuring the decrease, which is related to the fact that we will work with the (sub-)harmonic function $\log|f|$, and that the sequences are separated.

Definition 1.3. For any non-increasing function g from $[0,1)$ to $(0,1]$, tending to 0 as x tends to 1, we write

$$\tilde{g}(\lambda) := \log \frac{1}{g(1-2^{-\lambda})}, \quad \lambda \geqslant 0.$$

It is then elementary to see that condition (1.2) in Theorem B can be restated as

$$\sum_0^\infty \frac{1}{\tilde{g}(n)} < \infty. \tag{1.3}$$

2. The main theorem

We aim to bridge the gap between Theorems A and B by introducing classes of sequences which mediate between those involved in each of those results.

Notice first that one could define a set $\mathrm{NT}_\alpha(a)$ by fixing the number α in (1.1). Two such choices would differ from each other (and thus from the whole $\mathrm{NT}(a)$) only by a set of measure 0 [**10**]. Hence we will neglect sets of linear measure 0, fix a value of α, and write $\mathrm{NT}(a)$ for what is really $\mathrm{NT}_\alpha(a)$.

In order to measure the density of a sequence $a = \{a_k\}$ as it accumulates to the unit circle, we will consider the following function on the circle:

$$\phi_a(\mathrm{e}^{\mathrm{i}\theta}) := \#\{a_k\} \cap \Gamma_\alpha(\mathrm{e}^{\mathrm{i}\theta}).$$

This is the same function that was denoted by Γ_γ in [**9**, Equation (0.1)]. One easily sees that $\{a_k\}$ satisfies the Blaschke condition if and only if $\phi_a \in L^1(\partial\mathbb{D})$; and that the set $\mathrm{NT}(a)$ is the set of $\mathrm{e}^{\mathrm{i}\theta}$ such that $\phi_a(\mathrm{e}^{\mathrm{i}\theta}) = \infty$ [**9**, **10**]. Thus if a is Blaschke, $|\mathrm{NT}(a)| = 0$.

Because of the rotation invariance of the essential minorants we are considering, most of the relevant information about ϕ_a is given by

$$m_a(n) := |\{\zeta \in \partial\mathbb{D} : \phi_a(\zeta) \geqslant n\}|, \quad n \in \mathbb{Z}_+.$$

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For instance,

$$\int_{\partial\mathbb{D}} \phi_a = \sum_{n\geqslant 1} m_a(n) \quad \text{and} \quad |\{\phi_a = \infty\}| = \lim_{n\to\infty} m_a(n) = \inf_n m_a(n) = |\mathrm{NT}(a)|.$$

We will now introduce classes of non-Blaschke sequences, the size of which is measured by the behaviour of their associated function ϕ_a.

Definition 2.1. Let $\{v_n\}$, $\{w_n\}$ be non-increasing bounded sequences with non-negative values such that $\sum_0^\infty v_n = \infty$, $\sum_0^\infty w_n = \infty$. Define

$$\mathcal{S}_w := \biggl\{a \text{ separated} : \sum_0^\infty m_a(n) w_n = \infty\biggr\},$$

and

$$\mathcal{L}_v := \Bigl\{a \text{ separated} : \liminf_{n\to\infty} m_a(n)/v_n > 0\Bigr\}.$$

Clearly, if there is a $C > 0$ such that $w \leqslant Cw'$, then $\mathcal{S}_w \subset \mathcal{S}_{w'}$ and $\mathcal{L}_{w'} \subset \mathcal{L}_w$. Note that the class of all separated non-Blaschke sequences coincides with $\mathcal{S}_1$, where 1 stands for the constant sequence $w_n = 1$ for all n, and the set of separated sequences with $|\mathrm{NT}(a)| > 0$ coincides with $\mathcal{L}_1$.

Also, for any v, w satisfying the conditions above, $\mathcal{L}_1 \subset \mathcal{L}_v \subset \mathcal{S}_1$, and $\mathcal{L}_1 \subset \mathcal{S}_w \subset \mathcal{S}_1$, in particular, the classes we have defined are never empty and never contain any Blaschke sequence. Finally, it is also easy to see that

$$\mathcal{L}_1 = \bigcap \mathcal{L}_v = \bigcap \mathcal{S}_w \quad \text{and} \quad \mathcal{S}_1 = \bigcup \mathcal{L}_v = \bigcup \mathcal{S}_w.$$

More generally, we have the following properties.

(1) $\mathcal{L}_v \subset \mathcal{S}_w$ if and only if $\sum_0^\infty v_n w_n = \infty$.

(2) $\mathcal{S}_w = \bigcup\{\mathcal{L}_v : \sum_0^\infty v_n w_n = \infty\}$.

