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ON THE NUMBER OF CRITICAL POINTS OF A C^1 FUNCTION ON THE SPHERE

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To my mother, Isa

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Abstract. For a C^1 function $f : \mathbb{R}^n \to \mathbb{R}$ ($n \geq 2$), we consider the least number k of distinct critical points that f must possess when restricted to the sphere $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$. Clearly $k \geq 2$ (for f attains its absolute minimum and maximum on S), and a result of Lusternik and Schnirelmann establishes that $k = n$ if f is even. Here we prove that $k = n$ if, for a given orthonormal system (e_i), $\max\limits_{S\cap V_i} f < \min\limits_{S\cap V_i^\perp} f$, for all $i = 1, \ldots n-1$, where V_i is the subspace spanned by $e_1, \ldots, e_i$ and $V_i^\perp$ its orthogonal complement. It is shown that this criterion is satisfied by suitably restricted perturbations of quadratic forms having n distinct eigenvalues.

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Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function, and let $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$ be the unit sphere in $\mathbb{R}^n$; we denote by (x, y) the usual scalar product of $x, y \in \mathbb{R}^n$ and by $\|x\| = (x, x)^{1/2}$ the euclidian norm. If f is *even*, i.e. $f(-x) = f(x)$ for $x \in \mathbb{R}^n$, then according to a famous result of Lusternik and Schnirelmann [**4**] - reviewed, for example, in the surveys [**7**], [**8**], [**9**] and [**11**] - $f_S \equiv f|_S$ has at least n distinct (pairs of) critical points, i.e. n points $x_i \in S$ at which the derivative $f'_S(x_i) = 0$ or equivalently $\nabla f(x_i) = \lambda_i x_i$, $\lambda_i = (\nabla f(x_i), x_i)$, with ∇f denoting the gradient of f. (Note that critical points occur in antipodal pairs $(x, -x)$ because of the evenness of f.) Moreover if $(e_i)_{1\leq i\leq n}$ is an orthonormal basis of $\mathbb{R}^n$, if $V_i = [e_1, \ldots, e_i]$ denotes the subspace spanned by $e_1, \ldots, e_i$ ($1 \leq i \leq n$) and $V_i^\perp$ the subspace orthogonal to V_i, then the corresponding critical values $c_i = f(x_i)$ satisfy

$$\beta_{i-1} \leq c_i \leq \alpha_i \qquad (1 \leq i \leq n) \tag{1.1}$$

where

$$\alpha_i = \max_{S\cap V_i} f, \qquad \beta_{i-1} = \min_{S\cap V_{i-1}^\perp} f \tag{1.2}$$

with the understanding that $\beta_0 = \min\limits_S f$ (while $\alpha_n = \max\limits_S f$). The estimate (1.1) follows easily by the minimax expression of the c_i, which will be recalled later.

It is natural to ask what can be said for an $f \in C^1$ which is *not* even. In this note we use the same ideas introduced by Lusternik and Schnirelmann, and later extended by Krasnoselskii [**3**], Palais [**6**], Schwartz [**10**], Rabinowitz [**7**, **8**, **9**], to give a simple criterion for the existence of n distinct critical points of a general $f \in C^1$.

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THEOREM 1. *Let* $f \in C^1(\mathbb{R}^n)$, $n > 2$, *let* $(e_i)_{1 \le i \le n}$ *be an orthonormal basis of* $\mathbb{R}^n$, *and let* α_i, β_i *be defined as in (1.2). If* $\alpha_i < \beta_i$ *for* $1 \le i \le n-1$, *then* f_S *has n distinct critical values* c_i $(1 \le i \le n)$, *which moreover satisfy the inequality (1.1).*

We can say this in other terms: if the n subintervals $J_i \equiv [\beta_{i-1}, \alpha_i]$ $\quad (i = 1, \ldots, n)$ of the range $[\beta_0, \alpha_n]$ of f do not overlap, then f has (at least) n critical values, one in each J_i.

Theorem 1 is an easy consequence of the following result.

THEOREM 2. *Let* $f \in C^1(\mathbb{R}^n)$ *and let* V, W *be complementary subspaces of* $\mathbb{R}^n$ *so that* $\mathbb{R}^n = V \oplus W$. *Set*

$$\alpha = \max_{S \cap V} f, \qquad \beta = \min_{S \cap W} f. \tag{1.3}$$

Then if $\alpha < \beta$, f_S *has a critical value* $c \le \alpha$. *Moreover, given any subspace* V_0 *of* V, *we have* $c \ge \theta \equiv \min\limits_{S \cap (V_0 \oplus W)} f$.

