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A DETERMINANT FORMULA FOR RELATIVE CONGRUENCE ZETA FUNCTIONS FOR CYCLOTOMIC FUNCTION FIELDS

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Abstract

Rosen gave a determinant formula for relative class numbers for cyclotomic function fields, which may be regarded as an analogue of the classical Maillet determinant. In this paper, we give a determinant formula for relative congruence zeta functions for cyclotomic function fields. Our formula may be regarded as a generalization of the determinant formula for the relative class number.

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1. Introduction

Let h_p^- be the relative class number of the cyclotomic field of pth roots of unity. Carlitz and Olson [CO] computed the number h_p^- in terms of a certain classical determinant, known as the Maillet determinant.

In the cyclotomic function field case, several authors gave analogues of the Maillet determinant. Let k be the field of rational functions over the finite field $\mathbb{F}_q$ with q elements. Fix a generator T of k, and let A be the polynomial subring $\mathbb{F}_q[T]$ of k. Let m be a monic polynomial of A, and Λ_m be the set of all m-torsion points of the Carlitz module. The field K_m obtained by adjoining the points of Λ_m to k is called the mth cyclotomic function field. For the definition of the Carlitz module and the basic facts about cyclotomic function fields, see Section 2 below. Let K_m^+ be the decomposition field of the infinite prime of k in K_m/k, which is called the 'maximal real subfield' in K_m.

Let h_m and h_m^+ be the orders of the divisor class group of degree zero for K_m and K_m^+. Define the relative class number h_m^- of K_m by $h_m^- = h_m/h_m^+$.

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Rosen [Ro1] gave a determinant formula for h_P^- in the case of the monic irreducible polynomial P, which is regarded as an analogue of the Maillet determinant. Recently, several authors generalized Rosen's formula and gave class number formulas (see, for instance, [ACJ, BK]).

Let $\zeta(s, K_m)$ be the congruence zeta function for K_m. The function $\zeta(s, K_m)$ can be expressed in the form

$$\zeta(s, K_m) = \frac{Z_m(q^{-s})}{(1-q^{-s})(1-q^{1-s})},$$

where $Z_m(X)$ is a polynomial with integral coefficients. Then we have the decomposition

$$Z_m(X) = Z_m^{(+)}(X) Z_m^{(-)}(X),$$

where $Z_m^{(+)}(X)$ is the polynomial corresponding to the congruence zeta function $\zeta(s, K_m^+)$ for K_m^+. For the polynomial $Z_m^{(+)}(X)$, the author gave the determinant formula in the paper [Sh]. We see that

$$Z_m^{(-)}(q^{-s}) = \frac{\zeta(s, K_m)}{\zeta(s, K_m^+)};$$

this is called the relative congruence zeta function for K_m.

The main result of this paper is a determinant formula for $Z_m^{(-)}(X)$. Since $Z_m^{(-)}(1) = h_m^-$, our formula may be regarded as a generalization of the determinant formula for the relative class number.

As an application of our determinant formula, we will give an explicit formula for some coefficients of low-degree terms for $Z_m^{(-)}(X)$.

2. Basic facts

In this section, we outline several basic facts about cyclotomic function fields and their zeta functions. For the proofs of these facts, see [GR, Ha, Ro2, Wa].

2.1. Cyclotomic function fields. Let K^{ac} be the algebraic closure of k. For $x \in K^{\mathrm{ac}}$ and $m \in A$, we define the action

$$m \cdot x = m(\varphi + \mu)(x),$$

where φ and μ are the $\mathbb{F}_q$-linear maps of K^{ac} defined by

$$\varphi : K^{\mathrm{ac}} \longrightarrow K^{\mathrm{ac}} \quad x \mapsto x^q,$$
$$\mu : K^{\mathrm{ac}} \longrightarrow K^{\mathrm{ac}} \quad x \mapsto T \cdot x.$$