In addition to the extreme points, mentioned above, of the range of classes we have just defined, it might be useful for the reader to have in mind a few more examples of sequences which are or are not in a given $\mathcal{S}_w$ or $\mathcal{L}_v$. Let $\{v_n\}$ be a non-increasing sequence of positive numbers, as above, such that $v_n \geqslant 2^{-n}$ for any n. Put $Z(v) := \{z_{n,k}\}$,

$$z_{n,k} := (1 - 2^{-n}) \exp(2\pi \mathrm{i} k 2^{-n}), \quad 0 \leqslant n, \quad 0 \leqslant k < [2^n v_n],$$

where $[\cdot]$ stands for the integer part of a real number. Then $m_{Z(v)}(n) \asymp v_n$, so $Z(v)$ is an example of a sequence in $\mathcal{L}_v$ (with a single cluster point on the circle). Note that other examples of sequences in $\mathcal{L}_v$ for a given v, which cluster on Cantor sets, are constructed in Lemma 6.1 below.

The sequence $Z(v)$ will lie in $\mathcal{S}_w$ if and only if $\sum v_n w_n = \infty$. For instance, if $v_n = (\log n)^{-1}$, $n \geqslant 2$, then $Z(v) \in \mathcal{S}_{\{1/n\}}$. More generally, this family of examples has been studied in [**9**]. The motivation there was to study *thin* sequences, i.e. sequences $\{a_k\}$ such that there exists a non-trivial bounded holomorphic function f such that

$$\sum_k (1-|a_k|)|f(a_k)| < \infty.$$

It is proved in [**9**] that a sequence of the type $Z(v)$ is thin if and only if $\sum_n (1/n)v_n < \infty$, that is, if and only if $Z(v) \notin \mathcal{S}_{\{1/n\}}$.

Our main result is the following theorem.

Theorem 2.2.

(1) *g is an essential minorant for the class $\mathcal{S}_w$ if and only if*

$$\sum_0^\infty \frac{w_n}{\tilde{g}(n)} < \infty.$$

(2) *g is an essential minorant for the class $\mathcal{L}_v$ if and only if for any subset E of $\mathbb{Z}_+$ such that $\sum_{n\in E} v_n < \infty$, and for any positive integer C,*

$$\limsup_{n\notin E} \tilde{g}\left(\left[\frac{n}{C}\right]\right) v_n = \infty.$$

Here $[x]$ denotes, as usual, the largest integer which is smaller than or equal to the real number x.

Part (1) with $w_n = 1$ gives Theorem B [**8**]. On the other hand, part (2) shows that given any g satisfying the hypotheses in Definition 1.2, we can find a class of sequences strictly greater than $\mathcal{L}_1$ such that g is still an essential minorant. This sharpens Theorem A (for separated sequences only).

To take into account zero-sets of bounded analytic functions, we will have to introduce the following modified versions of the classes $\mathcal{L}_v$.

Definition 2.3.

$$\mathcal{L}'_v := \{a : \exists b,\ \text{a Blaschke sequence, such that } a \cup b \in \mathcal{L}_v\}.$$

Note that this makes the class stable under removal of a Blaschke sequence: if $a \in \mathcal{L}'_v$ and b is a Blaschke sequence, then $a \setminus b \in \mathcal{L}'_v$. In many cases, this makes no difference.

Lemma 2.4. *If $\{v_n\}$ verifies that there exists $\eta_1 > 1$, $\eta_2 > 0$ such that $v_{[\eta_1 n]} \geqslant \eta_2 v_n$ (which implies $v_n \geqslant (C/n)$ for all $n \geqslant 1$), then $\mathcal{L}_v = \mathcal{L}'_v$.*

When this is the case, the unpleasantly complicated statement of Theorem 2.2 (2) can be simplified.

Corollary 2.5. *Under the hypotheses of Lemma 2.4, g is an essential minorant for the class $\mathcal{L}_v$ if and only if*

$$\limsup_{n\to\infty} \tilde{g}(n) v_n = \infty.$$

3. Reduction to the zero-free case

It will be useful to have some preliminary result on the stability of the classes of sequences we have defined above.

Lemma 3.1. *Suppose that $a, b \notin \mathcal{S}_w$. Then $a \cup b \notin \mathcal{S}_w$.*
In particular, suppose we remove a Blaschke sequence b from $a \in \mathcal{S}_w$. Then $a \setminus b \in \mathcal{S}_w$.

Proof. We have $\phi_{a\cup b} = \phi_a + \phi_b$, so

$$\{\zeta \in \partial\mathbb{D} : \phi_{a\cup b}(\zeta) \geqslant \lambda\} \subset \{\zeta \in \partial\mathbb{D} : \phi_a(\zeta) \geqslant \tfrac{1}{2}\lambda\} \cup \{\zeta \in \partial\mathbb{D} : \phi_a(\zeta) \geqslant \tfrac{1}{2}\lambda\},$$

therefore $m_{a\cup b}(n) \leqslant m_a([n/2]) + m_b([n/2])$, and by the monotonicity of w,

$$\begin{aligned}\sum_0^\infty m_{a\cup b}(n) w_n &\leqslant \sum_0^\infty m_a([n/2]) w_{[n/2]} + \sum_0^\infty m_b([n/2]) w_{[n/2]} \\ &= 2\sum_0^\infty m_a(n) w_n + 2\sum_0^\infty m_b(n) w_n < \infty.\end{aligned}$$

□

The next step is to reduce our problem to one about positive harmonic functions. There is no loss of generality in assuming that the function g is always bounded above by 1.