Proof. Set

$$c = \inf_{F \in \mathcal{F}} \max_{x \in F} f(x) \tag{1.4}$$

where

$$\mathcal{F} = \{F \subset S \setminus W : F \text{ compact}, \ \mathrm{Cat}(F; S \setminus W) > 1\} \tag{1.5}$$

and $\mathrm{Cat}(F; S \setminus W)$ denotes the Lusternik-Schnirelmann category of the set F in $S \setminus W$. That is to say, $\mathcal{F}$ is the family of all compact subsets of $S \setminus W$ that are not contractible to a point by a continuous deformation *in* $S \setminus W$.

To prove that c is a critical value having the required properties, we first use the following two facts, proofs of which can be found, for example, in [**3**, Chapter 6, Lemma 2.6 and 2.7].

(i) $S \cap V \in \mathcal{F}$;
(ii) If $F \in \mathcal{F}$, then $F \cap (V_0 \oplus W) \neq \emptyset$ for any subspace V_0 of V.

Using (i) and (ii), the estimate $\theta \le c \le \alpha$ follows readily from the definitions. It remains to show that c is critical. We argue by contradiction, using the standard deformation technique due to Palais [**6**], Rabinowitz [**8**, **9**] et al. (though for our purposes, the simpler version given in [**7**] would suffice). Indeed if not, then there would be a (continuous) deformation $H : [0, 1] \times S \to S$ with the following properties:

(a) $f(H(t, x)) \le f(x), \forall (t, x) \in [0, 1] \times S$;
(b) there exists an ϵ with $0 < \epsilon < \beta - \alpha$ such that $f(x) \le c + \epsilon$ implies that $f(H(1, x) \le c - \epsilon$.

Let $F_\epsilon \in \mathcal{F}$ be such that $f(x) < c + \epsilon$ for all $x \in F_\epsilon$; then by (a) we have

$$f(H(t, x)) \le f(x) < c + \epsilon < \beta \qquad ((t, x) \in [0, 1] \times F_\epsilon),$$

which implies that $H(t, x) \notin W$, for all $(t, x) \in [0, 1] \times F_\epsilon$. Therefore H deforms F_ϵ *in* $S \setminus W$ to the set $G_\epsilon \equiv \{H(1, x) : x \in F_\epsilon\}$, and since the category of a set does not decrease under deformations, it follows that $G_\epsilon \in \mathcal{F}$. Also by (b), $\max_{G_\epsilon} f \leq c - \epsilon$ and so $c = \inf_{\mathcal{F}} \max_{F} f \leq c - \epsilon$, a contradiction.

Theorem 1 follows from Theorem 2 on taking, for $2 \leq i \leq n-1$, $V = V_i = [e_1, \ldots, e_i]$, $W = V_i^\perp$ and $V_0 = [e_i]$, so that $V_0 \oplus W = V_{i-1}^\perp$ and $\theta = \beta_{i-1}$. The corresponding critical values c_i are thus defined by

$$c_i = \inf_{F \in \mathcal{F}_i} \max_{x \in F} f(x) \tag{1.6}$$

where

$$\mathcal{F}_i = \{F \subset S \setminus V_i^\perp : F \text{ compact, } \operatorname{Cat}(F; S \setminus V_i^\perp) > 1\}. \tag{1.7}$$

For $i = 1$ and $i = n$, the existence of a critical value in $[\beta_0, \alpha_1]$ and $[\beta_{n-1}, \alpha_n]$, respectively, is trivial since $\beta_0 = \min_S f$ and $\alpha_n = \max_S f$. However, the conditions $\alpha_1 < \beta_1$ and $\alpha_{n-1} < \beta_{n-1}$ ensure that β_0 and α_n "do not mix" with the remaining c_i's. Let us further remark that $\alpha_n = c_n$, with c_n defined by (1.6), (1.7) for $i = n$. Indeed, it is easily seen that $\mathcal{F}_n = \{S\}$; i.e. the only subset of S which is not contractible to a point in S is S itself. We are unable to prove in general the corresponding equality $\beta_0 = c_1$.

Remark 1. More generally, f_S has (at least) $s + 1$ distinct critical values, where s is the total number of indices $i \in \{1, \ldots, n-1\}$ such that $\alpha_i < \beta_i$ (of course, the case $s = 0$ is included). It remains an open question as to what properties (if any) the numbers c_i - which are well-defined anyway by the formulas (1.6),(1.7) - have in general and what happens in particular when $c_i = c_{i+1} = \cdots = c_{i+p}$, for some i and $p > 0$. One obvious remark is that if the stronger condition

$$(\min_{S \cap V_{i-1}^\perp} f =)\ \beta_{i-1} = \alpha_i = \alpha_{i+1} = \cdots = \alpha_{i+p}\ (= \max_{S \cap V_{i+p}} f)$$

holds, then f is constant on $S \cap [e_i, \ldots e_{i+p}]$, and so there is a p-dimensional continuum of critical points of f_S corresponding to this constant value.