Under the above action, K^{ac} becomes an A-module, called the Carlitz module. Let Λ_m be the set of all x such that $m \cdot x = 0$; this is a cyclic sub-A-module of K^{ac}. Fix a generator λ_m of Λ_m. Then we have the following isomorphism of A-modules:

$$A/(m) \longrightarrow \Lambda_m \quad a \mod m \mapsto a \cdot \lambda_m,$$

where (m) is the principal ideal mA generated by m. Let $(A/(m))^\times$ be the group of units of $A/(m)$, and $\Phi(m)$ be the order of $(A/(m))^\times$. Let K_m be the field obtained by

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adjoining all the elements of Λ_m to k. We call K_m the mth cyclotomic function field. The extension K_m/k is abelian, and the following isomorphism is valid:

$$(A/(m))^\times \longrightarrow \mathrm{Gal}(K_m/k) \quad a \mod m \mapsto \sigma_{a \bmod m}, \tag{2.1}$$

where $\mathrm{Gal}(K_m/k)$ is the Galois group of K_m/k, and $\sigma_{a \bmod m}$ is the isomorphism given by $\sigma_{a \bmod m}(\lambda_m) = a \cdot \lambda_m$. By using isomorphism (2.1), we find that the extension degree of K_m/k is $\Phi(m)$. We see that $\mathbb{F}_q^\times$ is contained in $(A/(m))^\times$. Let K_m^+ be the subfield of K_m corresponding to $\mathbb{F}_q^\times$. Again from isomorphism (2.1), we find that the extension degree of K_m^+/k is $\Phi(m)/(q-1)$. Let P_∞ be the unique prime of k which corresponds to the valuation v_∞ with $v_\infty(T) < 0$. The prime P_∞ splits completely in K_m^+/k, and any prime of K_m^+ over P_∞ is totally ramified in K_m/K_m^+. Hence $K_m^+ = K_m \cap k_\infty$ where k_∞ is the completion of k by v_∞. The field K_m^+ is called the maximal real subfield of K_m; it is an analogue of the maximal real subfield of a cyclotomic field.

Next, we review some basic facts about Dirichlet characters. For a monic polynomial $m \in A$, let X_m be the group of all primitive Dirichlet characters of $(A/(m))^\times$. Let X_m^+ be the set of all characters in X_m such that $\chi(a) = 1$ for any $a \in \mathbb{F}_q^\times$. Put

$$\widetilde{K} = \bigcup_{m \text{ monic}} K_m$$

where m runs through all monic polynomials of A. Let $\mathbb{D}$ be the group of all primitive Dirichlet characters. By the same argument as in [Wa, Ch. 3], we have a one-to-one correspondence between finite subgroups of $\mathbb{D}$ and finite subextension fields of $\widetilde{K}/k$. The following theorem is useful for obtaining information about primes.

THEOREM 2.1 (See [Wa, Theorem 3.7]). *Let X be a finite subgroup of $\mathbb{D}$, and K_X the associated field. For an irreducible monic polynomial $P \in A$, put*

$$Y = \{\chi \in X \mid \chi(P) \neq 0\}, \qquad Z = \{\chi \in X \mid \chi(P) = 1\}.$$

Then the following hold.

- *X/Y is isomorphic to the inertia group of P in K_X/k.*
- *Y/Z is isomorphic to the cyclic group of order f_P; where f_P is the residue class degree of P in K_X/k.*
- *X/Z is isomorphic to the decomposition group of P in K_X/k.*

2.2. The relative congruence zeta function. Our next task is to investigate congruence zeta functions for cyclotomic function fields. Let K be a geometric extension of k of finite degree. We define the congruence zeta function of K as

$$\zeta(s, K) = \prod_{\mathcal{P} \text{ prime}} \left(1 - \frac{1}{\mathcal{NP}^s}\right)^{-1}$$

where $\mathcal{P}$ runs through all primes of K, and $\mathcal{NP}$ is the number of elements of the residue class field of the prime $\mathcal{P}$. We see that $\zeta(s, K)$ converges absolutely when $\mathrm{Re}(s) > 1$.