Lemma 3.2. *A function g is an essential minorant for the class $\mathcal{S}_w$ (respectively, $\mathcal{L}_v$) if and only if for any sequence a in $\mathcal{S}_w$ (respectively, $\mathcal{L}'_v$) there exist no (positive) harmonic function h on the disc such that*

$$h(a_k) \geqslant \log \frac{1}{g(|a_k|)}, \quad \textit{whenever } k \in \mathbb{Z}_+.$$

Proof. One direction is clear: if for some a in $\mathcal{S}_w$, a harmonic function h as above exists, then the function $f := \exp(-h - \mathrm{i}\tilde{h})$, where $\tilde{h}$ denotes the Hilbert transform of h, is holomorphic and bounded, and $|f(a_k)| \leqslant g(|a_k|)$, so that g is not an essential minorant for $\mathcal{S}_w$.

If on the other hand $a \in \mathcal{L}'_v$, choose a Blaschke sequence b so that $a \cup b \in \mathcal{L}_v$, denote by B the Blaschke product with zeros on the sequence b, then $f := B\exp(-h - \mathrm{i}\tilde{h})$ will verify $|f(a_k)| \leqslant g(|a_k|)$.

The proof of the converse is essentially the same as that of Lemma 4 in [**9**, p. 123].

Suppose that g is not an essential minorant. Then there exist a' in $\mathcal{S}_w$ (respectively, $\mathcal{L}_v$) and f in $H^\infty \setminus \{0\}$ such that $\|f\|_\infty \leqslant 1$ and $|f(a'_k)| \leqslant g(|a'_k|)$ for all k. Then $f = Bf_1$, where f_1 is zero-free and B is a Blaschke product,

$$B(z) = z^m \prod_k \frac{|b_k|}{b_k} \frac{b_k - z}{1 - \bar{b}_k z},$$

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where $b_k \in \mathbb{D}$, $\sum_k (1-|b_k|) < \infty$. Let $\delta = \inf\{d(a'_k, a'_j) : k \neq j\} > 0$, by the separation condition. First notice that for each k, there is at most one point a'_j such that $d_{\mathrm{G}}(b_k, a'_j) < \delta/2$, so the sequence

$$b' := \{a'_j : d_{\mathrm{G}}(\{b_k\}, a'_j) < \delta/2\}$$

is a Blaschke sequence. Define a sequence $\{a_k\}$ by $a := a' \setminus b'$. If $a' \in \mathcal{L}_v$, by construction $a \in \mathcal{L}'_v$. If $a' \in \mathcal{S}_w$, by Lemma 3.1, $a \in \mathcal{S}_w$.

We now need a harmonic function h_2 in the unit disc such that $h_2(z) \geqslant -\log|B(z)|$ for any z such that $d_{\mathrm{G}}(z, B^{-1}(0)) \geqslant \delta/2$. Then the function $h := -\operatorname{Re}\log f_1 - h_2$ will satisfy the conclusion of Lemma 3.2.

To construct h_2, we first associate with any point z in the disc its 'Privalov shadow':

$$I_z := \left\{\zeta \in \partial\mathbb{D} : \left|\zeta - \frac{z}{|z|}\right| \leqslant (1-|z|^2)\right\}. \tag{3.1}$$

Let χ_{I_z} stand for the characteristic function of I_z. On lines 3–17 on p. 124 of [**9**] it is proved that for an appropriate positive constant c_0, the unique harmonic function with boundary values equal to

$$c_0 \sum_k \chi_{I_{b_k}}(\zeta)$$

will verify the desired property.

Similar (but not identical) estimates have been used in [**6**, § 6, p. 138]. □

4. A dyadic partition

Consider the following partition of $\partial\mathbb{D}$ in dyadic arcs, for any n in $\mathbb{Z}_+$:

$$I_{n,k} := \{\mathrm{e}^{\mathrm{i}\theta} : \theta \in [2\pi k 2^{-n}, 2\pi(k+1)2^{-n})\}, \quad 0 \leqslant k < 2^n.$$

To this we associate the 'dyadic cubes'

$$Q_{n,k} := \{r\mathrm{e}^{\mathrm{i}\theta} : \mathrm{e}^{\mathrm{i}\theta} \in I_{n,k},\ 1 - 2^{-n} \leqslant r < 1 - 2^{-n-1}\}.$$

Recall that d_{G} stands for the Gleason distance in the disc (Definition 1.1). The following property is well known and easy to check:

$$\exists \delta_0 < 1 \quad \text{such that } \forall z_1, z_2 \in Q_{n,k},\ d_{\mathrm{G}}(z_1, z_2) \leqslant \delta_0. \tag{4.1}$$

Whenever we consider a sequence, it will be natural to consider the set $\mathcal{Q}$ of the dyadic cubes which meet the sequence. On the other hand, from Equation (4.1) and the fact that surface area is doubling with respect to the Gleason distance, we deduce that if a sequence $\{a_k\}$ is separated, then there exists an integer $N = N(\delta)$, where δ is the separation constant, so that

$$\#\{a_k\} \cap Q_{n,j} \leqslant N, \quad \text{whenever } n \geqslant 0,\ 0 \leqslant j < 2^n. \tag{4.2}$$

A Stolz angle could be defined as the union of all hyperbolic balls of a fixed size that are centred on a given radius (or that intersect it). Analogous facts hold for dyadic cubes. For any given M in $\mathbb{Z}_+$, let

$$Q_{n,j}^M := \bigcup_{l:j-M\leqslant l\leqslant j+M} Q_{n,l}$$

(where the integers j and l are taken mod 2^n). The proof of the following fact is best left to the reader.

