

ON SOME COMPLEX SUBMANIFOLDS IN KAEHLER MANIFOLDS

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1. Introduction. The purpose of this paper is to give some conditions for complex submanifolds in a Kaehler manifold of constant holomorphic sectional curvature to be Einstein.

For a complex hypersurface which is Einstein, Smyth [8] has obtained its classification and Chern [2] has proved the corresponding local result. Moreover, Takahashi [9] and Nomizu-Smyth [3] generalized this to a complex hypersurface with parallel Ricci tensor. We shall consider a condition weaker than the requirement that the Ricci tensor be parallel, that is we shall consider a complex submanifold with commuting curvature and Ricci operator, which condition was treated by Bishop-Goldberg [1]. For such a complex submanifold, we shall prove that it is Einstein if the Ricci operator commutes to the second fundamental form (Theorem 1). This condition is satisfied for a complex hypersurface automatically.

We shall also consider a complex submanifold with parallel second fundamental form in a Kaehler manifold of constant holomorphic sectional curvature by using Simons' type formula which was given by Simons [7] and studied by Ogiue [4] for a complex submanifold.

2. Preliminaries. Let \bar{M} be a Kaehler manifold of complex dimension $n + p$ with the structure tensor field J and the Kaehler metric $\langle \cdot, \cdot \rangle$, and let M be an n -dimensional complex submanifold of \bar{M} . The Riemannian metric induced on M is a Kaehler metric, which is denoted by the same $\langle \cdot, \cdot \rangle$ and all metric properties of M refer to this metric. The Kaehler structure of M is written by J as in \bar{M} . By $\bar{\nabla}$, we denote the covariant differentiation in \bar{M} and by ∇ the one in M determined by the induced metric. Then the Gauss-Weingarten formulas are given by

$$\begin{aligned}\bar{\nabla}_X Y &= \nabla_X Y + B(X, Y), & X, Y \in \mathcal{X}(M), \\ \bar{\nabla}_X N &= -A^N(X) + D_X N, & X \in \mathcal{X}(M), \quad N \in \mathcal{X}(M)^\perp\end{aligned}$$

where $\langle B(X, Y), N \rangle = \langle A^N(X), Y \rangle$ and D is the linear connection in the normal bundle $T(M)^\perp$. Both A and B are called the second fundamental form of M . The second fundamental form B is a vector valued bilinear form on each $T_m(M)$ taking values in $T_m(M)^\perp$ and the second fundamental form A is a cross-section of a vector bundle $\text{Hom}(T(M)^\perp, S(M))$ where $S(M)$ is the

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bundle whose fibre at each point is the space of symmetric linear transformations of $T_m(M) \rightarrow T_m(M)$, i.e., for $w \in T_m(M)^\perp, A^w : T_m(M) \rightarrow T_m(M)$.

M is called a totally geodesic submanifold of \bar{M} if its second fundamental form is identically zero. Since any complex submanifold of a Kaehler manifold is minimal, its mean curvature vanishes, i.e., $\sum B(e_i, e_i) = 0$ where e_1, \dots, e_{2n} is a frame for $T_m(M)$.

Let \bar{R} and R denote the curvature tensors of \bar{M} and M respectively. If we assume that \bar{M} is of constant holomorphic sectional curvature c , then we have

$$(2.1) \quad \bar{R}_{X,Y}Z = \frac{1}{4}c(\langle Y, Z \rangle X - \langle X, Z \rangle Y + \langle Z, JY \rangle JX - \langle Z, JX \rangle JY + 2\langle X, JY \rangle JZ),$$

$$(2.2) \quad \bar{R}_{X,Y}Z = R_{X,Y}Z - A^{B(Y,Z)}(X) + A^{B(X,Z)}(Y).$$

Let v_1, \dots, v_{2p} be a frame for $T_m(M)^\perp$, and let $x, y \in T_m(M)$. Then the Ricci tensor S of M is given by

$$(2.3) \quad S(x, y) = \frac{1}{2}(n + 1)c\langle x, y \rangle - \sum_{i=1}^{2p} \langle A^i A^i(x), y \rangle.$$

Here we write A^i instead of A^{v_i} to simplify the presentation. From (2.3), the scalar curvature K of M is represented by

$$(2.4) \quad K = n(n + 1)c - \|A\|^2$$

where $\|A\|$ denotes the length of the second fundamental form.