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THEOREM 2.2 (See [Ro2, Theorem 5.9]). *Let g_K be the genus of K and h_K be the order of the divisor class group of degree zero. Then there is a polynomial $Z_K(X) \in \mathbb{Z}[X]$ of degree $2g_K$ satisfying*

$$\zeta(s, K) = \frac{Z_K(q^{-s})}{(1-q^{-s})(1-q^{1-s})}, \tag{2.2}$$

and $Z_K(0) = 1$ and $Z_K(1) = h_K$.

Since the right-hand side of Equation (2.2) is meromorphic on the whole of $\mathbb{C}$, this equation provides the analytic continuation of $\zeta(s, K)$ to the whole of $\mathbb{C}$.

Next, we explain the zeta function of $\mathcal{O}_K$, which is the integral closure of A in the field K. We define the zeta function $\zeta(s, \mathcal{O}_K)$ for the ring $\mathcal{O}_K$ by

$$\zeta(s, \mathcal{O}_K) = \prod_{\mathcal{P}} \left(1 - \frac{1}{\mathcal{N}\mathcal{P}^s}\right)^{-1},$$

where the product runs over all primes of $\mathcal{O}_K$. Let X be a finite subgroup of $\mathbb{D}$, and K_X be the associated field. By the same argument as in the case of number fields (see [Wa]), we have the L-function decomposition

$$\zeta(s, \mathcal{O}_{K_X}) = \prod_{\chi \in X} L(s, \chi)$$

where the L-function is defined by

$$L(s, \chi) = \prod_{P} \left(\frac{1 - \chi(P)}{\mathcal{N}P^s}\right)^{-1},$$

where P runs through all monic irreducible polynomials of A. Let f_∞ be the residue class degree of P_∞ in K_X/k and g_∞ be the number of primes in K_X over P_∞. Then

$$\zeta(s, K_X) = \zeta(s, \mathcal{O}_{K_X})(1 - q^{-sf_\infty})^{-g_\infty}.$$

From now on, we will focus on the cyclotomic function field case. For a monic polynomial $m \in A$, let K_m and K_m^+ be the mth cyclotomic function field and its maximal real subfield. The relative congruence zeta function for K_m is defined by

$$\zeta^{(-)}(s, K_m) = \frac{\zeta(s, K_m)}{\zeta(s, K_m^+)}.$$

By Theorem 2.2, there are polynomials $Z_m(X)$ and $Z_m^{(+)}(X)$ with integral coefficients such that

$$\zeta(s, K_m) = \frac{Z_m(q^{-s})}{(1-q^{-s})(1-q^{1-s})},$$

$$\zeta(s, K_m^+) = \frac{Z_m^{(+)}(q^{-s})}{(1-q^{-s})(1-q^{1-s})}.$$

Put

$$Z_m^{(-)}(X) = \frac{Z_m(X)}{Z_m^{(+)}(X)};$$

then

$$\zeta^{(-)}(s, K_m) = Z_m^{(-)}(q^{-s}).$$

Notice that the fields K_m and K_m^+ are associated with X_m and X_m^+ respectively. Since any prime in K_m^+ above P_∞ is totally ramified in K_m/K_m^+,

$$Z_m^{(-)}(q^{-s}) = \prod_{\chi \in X_m^-} L(s, \chi) \tag{2.3}$$

where $X_m^- = X_m - X_m^+$. The L-function associated with the nontrivial character can be expressed by a polynomial of q^{-s} with complex coefficients. Hence, we see that $Z_m^{(-)}(X)$ is a polynomial with integral coefficients.