Lemma 4.1. *For any positive number α, there exists $M_1 = M_1(\alpha)$ such that*

$$\Gamma_\alpha(\mathrm{e}^{\mathrm{i}\theta}) \subset \bigcup \{Q_{n,j}^{M_1} : Q_{n,j} \cap \{r\mathrm{e}^{\mathrm{i}\theta},\ 0 \leqslant r < 1\} \neq \emptyset\}.$$

5. Proof of the sufficient condition

We begin with a simple remark. Let F be any non-negative function on the disc. The non-tangential maximal function associated with F is

$$F^*(\mathrm{e}^{\mathrm{i}\theta}) := \sup_{\Gamma_\alpha(\mathrm{e}^{\mathrm{i}\theta})} F.$$

From the fact that if h is a positive harmonic function, then h^* is weak $L^1(\partial\mathbb{D})$ (see, for example, [**5**, Theorem 5.1, p. 28]), we deduce the following lemma.

Lemma 5.1. *If h is harmonic in the disc and $h(z) \geqslant F(z) \geqslant 0$, whenever $z \in \mathbb{D}$, then F^* is weak $L^1(\partial\mathbb{D})$, that is, for any positive number λ, $|\{F^* > \lambda\}| \leqslant C/\lambda$.*

Proof of sufficiency in Theorem 2.2 (1). Suppose that we have $f \in H^\infty \setminus \{0\}$, a function g as in Definition 1.3 such that $\sum_n w_n/\tilde{g}(n) < \infty$ and $a' \in \mathcal{S}_w$ such that $|f(a'_k)| \leqslant g(|a'_k|)$. Then by Lemma 3.2, we may assume that we have in fact $h(a_k) \geqslant \log(1/g(|a_k|))$, with h a harmonic function and a in $\mathcal{S}_w$.

We set $F(z) := \log(1/g(|a_k|)) \geqslant 0$ when $z = a_k$, $F(z) = 0$ otherwise. We now want to relate the sets where F^* is large to those where ϕ_a is large.

Lemma 5.2. *There exists a constant $M \geqslant 1$ (depending only on α and the constant N in the hypotheses above) such that*

$$\{\phi_a \geqslant n\} \subset \{F^* \geqslant \tilde{g}([n/M])\}. \tag{5.1}$$

Proof. We prove the contrapositive inclusion. Suppose that $F^*(\mathrm{e}^{\mathrm{i}\theta}) < \tilde{g}(m)$. Then, for any a_k in $\Gamma_\alpha(\mathrm{e}^{\mathrm{i}\theta})$, $\log(1/g(|a_k|)) < \tilde{g}(m)$, that is to say, $|a_k| < 1 - 2^{-m}$. Then, by Lemma 4.1, we see that $a \cap \Gamma_\alpha(\mathrm{e}^{\mathrm{i}\theta})$ is contained in the union of sets $Q_{n,j_n}^{M_1}$ for $0 \leqslant n < m$, and therefore, because of (4.2),

$$\phi_a(\mathrm{e}^{\mathrm{i}\theta}) = \#(a \cap \Gamma_\alpha(\mathrm{e}^{\mathrm{i}\theta})) \leqslant N(2M_1(\alpha) + 1)m =: M(N, \alpha)m.$$

Choosing m approximately equal to $n/M(N, \alpha)$, we obtain that $\phi_a(\mathrm{e}^{\mathrm{i}\theta}) < n$, therefore, with a slightly modified M,

$$\{F^* < \tilde{g}([n/M])\} \subset \{\phi_a < n\}.$$

□

By Lemma 5.1, which applies because of the assumptions made at the start of the proof of Theorem 2.2 (1), the inclusion (5.1) implies that

$$m_a(n) \leqslant \frac{C}{\tilde{g}([n/M])}. \tag{5.2}$$

Then, by the monotonicity of $\{w_n\}$,

$$\sum_n w_n m_a(n) \leqslant C \sum_n \frac{w_n}{\tilde{g}([n/M])} \leqslant C \sum_n \frac{w_{[n/M]}}{\tilde{g}([n/M])} \leqslant CM \sum_n \frac{w_n}{\tilde{g}(n)} < \infty,$$

which contradicts the fact that $a \in \mathcal{S}_w$. □

Proof of sufficiency in Theorem 2.2 (2). Suppose that we have $f \in H^\infty \setminus \{0\}$, a function g as in Definition 1.3 and $a' \in \mathcal{L}_v$ such that $|f(a'_k)| \leqslant g(|a'_k|)$. Then by Lemma 3.2, we may assume that we have in fact $h(a_k) \geqslant \log(1/g(|a_k|))$, with h a harmonic function and $a \in \mathcal{L}'_v$. Lemma 5.2 and its consequence (5.2) then apply to the separated sequence a and the decreasing function g.