Remark 2. Returning to Theorem 1 - whose assumptions imply *a priori* that the c_i's are all distinct - an independent open problem is to see how many critical points of f_S correspond to the same critical value c_i. In analogy with the case in which $f(x) = \frac{1}{2}(Ax, x)$ is the quadratic form associated with a symmetric $n \times n$ matrix A, or more generally the case in which f is an even function, one expects that there are at least two.

Remark 3. In case f is even, f_S possesses at least n distinct (pairs of) critical points. They are associated with the critical values $c_i = \inf_{\mathcal{Z}_i} \max_F f$ where

$$\mathcal{Z}_i = \{F \subset S : F \text{ compact, symmetric, } \gamma(F) \geq \text{i}\},$$

$\gamma(F)$ denoting the *genus* of F; for this matter, see in particular Rabinowitz [**7**, **8**, **9**] and Szulkin [**11**]. (In the original work of Lusternik and Schnirelmann [**4**], the c_i's were defined via the category of sets in the projective space obtained by identifying

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antipodal points of S; that the two approaches are equivalent was proved in [**7**].) The estimate $\beta_{i-1} \le c_i \le \alpha_i \, (i = 1, \ldots, n-1)$ mentioned in the Introduction follows easily from the properties of γ; in particular the following two:

(i) $\gamma(S \cap V) = i$ if V is a subspace of dimension i;
(ii) if $F \in \mathcal{F}_i$ and $\dim W < i$, then $F \cap W^{\perp} \neq \emptyset$.

REMARK 4. A more general version of Theorem 2 is given in [**1**] (see also [**2**]), where it is used to prove the existence of normed eigenfunctions for a class of semi-linear elliptic problems. It was suggested by a result of Krasnoselskii ([**3**, Chapter 6, Theorem 2.2]) and provides a sort of constrained version of Rabinowitz' Saddle Point Theorem ([**9**, Theorem 4.6]; see also [**5**, Theorem 4.7]).

REMARK 5. The above results hold unaltered if S is replaced by $rS = \{x \in \mathbb{R}^n : \|x\| = r\}$ or more generally by a C^1 submanifold of $\mathbb{R}^n$ that is *sphere-like*; i.e. diffeomorphic to S via the radial projection $p(x) = x/\|x\| \; (x \neq 0)$.

One disadvantage of Theorem 1 is that it relates the existence of critical points of f to its behaviour along a given orthonormal system. Sometimes however, the choice of this system is transparent from the problem itself, as shown by the following example.

PPROPOSITION 1. *Let $A = (a_{ij})$ be a symmetric $n \times n$ matrix having n simple eigenvalues $\lambda_1^0 < \lambda_2^0 < \ldots < \lambda_n^0$, and let*

$$d = \min\{\lambda_{i+1}^0 - \lambda_i^0 : 1 \le i \le n-1\}.$$

Also, let $F \in C(\mathbb{R}^n, \mathbb{R}^n)$ be a gradient vector field such that $F(0) = 0$ and, for some $p, q \in \mathbb{R}$,

$$p \le \frac{(F(x), x)}{(x, x)} \le q \qquad (0 < \|x\| \le 1). \tag{H}$$

If $q - p < d$, then $A + F$ has n distinct norm-one eigenvectors; i.e. n points $x_i \in S$ such that

$$Ax_i + F(x_i) = \lambda_i x_i.$$

Proof. We use Theorem 1, taking as orthonormal basis the normalized eigenvectors e_i of A, $Ae_i = \lambda_i^0 e_i$. Then for $i = 1, \ldots, n$ we have

$$\lambda_i^0 = \max_{x \in S \cap V_i} (Ax, x) \quad = \min_{x \in S \cap V_{i-1}^{\perp}} (Ax, x). \tag{1.8}$$

Set $f_0(x) = \frac{1}{2}(Ax, x)$ and $f(x) = f_0(x) + h(x)$, where

$$h(x) = \int_0^1 (F(tx), x)\, dt \tag{1.9}$$

is the primitive of F vanishing at $x = 0$; thus $\nabla f(x) = Ax + F(x)$, for $x \in \mathbb{R}^n$. Then using (1.9) and (H), it follows easily that for $x \in S$ we have

$$\frac{1}{2}p \leq h(x) \leq \frac{1}{2}q. \tag{1.10}$$

Given $1 \leq i \leq n-1$, (1.8) and (1.10) imply that

$$\begin{cases} f(x) \leq \frac{1}{2}(\lambda_i^0 + q), & x \in S \cap V_i, \\ f(x) \geq \frac{1}{2}(\lambda_{i+1}^0 + p), & x \in S \cap V_i^{\perp}, \end{cases}$$

and the result follows, since by assumption $\lambda_i^0 + q < \lambda_{i+1}^0 + p$. It also follows from Theorem 1 that

$$\lambda_i^0 + p \leq c_i = f(x_i) \leq \lambda_i^0 + q \qquad (1 \leq i \leq n). \tag{1.11}$$

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