3. The determinant formula for $Z_m^{(-)}(X)$

In the previous section, we defined the relative congruence zeta function $\zeta^{(-)}(s, K_m)$ for the mth cyclotomic function field, and showed that $\zeta(s, K_m)$ is given by a polynomial $Z_m^{(-)}(X)$ with integral coefficients. The goal of this section is to give a determinant formula for $Z_m^{(-)}(X)$. First, we need some notation to construct the determinant formula. Let m be a monic polynomial of degree d. For $\alpha \in (A/(m))^\times$, there is a unique element $r_\alpha \in A$ satisfying

$$\begin{aligned} r_\alpha &= a_n T^n + a_{n-1} T^{n-1} + \cdots + a_0 \quad \text{where } n = \deg r_\alpha < d, \\ r_\alpha &\equiv \alpha \mod m, \end{aligned}$$

where $\deg f$ denotes the degree of the polynomial f. Then we define

$$\mathrm{Deg}(\alpha) = n, \qquad L(\alpha) = a_n \in \mathbb{F}_q^\times,$$

and $c^\lambda(\alpha) = \lambda^{-1}(L(\alpha))$, where λ is a character of $\mathbb{F}_q^\times$. Put $N_m = \Phi(m)/(q-1)$. Let $\alpha_1, \alpha_2, \ldots, \alpha_{N_m}$ be all of the elements of $(A/(m))^\times$ such that $L(\alpha) = 1$; these form a complete system of representatives for $\mathcal{R}_m = (A/(m))^\times/\mathbb{F}_q^\times$. We put

$$\begin{aligned} c_{ij}^\lambda &= c^\lambda(\alpha_i \alpha_j^{-1}) \quad \forall i, j = 1, 2, \ldots, N_m, \\ d_{ij} &= \mathrm{Deg}(\alpha_i \alpha_j^{-1}) \quad \forall i, j = 1, 2, \ldots, N_m. \end{aligned}$$

For any character λ of $\mathbb{F}_q^\times$, we define the matrix

$$D_m^{(\lambda)}(X) = (c_{ij}^\lambda X^{d_{ij}})_{i,j=1,2,\ldots,N_m}.$$

This matrix plays an essential role in our argument. Note that $d_{ij} > 0$ when $i \neq j$, and $d_{ij} = 0$ and $c_{ij}^\lambda = 1$ when $i = j$. Thus $D_m^{(\lambda)}(0)$ is the unit matrix. We put

$$D_m^{(-)}(X) = \prod_{\lambda \neq 1} \det D_m^{(\lambda)}(X),$$

where the product runs over all nontrivial characters of $\mathbb{F}_q^{\times}$. To be able to state the main result, we define the polynomial $J_m^{(-)}(X)$ by

$$J_m^{(-)}(X) = \prod_{\chi \in X_m^-} \prod_{Q|m} (1 - \chi(Q) X^{\deg Q}),$$

where Q is an irreducible monic polynomial dividing m. First, we prove the following proposition.

PROPOSITION 3.1. *With the notation above,*

$$J_m^{(-)}(X) = \prod_{Q|m} \frac{(1 - X^{f_Q \deg Q})^{g_Q}}{(1 - X^{f_Q^+ \deg Q})^{g_Q^+}},$$

where f_Q and f_Q^+ are the residue class degrees of Q in K_m/k and K_m^+/k, and g_Q and g_Q^+ are the numbers of primes in K_m and K_m^+ over Q.

PROOF. Notice that X_m and X_m^+ are associated with the mth cyclotomic function field K_m and its maximal real subfield K_m^+ respectively. Let Q be an irreducible monic polynomial dividing m. Put

$$Y_Q = \{\chi \in X_m \mid \chi(Q) \neq 0\} \quad \text{and} \quad Z_Q = \{\chi \in X_m \mid \chi(Q) = 1\}.$$

From Theorem 2.1,

$$\begin{aligned} \prod_{\chi \in X_m} (1 - \chi(Q) X^{\deg Q}) &= \prod_{\chi \in Y_Q} (1 - \chi(Q) X^{\deg Q}) \\ &= \prod_{\chi \in Y_Q/Z_Q} \prod_{\psi \in Z_Q} (1 - \chi\psi(Q) X^{\deg Q}) \\ &= \Bigg(\prod_{\chi \in Y_Q/Z_Q} (1 - \chi(Q) X^{\deg Q}) \Bigg)^{g_Q}. \end{aligned}$$