To get a contradiction, we assume that for any positive integer M and any subset E of $\mathbb{Z}_+$ such that

$$\sum_{n \in E} v_n < \infty, \qquad \limsup_{n \notin E} \tilde{g}([n/M]) v_n = \infty.$$

Let b be a Blaschke sequence such that $m_{a \cup b}(n) \geqslant c_0 v_n$ for n large enough. As in the proof of Lemma 3.1, $m_{a\cup b}(n) \leqslant m_a([n/2]) + m_b([n/2])$. Let E be the set of indices n such that $m_b([n/2]) \geqslant \frac{1}{2} m_{a\cup b}(n)$. Since by hypothesis $\sum_n m_b(n) < \infty$, we do have $\sum_{n\in E} v_n < \infty$.

For $n \notin E$,

$$m_a([n/2]) \geqslant \tfrac{1}{2} m_{a\cup b}(n) \geqslant \tfrac{1}{2} c_0 v_n.$$

Therefore, applying (5.2),

$$\tilde{g}\left(\left[\frac{n}{M}\right]\right) \leqslant \frac{C}{m_a([n/2])} \leqslant \frac{C'}{v_n},$$

which contradicts our assumption. □

6. Proof of the necessary condition

Lemma 6.1. *Given a function g as in Definition 1.3 such that $\sum_n 1/\tilde{g}(n) = \infty$, there exist a constant $C > 0$, a function h harmonic in the disc, a non-empty sequence p and a Blaschke sequence b such that*

(1) *for any p_k in p, $h(p_k) \geqslant C \log(1/g(|p_k|))$;*

(2) *for all n, $m_{p\cup b}(n) \geqslant 1/\tilde{g}(n)$.*

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Proof of Lemma 6.1. This proof is patterned after that in [**8**, pp. 51, 52], where instead of Conclusion (2), only the fact that p was not Blaschke was to be obtained.

We define points $p_{n,j}$ associated with the arcs $I_{n,j}$:

$$p_{n,j} := (1 - 2^{-n}) \exp(2^{-n} 2\pi (j + \tfrac{1}{2}) \mathrm{i})$$

(their radial projection is at the centre of the corresponding interval).

We will define the sequence through the choice of a certain subfamily $\mathcal{F}$ of the family of all the $I_{n,j}$. We shall define simultaneously a sequence of probability measures μ_n, supported on the union of all the $I_{n,j}$ for a given index n, with the same uniform density on each of the intervals where it is supported.

First we define a sequence of integers by

$$l_n := \max\{k \in \mathbb{Z}_+ : k \leqslant \log_2 \tilde{g}(n-j) + j,\ 0 \leqslant j \leqslant n\}. \tag{6.1}$$

One verifies that $l_n \leqslant l_{n+1} \leqslant l_n + 1$, and in fact the sequence we have defined is the largest verifying that property together with $l_n \leqslant \log_2 \tilde{g}(n)$, for any non-negative n.

The family $\mathcal{F}_0 := \{I_{n,j} : n \geqslant 0,\ j \in J_n\}$, where J_n is a subset of $\{0, \dots, 2^n - 1\}$, which is defined recursively in the following way: if $l_{n+1} = l_n$, we want

$$\bigcup \{I_{n+1,j} : j \in J_{n+1}\} = \bigcup \{I_{n,j} : j \in J_n\},$$

which is ensured by $J_{n+1} := \{2j, 2j+1 : j \in J_n\}$; if on the contrary $l_{n+1} = l_n + 1$, we only select the first half of each interval at the nth level, i.e. $J_{n+1} := \{2j : j \in J_n\}$ and

$$\Big| \bigcup \{I_{n+1,j} : j \in J_{n+1}\} \Big| = \frac{1}{2} \Big| \bigcup \{I_{n,j} : j \in J_n\} \Big|. \tag{6.2}$$

We will denote $p^0 := \{p_{n,j} : I_{n,j} \in \mathcal{F}_0\}$.

By (6.2), we see that

$$\frac{1}{2\pi} \Big| \bigcup \{I_{n,j} : j \in J_n\} \Big| = 2^{-n} \# J_n = 2^{-l_n}, \tag{6.3}$$

and we thus define the probability measure μ_n as having uniform density $2^{l_n}/2\pi$ on the set $\bigcup \{I_{n,j} : j \in J_n\}$.

Observe that the way we have defined the families J_n implies that

$$\mu_m(I_{n,j}) = \mu_n(I_{n,j}), \quad \forall j \in J_n,\ m \geqslant n,$$

and therefore if we denote by μ a weak limit point of the sequence $\{\mu_n\}$, $\mu(I_{n,j}) = \mu_n(I_{n,j}) = 2^{l_n - n}$.