On the other hand, we have the following relations between the second fundamental form A and the complex structure J :

$$(2.5) \quad A^N J + J A^N = 0 \quad \text{and} \quad A^{JN} - J A^N = 0 \quad \text{for any } N \in \mathcal{X}(M)^\perp.$$

We also have

$$(2.6) \quad S(Jx, Jy) = S(x, y) \quad \text{and} \quad JQ = QJ$$

where Q is the Ricci operator of M defined by $S(x, y) = \langle Qx, y \rangle$.

Next we define operators which we later use. Simons [7] defined the following symmetric, positive semi-definite operators:

$$(2.7) \quad A^\sim = {}^t A \cdot A \quad \text{and} \quad A_\sim = \sum_{i=1}^{2p} \text{ad} A^i \text{ad} A^i.$$

And we define the operator A^* by setting

$$(2.8) \quad A^* = \sum_{i=1}^{2p} (A^i)^2.$$

Clearly A^* is symmetric, positive semi-definite operator. And we have $\text{Tr } A^* = \|A\|^2$ where Tr is the trace of an operator.

3. Complex submanifolds with certain Ricci tensor. Let \bar{M} be a Kaehler manifold of complex dimension $n + p$ and constant holomorphic

sectional curvature c , which will be denoted by $\bar{M}^{n+p}(c)$. Let M be a complex submanifold of \bar{M} of complex dimension n . In this section we consider a condition weaker than the requirement that Q be parallel ($\nabla Q = 0$):

$$(P) \quad R_{X,Y}(Q) = 0,$$

which is equivalent to $R_{X,Y} \cdot Q = Q \cdot R_{X,Y}$ (cf. [1]). This condition is also equivalent to

$$(T) \quad R_{X,Y}(S) = 0.$$

We also consider a condition $QA^j = A^jQ$, that is, Q and A^j are commuting as operators. This condition is satisfied obviously when M is an Einstein manifold and we shall prove that this condition is satisfied for any complex hypersurface in a Kaehler manifold of constant holomorphic sectional curvature.

Let e_1, \dots, e_{2n} be a frame for $T_m(M)$ such that $e_{n+i} = Je_i$, and let v_1, \dots, v_{2p} be a frame for $T_m(M)^\perp$ such that $v_{p+s} = Jv_s$.

THEOREM 1. *Let \bar{M} be a Kaehler manifold of complex dimension $n + p$ ($n > 1$) and constant holomorphic sectional curvature c , and let M be an n -dimensional complex submanifold in \bar{M} with a condition (P). If $c < 0$, then M is an Einstein manifold. If $c > 0$, then M is an Einstein manifold if and only if Q is commuting with A^j ($j = 1, \dots, p$).*

Proof. Let $x, y \in T_m(M)$. By the condition (T), we obtain

$$\sum_{i=1}^{2n} (S(R_{e_i,xe_i}y) + S(e_i, R_{e_i,xy})) = 0.$$

By equation (2.2), this becomes

$$\begin{aligned} \sum_{i=1}^{2n} \{S(\bar{R}_{e_i,xe_i}y) + S(A^{B(x,e_i)}(e_i), y) + S(\bar{R}_{e_i,xy}, e_i) \\ + S(A^{B(x,y)}(e_i), e_i) - S(A^{B(e_i,y)}(x), e_i)\} = 0. \end{aligned}$$

In the following, we calculate this equation. First we have, by (2.1),

$$\sum_{i=1}^{2n} (S(\bar{R}_{e_i,xe_i}y) + S(\bar{R}_{e_i,xy}, e_i)) = \frac{1}{2}nc \left(\frac{K}{2n} \langle x, y \rangle - S(x, y) \right).$$

We obtain $\sum S(A^{B(x,y)}(e_i), e_i) = 0$, by using (2.5) and (2.6), i.e.,

$$-S(A^{B(x,y)}(e_i), e_i) = -S(JA^{B(x,y)}(e_i), Je_i) = S(A^{B(x,y)}(Je_i), Je_i).$$

We have

$$\begin{aligned} \sum_{i=1}^{2n} (S(A^{B(x,e_i)}(e_i), y) - S(A^{B(e_i,y)}(x), e_i)) \\ = \sum_{i=1}^{2n} \sum_{j=1}^{2p} (\langle A^j(e_i), Qy \rangle \langle A^j(x), e_i \rangle - \langle A^j(x), Qe_i \rangle \langle A^j(y), e_i \rangle) \\ = \sum_{j=1}^{2p} (\langle QA^jA^j(x), y \rangle - \langle A^jQA^j(x), y \rangle). \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} \frac{1}{2}nc \left(\frac{K}{2n} \langle x, y \rangle - S(x, y) \right) \\ + \sum_{j=1}^{2p} (\langle QA^jA^j(x), y \rangle - \langle A^jQA^j(x), y \rangle) = 0. \end{aligned}$$

From this, we can see that

$$\begin{aligned} (3.1) \quad nc \left(\frac{K^2}{2n} - \|Q\|^2 \right) &= 2 \sum_{j=1}^{2p} \text{Tr}(QA^jQA^j - QQA^jA^j) \\ &= - \sum_{j=1}^{2p} \|[Q, A^j]\|^2 \leq 0, \end{aligned}$$

where $[Q, A^j] = QA^j - A^jQ$.