Since Y_Q/Z_Q is a cyclic group of order f_Q,

$$\prod_{\chi \in Y_Q/Z_Q} (1 - \chi(Q) X^{\deg Q}) = (1 - X^{f_Q \deg Q}).$$

Hence we obtain the formula

$$\prod_{\chi \in X_m} (1 - \chi(Q) X^{\deg Q}) = (1 - X^{f_Q \deg Q})^{g_Q}.$$

By the same argument,

$$\prod_{\chi \in X_m^+} (1 - \chi(Q) X^{\deg Q}) = (1 - X^{f_Q^+ \deg Q})^{g_Q^+}.$$

Noting that $X_m^- = X_m - X_m^+$, we can deduce the proposition from the last two equations. □

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There are several consequences of this proposition. First of all, by Proposition 3.1, we see that $J_m^{(-)}(X)$ is a polynomial with integral coefficients. Second, if m is a power of an irreducible polynomial P, the prime P is totally ramified in K_m/k (see [Ro2]). Hence $J_m^{(-)}(X) = 1$ in this case.

The next theorem is the main result of this paper.

THEOREM 3.2. *Let $m \in A$ be a monic polynomial. Then*

$$D_m^{(-)}(X) = Z_m^{(-)}(X) J_m^{(-)}(X).$$

PROOF. For any $\chi \in X_m$, let the monic polynomial f_χ be the conductor of χ. Define $\tilde{\chi}$ by

$$\tilde{\chi} = \chi \circ \pi_\chi$$

where $\pi_\chi : (A/(m))^\times \to (A/(f_\chi))^\times$ is the natural homomorphism. Then

$$L(s, \tilde{\chi}) = L(s, \chi) \cdot \prod_{Q|m} (1 - \chi(Q) q^{-s \deg Q}).$$

Fix a nontrivial character λ of $\mathbb{F}_q^\times$ and $\psi \in X_m^-(\psi|_{\mathbb{F}_q^\times} = \lambda)$. Then

$$\psi \cdot X_m^+ = \{\chi \in X_m^- \mid \chi|_{\mathbb{F}_q^\times} = \lambda\}.$$

For each character $\chi \in X_m^-(\chi|_{\mathbb{F}_q^\times} = \lambda)$, there is a unique character $\phi \in X_m^+$ with $\chi = \psi \cdot \phi$. By the same argument as in [GR, Lemma 3],

$$\begin{aligned} L(s, \tilde{\chi}) &= \sum_{i=1}^{N_m} \tilde{\chi}(\alpha_i) q^{-\mathrm{Deg}(\alpha_i)s} \\ &= \sum_{i=1}^{N_m} \tilde{\phi}(\alpha_i) \tilde{\psi}(\alpha_i) c^\lambda(\alpha_i) q^{-\mathrm{Deg}(\alpha_i)s}. \end{aligned}$$

Notice that $\tilde{\psi}(\alpha) c^\lambda(\alpha)$ and Deg are functions over $\mathcal{R}_m$, and $\tilde{\phi}$ runs through all characters of $\mathcal{R}_m$ when ϕ runs through all characters of X_m^+. By the Frobenius determinant formula (see [Wa, Lemma 5.26]),

$$\begin{aligned} \prod_{\chi|_{\mathbb{F}_q^\times} = \lambda} L(s, \tilde{\chi}) &= \prod_{\phi \in X_m^+} \sum_{i=1}^{N_m} \tilde{\phi}(\alpha_i) \tilde{\psi}(\alpha_i) c^\lambda(\alpha_i) q^{-\mathrm{Deg}(\alpha_i)s} \\ &= \det(\psi(\alpha_i \alpha_j^{-1}) c_{ij}^\lambda q^{-s d_{ij}})_{i,j=1,2,\ldots,N_m} \\ &= \det D_m^{(\lambda)}(q^{-s}). \end{aligned}$$