We let h_0 be the Poisson integral of the measure μ. This is a positive harmonic function which satisfies

$$h_0(p_{n,j}) \geqslant c \frac{\mu_n(I_{n,j})}{|I_{n,j}|} = c 2^{l_n}, \quad \forall I_{n,j} \in \mathcal{F}_0. \tag{6.4}$$

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In order to ensure that our sequence satisfies Conclusion (1), we need to reduce the sequence p^0. First, define a subset of $\mathcal{F}_0$ by $\mathcal{F} := \{I_{n,j} : j \in J_n,\ l_n \geqslant \log_2 \tilde{g}(n) - 1\}$, then define a subset p of p^0 by

$$p := \{p_{n,j} : I_{n,j} \in \mathcal{F}\}, \quad \text{thus } 2^{l_n} \leqslant \tilde{g}(n) \leqslant 2^{l_n+1} \text{ for } p_{n,j} \in p. \tag{6.5}$$

Equations (6.5) and (6.4) imply that for an appropriate positive constant C, $h := Ch_0(p_{n,j}) \geqslant \tilde{g}(n)$, for any $p_{n,j}$ in p, which yields Conclusion (1).

Let $b := p^0 \setminus p$, where p^0 was defined after Equation (6.2). Let us check Conclusion (2).

Since for any (n, j), $j \in J_n$, there exists $j_k \in J_k$, $0 \leqslant k \leqslant n - 1$, such that

$$I_{n,j} \subset I_{n-1,j_{n-1}} \subset \cdots \subset I_{0,j_0},$$

we see that if α (the aperture of the Stolz angles) is larger than an absolute constant, then $\phi_{p^0}(x) \geqslant n$ for any x in $I_{n,j}$. (To deal with smaller values of α, we would have to 'thicken' the sequence by putting N equidistant points in each interval $I_{n,j}$, N depending on α. This has no detrimental effects on the properties that we are interested in. Details are left to the reader.) We then have

$$m_{p^0}(n) \geqslant \left| \bigcup \{I_{n,j} : j \in J_n\} \right| = 2\pi 2^{-l_n} \geqslant \frac{1}{\tilde{g}(n)},$$

by the definition of l_n and (6.3).

Claim 6.2. *The sequence $b = p^0 \setminus p$ is Blaschke.*

Proof. Let $A := \{n : l_n < \log_2 \tilde{g}(n) - 1\}$; then

$$p^0 \setminus p = \{p_{n,j} : j \in J_n,\ n \in A\}.$$

We will show that if $n \neq m$ and $n, m \in A$, then $l_n \neq l_m$. Accepting this, we then have, using (6.3),

$$\sum_{p^0 \setminus p} (1 - |p_k|) = \sum_{n \in A} 2^{-n} \# J_n = \frac{1}{2\pi} \sum_{n \in A} 2^{-l_n} \leqslant \frac{1}{2\pi} \sum_{k} 2^{-k} < \infty, \quad \text{QED.}$$

To prove our first assertion: if $n \in A$, then

$$l_n + 1 < \log_2 \tilde{g}(n) \leqslant \log_2 \tilde{g}(n + 1),$$

and (6.1) implies that, for any n and $0 \leqslant j \leqslant n$,

$$l_n + 1 \leqslant \log_2 \tilde{g}(j) + (n - j) + 1;$$

therefore, by the definition of l_{n+1} as an upper bound, $l_{n+1} \geqslant l_n + 1$. This implies the desired property. □

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If the sequence p is empty, then by the above $l_{n+1} = l_n + 1$, for any n, so $l_n = n + l_0$, and

$$\sum_n 1/\tilde{g}(n) \leqslant C \sum_n 2^{-n} < \infty,$$

in contradiction to our assumption. □

Proof of necessity in Theorem 2.2 (1). Given a function g so that

$$\sum_0^\infty \frac{w_n}{\tilde{g}(n)} = \infty,$$

consider the sequence p given by Lemma 6.1. Conclusion (2) of the lemma then implies that $p \cup b \in \mathcal{S}_w$, and Lemma 3.1 then yields that $p \in \mathcal{S}_w$. Conclusion (1) and Lemma 3.2 then show that g is not an essential minorant for $\mathcal{S}_w$. □

Proof of necessity in Theorem 2.2 (2). Suppose that g is such that there exists a positive integer C and a subset E of $\mathbb{Z}_+$ such that $\sum_{n\in E} v_n < \infty$ and

$$\sup_{n \notin E} \tilde{g}\left(\left[\frac{n}{C}\right]\right) v_n =: A < \infty.$$

Without loss of generality, we may assume that $v_n \leqslant 1$ for all n, and apply Lemma 6.1 to the unique function g_1 such that g_1 is constant on the intervals of the form $[1 - 2^{-n}, 1 - 2^{-n-1})$ and

$$\tilde{g}_1(m) = \frac{A}{v_{C(m+1)-1}} = \max\left\{\frac{A}{v_n} : \left[\frac{n}{C}\right] = m\right\}.$$

We get sequences p and b such that b is Blaschke and $m_{p\cup b}(n) \geqslant v_{C(n+1)}/A$, and a harmonic function h.