On the other hand, we always have $K^2 \leq 2n\|Q\|^2$, and equality holding if and only if M is Einstein. If $c < 0$, then M is an Einstein manifold by (3.1). Let $c > 0$. If $QA^j = A^jQ$, then $QJA^j = JQA^j = JA^jQ$ by using (2.6). Therefore if $QA^j = A^jQ$ ($j = 1, \dots, p$), (3.1) implies that M is Einstein.

COROLLARY 1. *Let M be a complex hypersurface in $\bar{M}^{n+1}(c)$, $c > 0$, $n > 1$. If M satisfies (P), then either M is totally geodesic, or M is a locally symmetric Einstein manifold with scalar curvature $K = n^2c$.*

Proof. Let v, Jv be a frame for $T_m(M)^\perp$. Then we have

$$S(x, y) = \frac{1}{2}(n + 1)c\langle x, y \rangle - 2\langle A^vA^v(x), y \rangle.$$

Hence we have $S(A^v(x), y) = S(x, A^v(y))$, which shows that $QA^v = A^vQ$. By the above theorem, M is an Einstein manifold and we have our assertion by Theorem C of Takahashi [9].

COROLLARY 2. *Let M be a complex hypersurface in $\bar{M}^{n+1}(c)$, $c < 0$, $n > 1$. If M has the property (P), then M is totally geodesic.*

Proof. By Theorem 1, M is Einstein and we have our result by the theorem of Chern [2], or Takahashi [9].

Remark 1. Let M be a complex hypersurface of $\bar{M}(c)$. If the Ricci tensor of M is parallel, then M is Einstein [3; 9]. Our results are the partial generalization of these results.

4. Simons' type formula of complex submanifolds. Let \bar{M} be a Kaehler manifold of complex dimension $n + p$, and let M be an n -dimensional complex submanifold of \bar{M} . We can take a frame e_1, \dots, e_{2n} for $T_m(M)$ such that $e_{n+i} = Je_i$ and a frame v_1, \dots, v_{2p} for $T_m(M)^\perp$ such that $v_{p+s} = Jv_s$.

For the operator A^\sim , $JA^\sim = A^\sim J$ (see Simons [7, p. 76]). By the similar way as in the case of A^* , A^\sim is represented by the matrix form, for a suitable basis,

$$A^\sim \equiv \begin{bmatrix} \mu_1 & & & & 0 \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & & \mu_{2p} \end{bmatrix}, \quad \mu_{p+l} = \mu_l, \quad \mu_l \geq 0.$$

Then we have

$$\langle A \cdot A^\sim, A \rangle = \sum_{i=1}^{2p} \langle A \cdot A^\sim(v_i), A^i \rangle = \sum_{i=1}^{2p} \mu_i \langle A^\sim(v_i), v_i \rangle = \sum_{i=1}^{2p} \mu_i^2.$$

From this we obtain the following:

$$(4.2) \quad \frac{1}{2p} \|A\|^4 \leq \langle A \cdot A^\sim, A \rangle \leq \frac{1}{2} \|A\|^4.$$

LEMMA 1. *Let M be a complex submanifold of complex dimension n ($n > 1$) in $\bar{M}^{n+p}(c)$. Then M is Einstein if and only if $\langle A_\sim \cdot A, A \rangle = (1/n)\|A\|^4$.*

Proof. We have already that

$$\langle A_\sim \cdot A, A \rangle = 2 \sum_{i=1}^{2n} \langle (A^*)^2(e_i), e_i \rangle \quad \text{and} \quad Q = \frac{1}{2}(n+1)cI - A^*.$$

Therefore we obtain

$$\langle A_\sim \cdot A, A \rangle = \frac{1}{n} \|A\|^4 - \frac{1}{n} K^2 + 2\|Q\|^2,$$

where $\|Q\|$ denotes the length of the Ricci operator Q . Generally, $K^2 \leq 2n\|Q\|^2$ and equality holding if and only if M is Einstein. Hence M is Einstein if and only if $\langle A_\sim \cdot A, A \rangle = (1/n)\|A\|^4$.