From the decomposition

$$X_m^- = \bigcup_{\lambda \neq 1} \{\chi \in X_m \mid \chi|_{\mathbb{F}_q^\times} = \lambda\},$$

we see that

$$D_m^{(-)}(q^{-s}) = \left(\prod_{\chi \in X_m^-} L(s, \chi) \right) \times J_m^{(-)}(q^{-s}).$$

By Equation (2.3), we obtain the formula

$$D_m^{(-)}(q^{-s}) = Z_m^{(-)}(q^{-s}) J_m^{(-)}(q^{-s}).$$

Putting $X = q^{-s}$, we obtain the desired result. □

We offer two remarks about this theorem. First, $Z_m^{(-)}(X) = 1$ when m is a monic polynomial of degree one. In fact, we can calculate that $D_m^{(-)}(X) = 1$ in this case. Second, recall that $J_m^{(-)}(X) = 1$ when m is a power of an irreducible polynomial. Hence $D_m^{(-)}(X) = Z_m^{(-)}(X)$ in this case.

As a special case of our result, we obtain the following determinant formula for relative class numbers.

COROLLARY 3.3 (See [ACJ, BK]). *Let h_m^- be the relative class number of K_m. Put $f_Q^- = f_Q/f_Q^+$ and $g_Q^- = g_Q/g_Q^+$. Then*

$$\prod_{\lambda \neq 1} \det(c_{ij}^{\lambda})_{i,j=1,2,\ldots,N_m} = W_m^- \cdot h_m^-,$$

where

$$W_m^- = \begin{cases} \prod_{Q|m} (f_Q^-)^{g_Q^+} & \text{if } g_Q^- = 1 \text{ for every prime } Q \text{ dividing } m, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Putting $X = 1$ in Theorem 3.2, we see that

$$D_m^{(-)}(1) = \prod_{\lambda \neq 1} \det(c_{ij}^{\lambda}),$$

and $J_m^{(-)}(1) = W_m^-$ by Proposition 3.1. Since $Z_m^{(-)}(1) = h_m^-$, we obtain the desired result. □

If m is a power of an irreducible polynomial, we see that $W_m^- = 1$. Otherwise, each finite prime in K_m^+ is not ramified in K_m/K_m^+. Thus we see that $f_Q^- = q - 1$ for a prime Q with $g_Q^- = 1$.

4. Some coefficients of the low degree terms of $D_m^{(-)}(X)$

In this section, we will calculate the coefficients of $D_m^{(-)}(X)$ of degrees one and two, by using the derivative of the determinant. Let $m \in A$ be a monic polynomial. Noting that $D_m^{(-)}(0) = 1$, we see that $D_m^{(-)}(X)$ may be written in the form

$$D_m^{(-)}(X) = 1 + a_1 X + a_2 X^2 + \cdots,$$

where each a_i is an integer ($i = 1, 2, \ldots$).

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PROPOSITION 4.1. *Let $m \in A$ be a monic polynomial of degree d, where $d > 1$. Then*

$$a_1 = 0, \tag{4.4}$$

$$a_2 = 0 \quad \textit{if} \deg m > 2, \tag{4.5}$$

$$a_2 = \frac{N_m}{2}\{(q-1)(1-C_m) + N_m - 1\} \quad \textit{if} \deg m = 2,$$

where

$$C_m = \#\{i = 1, 2, \ldots, N_m \mid L(\alpha_i^{-1}) = 1\}.$$

Here $\#A$ is the number of elements of a set A.

By Proposition 3.1, we can find $J_m^{(-)}(X)$. Hence we can also calculate the coefficients of the low-degree terms of $Z_m^{(-)}(X)$. As a preliminary to Proposition 4.1, we first state the next lemma, which can be proved by simple calculations.