Let $E_1 := \{m = [n/C]$ for some $n \in E\}$. For any $p_{m,j}$ in p, $m \notin E_1$, $h(p_{m,j}) \geqslant \tilde{g}_1(m) \geqslant \tilde{g}(m)$. Define

$$a := \{p_{m,j} : m \in E_1\}, \quad q := p \setminus a.$$

Thus $h(p_{m,j}) \geqslant \tilde{g}(m)$ whenever $p_{m,j} \in q$. We still need to prove that a is Blaschke. For any m such that $p_{m,j} \in p$, by (6.3) and (6.5),

$$\sum_j (1 - |p_{m,j}|) \asymp 2^{-l_m} \asymp \frac{1}{\tilde{g}_1(m)} = \frac{v_{C(m+1)-1}}{A},$$

so that

$$\sum_{p_{m,j}\in a} (1 - |p_{m,j}|) \preceq \sum_{n\in E} v_n < \infty,$$

and a is a Blaschke sequence, therefore $a \cup b$ is Blaschke too.

Let $\tilde{q}$, $\tilde{b}$, $\tilde{a}$, $\tilde{p}$ be the sequences obtained, respectively, from q, b, a, p by adjoining to each point of the sequence a set of M separated points located in a hyperbolic neighbourhood of fixed size so that there exists C' greater than C such that

$$m_{\tilde{p}\cup\tilde{b}}(C'n) \geqslant m_{p\cup b}(n) \geqslant v_{C(n+1)}/A.$$

It is easy to see that $\tilde{a} \cup \tilde{b}$ is a Blaschke sequence and, using Harnack's inequality, that there exists a positive constant C_1 such that $h(\tilde{q}_k) \geqslant C_1 \log(1/g(\tilde{q}_k))$, for any k. Now we want to show that $\tilde{p} \cup \tilde{b} = \tilde{q} \cup (\tilde{a} \cup \tilde{b}) \in \mathcal{L}_v$. Take any m, and an integer n such that $C(n+1) \leqslant m \leqslant C(n+2)$. For m large enough, $m \leqslant C(n+2) < C'n$, so

$$m_{\tilde{p}\cup\tilde{b}}(m) \geqslant m_{\tilde{p}\cup\tilde{b}}(C'n) \geqslant v_{C(n+1)}/A \geqslant v_m/A,$$

therefore $q \cup (\tilde{a} \cup \tilde{b}) \in \mathcal{L}_v$, and since $\tilde{a} \cup \tilde{b}$ is Blaschke, $\tilde{q} \in \mathcal{L}'_v$. Lemma 3.2 then shows that g is not an essential minorant for $\mathcal{L}_v$. □

7. Comparison with summatory conditions

The classes of sequences we have defined may seem somewhat arbitrary, and perhaps one would like to express results about the decrease of holomorphic functions in terms of classes given by more usual summatory conditions such as

$$\sum_k f(1 - |a_k|) = \infty,$$

where f is a positive increasing function on the interval $(0, 1]$ such that $\lim_{x\to 0} f(x)/x = 0$ (so that the condition is stronger than being non-Blaschke), and $\int_0^1 x^{-2} f(x)\,\mathrm{d}x = \infty$ (so that the condition can be satisfied by some separated sequences).

There is no loss of generality in supposing that f is constant on intervals of the form $(2^{-n-1}, 2^{-n}]$, and if we set

$$w(n) := w_n := \frac{f(2^{-n})}{2^{-n}},$$

then the above conditions become exactly those we have imposed on $\{w_n\}$ in Definition 2.1. Therefore, rather than reasoning in terms of a function f, for a sequence $\{w_n\}$ as in Definition 2.1, we define

$$\mathcal{P}_w := \left\{ a \text{ separated sequence} : \sum_k (1 - |a_k|) w\left(\left[\log_2 \frac{1}{1 - |a_k|}\right]\right) = \infty \right\}. \tag{7.1}$$

Clearly, this class is stable under removal of Blaschke sequences.

Lemma 7.1. *$\mathcal{P}_w \subset \mathcal{S}_w$, but for any v, and w such that $\lim_{n\to\infty} w_n = 0$, $\mathcal{S}_v \not\subset \mathcal{P}_w$.*

On the other hand, it is also possible to determine the essential minorants for the class $\mathcal{P}_w$.

Theorem 7.2. *g is an essential minorant for the class $\mathcal{P}_w$ if and only if*

$$\sum_0^\infty \frac{w_n}{\tilde{g}(n)} < \infty.$$

So the classes $\mathcal{P}_w$ and $\mathcal{S}_w$ have the same essential minorants, while the former is quite a bit narrower than the latter. Since we gain information about the decrease of bounded holomorphic functions every time we can exhibit a sequence of points and a function g which is an H^∞-essential minorant for that sequence, Theorem 2.2, by providing more sequences admitting a given essential minorant, seems a more interesting generalization of [**8**] than Theorem 7.2. Of course, it remains an open problem (and perhaps one which cannot admit any manageable answer) to determine the essential minorants for H^∞ over a given sequence in the disc (rather than a class of sequences).