If \bar{M} is of constant holomorphic sectional curvature c , then the Simons' type formula is given by (see [4] and [7, p. 81])

$$(4.3) \quad \nabla^2 A = \frac{1}{2}(n+2)cA - A \cdot A^\sim - A_\sim \cdot A.$$

Here we notice that if the length of the second fundamental form A is constant, then $\langle \nabla A, \nabla A \rangle = -\langle \nabla^2 A, A \rangle$.

PROPOSITION 2. *Let M be an Einstein complex hypersurface of $\bar{M}^{n+1}(c)$. Then the second fundamental form A of M is parallel ($\nabla A = 0$).*

Proof. By (4.2), (4.3) and Lemma 1, we get

$$\langle \nabla A, \nabla A \rangle = \frac{(n + 2)}{2n} (||A||^2 - nc) ||A||^2.$$

By Corollary 1 and (2.4), we get $||A||^2 = nc$, hence $\nabla A = 0$.

Remark 3. Let M be a complex hypersurface of $\bar{M}(c)$. If A is parallel, then Q is parallel. Hence by Corollary 1, M is Einstein. Consequently, M is Einstein if and only if $\nabla A = 0$. (Compare also Nomizu-Smyth [3, p. 507, Lemma 5].)

THEOREM 2. *Let M be an n -dimensional complex submanifold of $\bar{M}^{n+p}(c)$ with parallel second fundamental form ($\nabla A = 0$). If $c \leq 0$, then M is totally geodesic. If $c > 0$, then the scalar curvature K of M satisfies*

$$K \geq \frac{n^2c(n + p + 1)}{(n + 2p)}$$

and if equality holds, then M is an Einstein manifold.

Proof. From (4.1), (4.2) and (4.3), we get the following inequality:

$$0 = \langle \nabla A, \nabla A \rangle \geq \left(\frac{n + 2p}{2pn} ||A||^2 - \frac{(n + 2)c}{2} \right) ||A||^2.$$

Hence if $c \leq 0$, then M is totally geodesic in \bar{M} .

Let $c > 0$. Then we obtain $||A||^2 \leq pn(n + 2)c/(n + 2p)$. From this and (2.4), we can see that $K \geq n^2c(n + p + 1)/(n + 2p)$. If equality holds, then $||A||^2 = pn(n + 2)c/(n + 2p)$ and hence, by (4.1), (4.2) and (4.3),

$$\left\{ \frac{(n + 2)c}{2} - \frac{1}{2p} ||A||^2 \right\} ||A||^2 \geq \langle A_{\sim} \cdot A, A \rangle \geq \frac{1}{n} ||A||^4,$$

$$\frac{(n + 2)c}{2} - \frac{1}{2p} ||A||^2 = \frac{1}{n} ||A||^2,$$

which imply $\langle A_{\sim} \cdot A, A \rangle = (1/n)||A||^4$. Therefore, by Lemma 1, M is an Einstein manifold.

Remark 4. For an Einstein complex hypersurface, Chern [2] proved that if $c \leq 0$, then M is totally geodesic. By Proposition 2, our theorem is the extension of this. (See also Nomizu-Smyth [3] and Takahashi [9].)

REFERENCES

1. R. L. Bishop and S. I. Goldberg, *On conformally flat spaces with commuting curvature and Ricci transformations*, Can. J. Math. 24 (1972), 799–804.
2. S. S. Chern, *Einstein hypersurfaces in a Kaehler manifold of constant holomorphic curvature*, J. Differential Geometry 1 (1967), 21–31.
3. K. Nomizu and B. Smyth, *Differential geometry of complex hypersurfaces, II*, J. Math. Soc. Japan 20 (1968), 498–521.

4. K. Ogiue, *Complex submanifolds of the complex projective space with second fundamental form of constant length*, Kōdai Math. Sem. Rep. *21* (1969), 252–254.
5. ——— *On Kaehler immersions*, Can. J. Math. *24* (1972), 1178–1182.
6. B. O'Neill, *Isotropic and Kaehler immersions*, Can. J. Math. *17* (1965), 907–917.
7. J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math. *88* (1968), 62–105.
8. B. Smyth, *Differential geometry of complex hypersurfaces*, Ann. of Math. *85* (1967), 246–266.
9. T. Takahashi, *Hypersurface with parallel Ricci tensor in a space of constant holomorphic sectional curvature*, J. Math. Soc. Japan *19* (1967), 199–204.

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