LEMMA 4.2. *Let $F(X) = (f_{ij}(X))_{i,j}$ be a matrix with coefficients in the ring of functions of one variable. If $F(X)$ is twice differentiable and invertible when $X = X_0$, then*

$$\left.\frac{d \det F(X)}{dX}\right|_{X=X_0} = \det F(X_0) \cdot \mathrm{Tr}\left(F(X_0)^{-1}\frac{dF}{dX}(X_0)\right),$$

$$\left.\frac{d^2 \det F(X)}{dX^2}\right|_{X=X_0} = \det F(X_0) \cdot \left\{\mathrm{Tr}\left(F(X_0)^{-1}\frac{d^2F}{dX^2}(X_0)\right) - \mathrm{Tr}\left(F(X_0)^{-1}\frac{dF}{dX}(X_0)F(X_0)^{-1}\frac{dF}{dX}(X_0)\right) + \mathrm{Tr}\left(F(X_0)^{-1}\frac{dF}{dX}(X_0)\right)^2\right\},$$

where $\mathrm{Tr}(A)$ *denotes the trace of the matrix A.*

We now prove Proposition 4.1.

PROOF. Let λ be a nontrivial character of $\mathbb{F}_q^\times$, and write

$$\det D_m^{(\lambda)}(X) = 1 + a_1^\lambda X + a_2^\lambda X^2 + \cdots .$$

Note that $D_m^{(\lambda)}(0)$ is the unit matrix, and

$$\frac{dD_m^{(\lambda)}}{dX}(0) = (l_{ij})_{i,j=1,2,\ldots,N_m},$$

where

$$l_{ij} = \begin{cases} 0 & \text{if } d_{ij} = 0 \text{ or } d_{ij} > 1, \\ c_{ij}^\lambda & \text{if } d_{ij} = 1. \end{cases}$$

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By Lemma 4.2, $a_1^\lambda = 0$ and

$$a_2^\lambda = -\frac{1}{2}\,\mathrm{Tr}\Big(\Big(\frac{dD_m^{(\lambda)}}{dX}(0)\Big)^2\Big).$$

Thus we have shown assertion (4.4).

If $\deg m > 2$, there is no pair (i, j) such that $d_{ij} = 1$ and $d_{ji} = 1$. Thus $a_2^\lambda = 0$ in the case where $\deg m > 2$. Since $a_2 = \sum_{\lambda \neq 1} a_2^\lambda$, we obtain assertion (4.5).

Next we consider the case where $\deg m = 2$. In this case,

$$l_{ij} = \begin{cases} 0 & \text{if } i = j, \\ c_{ij}^\lambda & \text{if } i \neq j. \end{cases}$$

Thus

$$\begin{aligned}\sum_{\lambda \neq 1} a_2^\lambda &= \sum_{\lambda \neq 1}\Big(\frac{N_m}{2} - \frac{1}{2}\sum_{i=1}^{N_m}\sum_{j=1}^{N_m} \lambda^{-1}(L(\alpha_i\alpha_j^{-1})L(\alpha_j\alpha_i^{-1}))\Big) \\ &= \frac{N_m(q-2)}{2} - \frac{1}{2}\sum_{i=1}^{N_m}\sum_{j=1}^{N_m} e_{ij},\end{aligned}$$

where

$$e_{ij} = \begin{cases} q-2 & \text{if } L(\alpha_i\alpha_j^{-1})L(\alpha_j\alpha_i^{-1}) = 1, \\ -1 & \text{otherwise.} \end{cases}$$

For any $i, j \in \{1, 2, \ldots, N_m\}$, there exist $\gamma_{ij} \in \mathbb{F}_q^\times$ and $\beta_{ij} \in (A/(m))^\times$ such that $L(\beta_{ij}) = 1$ and $\alpha_i\alpha_j^{-1} = \gamma_{ij}\beta_{ij}$. Then

$$L(\alpha_i\alpha_j^{-1})L(\alpha_j\alpha_i^{-1}) = L(\beta_{ij}^{-1}).$$

By noting that

$$\{\beta_{ij} \mid j = 1, 2, \ldots, N_m\} = \{\alpha_j \mid j = 1, 2, \ldots, N_m\},$$

we see that

$$\sum_{j=1}^{N_m} e_{ij} = (q-1)C_m - N_m.$$

Thus we have completed the proof of Proposition 4.1. □

We consider the case where $m = T^2 + aT + b \in A$. If $\alpha = T - c$ satisfies $L(\alpha^{-1}) = 1$, then c is a root of the equation $T^2 + aT + b + 1$. Thus $C_m \leq 3$.