Proof of Theorem 7.2. The proof follows exactly the arguments in [**8**], so we only sketch it. First, the following 'weighted' version of [**8**, Theorem 2] can be proved in exactly the same way. For a real-valued function u on the disc, set $E_g(u) := \{z \in \mathbb{D} : u(z) > \log(1/g(|z|))\}$. We denote by λ_2 the two-dimensional Lebesgue measure.

Proposition 7.3. *For any g as in Definition 1.3, f as above, if*

$$\int_0^1 \frac{f(1-r)\,\mathrm{d}r}{(1-r)^2 \log(1/g(r))} < \infty,$$

then for any (super)harmonic function u on $\mathbb{D}$,

$$\int_{E_g(u)} \frac{f(1-|z|)\,\mathrm{d}\lambda_2(z)}{(1-|z|)^2} < \infty.$$

Proposition 7.3 implies, as in [**8**, pp. 50, 51], that the given condition is sufficient for g to be an essential minorant for $\mathcal{P}_w$ (we actually have an if and only if statement, but we do not need it right now).

Conversely, if we take g such that

$$\sum_0^\infty \frac{w_n}{\tilde{g}(n)} = \infty,$$

Lemma 6.1 yields a Blaschke sequence b, a separated sequence p and a harmonic function h so that $h(p_k) \geqslant \log(1/g(1-|p_k|))$. We also have, for the sequence $p \cup b$,

$$\sum_{\substack{z\in p\cup b,\\ 1-|z|=2^{-n}}} 1-|z| \geqslant \frac{C}{\tilde{g}(n)},$$

so that

$$\begin{aligned}\sum_n w_n \biggl(\sum_{k:1-|p_k|=2^{-n}} 1-|p_k|\biggr) &\geqslant \sum_n w_n \biggl(\frac{C}{\tilde{g}(n)} - \sum_{1-|b_k|=2^{-n}} 1-|b_k|\biggr)\\ &\geqslant C\sum_0^\infty \frac{w_n}{\tilde{g}(n)} - \sum_k 1-|b_k| = \infty,\end{aligned}$$

and therefore $p \in \mathcal{P}_w$. $\square$

Proof of Lemma 7.1. Let $a \notin \mathcal{S}_w$, a separated. We want to prove that $a \notin \mathcal{P}_w$, that is to bound the sum in (7.1).

Associate with any point z in $\mathbb{D}$ the arc $I_z := \{e^{i\theta} : z \in \Gamma_\alpha(e^{i\theta})\}$, and define

$$W_a(e^{i\theta}) := \sum_k \chi_k w\biggl(\biggl[\log_2 \frac{1}{1-|a_k|}\biggr]\biggr),$$

where χ_k stands for the characteristic function of the arc I_{a_k}. Since $|I_{a_k}| \asymp 1 - |a_k|$, the sum which we are trying to control is bounded by a constant multiple of $\int_{\partial\mathbb{D}} W_a$.

Consider any point $e^{i\theta}$ where $\phi_a(e^{i\theta}) = n$. Then

$$W_a(e^{i\theta}) = \sum_{j=1}^{n} w_{i_j},$$

where $\Gamma_\alpha(e^{i\theta}) \cap a = \{a_{k_1}, \dots, a_{k_n}\}$, and $i_j = [\log_2(1/1 - |a_{k_j}|)]$. As in the proof of Lemma 5.2, we know that in $\Gamma_\alpha(e^{i\theta}) \cap \{2^{-m} \geqslant 1 - |z| > 2^{-m-1}\}$ there can be no more than M points of a; furthermore, the monotonicity of w shows that the sum defining W_a is largest when we take the points a_{k_j} with the smallest possible moduli. We thus get

$$W_a(e^{i\theta}) \leqslant M \sum_{i=1}^{n/M} w_i \leqslant M \sum_{i=1}^{n} w_i.$$

Recalling the definition of $m_a(n)$, we conclude with an integration by parts:

$$\int_{\partial\mathbb{D}} W_a(\theta)\, d\theta \leqslant M \sum_{n\geqslant 1} \biggl(\sum_{i=1}^{n} w_i\biggr)(m_n - m_{n+1}) \preceq \sum_n m_a(n) w_n < \infty.$$

To see that $\mathcal{P}_w$ contains no $\mathcal{S}_v$ when w_n tends to 0, consider a sequence of the form

$$a_{k,j} := (1 - 2^{-n_k}) \exp(2\pi i j 2^{-n_k}), \quad 0 \leqslant j < 2^{n_k}.$$

Picking $\{n_k\}$ an increasing sequence of integers so that $\sum_k w(n_k) < \infty$, we see that $a \notin \mathcal{P}_w$; on the other hand, it is easy to see that $\phi_a(e^{i\theta}) = \infty$ almost everywhere on $\partial\mathbb{D}$ (for an α large enough, and therefore for any α). Thus $a \in \mathcal{S}_v$, for any v such that $\sum_n v_n = \infty$. □

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