5. Examples

We conclude this paper with some examples.

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EXAMPLE 5.1. When $q = 3$ and $m = T^2 + 1$, we see that the extension degree of K_m/k is 8 and $N_m = 4$. Since the polynomial m is irreducible, $D_m^{(-)}(X) = Z_m^{(-)}(X)$. Put

$$\alpha_1 = 1, \quad \alpha_2 = T, \quad \alpha_3 = T + 1, \quad \alpha_4 = T + 2.$$

Then

$$\begin{aligned} Z_m^{(-)}(X) &= D_m^{-}(X) \\ &= \begin{vmatrix} 1 & -X & X & X \\ X & 1 & -X & X \\ X & -X & 1 & -X \\ X & X & X & 1 \end{vmatrix} \\ &= 1 - 2X^2 + 9X^4. \end{aligned}$$

The relative class number h_m^- of K_m is $Z_m^{(-)}(1) = 8$.

EXAMPLE 5.2. When $q = 3$ and $m = T^3 + T^2$, we see that the extension degree of K_m/k is 12 and $N_m = 6$. Put

$$\begin{gathered} \alpha_1 = 1, \quad \alpha_2 = T^2 + 2T + 2, \quad \alpha_3 = T^2 + T + 1, \\ \alpha_4 = T + 2, \quad \alpha_5 = T^2 + 1, \quad \alpha_6 = T^2 + T + 2. \end{gathered}$$

Then

$$\begin{aligned} D_m^{(-)}(X) &= \begin{vmatrix} 1 & X & -X^2 & X^2 & X^2 & -X^2 \\ X^2 & 1 & -X^2 & -X^2 & -X^2 & -X \\ X^2 & X^2 & 1 & X & -X^2 & X^2 \\ X & X^2 & X^2 & 1 & X^2 & X^2 \\ X^2 & X^2 & -X & -X^2 & 1 & X^2 \\ X^2 & -X^2 & -X^2 & X^2 & X & 1 \end{vmatrix} \\ &= 1 - 6X^3 - 3X^4 - 6X^5 + 23X^6 + 30X^7 + 6X^8 - 18X^9 - 27X^{10} \end{aligned}$$

and

$$J_m^{(-)}(X) = 1 + X - X^3 - X^4.$$

Thus

$$\begin{aligned} Z_m^{(-)}(X) &= \frac{D_m^{(-)}(X)}{J_m^{(-)}(X)} \\ &= 1 - X + X^2 - 6X^3 + 3X^4 - 9X^5 + 27X^6. \end{aligned}$$

The relative class number h_m^- of K_m is $Z_m^{(-)}(1) = 16$.

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References

[ACJ] J. Ahn, S. Choi and H. Jung, 'Class number formulae in the form of a product of determinants in function fields', *J. Aust. Math. Soc.* **78**(2) (2005), 227–238.

[BK] S. Bae and P.-L. Kang, 'Class numbers of cyclotomic function fields', *Acta Arith.* **102**(3) (2002), 251–259.

[CO] L. Carlitz and F. R. Olson, 'Maillet's determinant', *Proc. Amer. Math. Soc.* **6** (1955), 265–269.

[GR] S. Galovich and M. Rosen, 'The class number of cyclotomic function fields', *J. Number Theory* **13**(3) (1981), 363–375.

[Ha] D. R. Hayes, 'Explicit class field theory for rational function fields', *Trans. Amer. Math. Soc.* **189** (1974), 77–91.

[Ro1] M. Rosen, 'A note on the relative class number in function fields', *Proc. Amer. Math. Soc.* **125**(5) (1997), 1299–1303.

[Ro2] M. Rosen, *Number Theory in Function Fields* (Springer, Berlin, 2002).

[Sh] D. Shiomi, 'A determinant formula of congruence zeta functions for maximal real cyclotomic function fields', *Acta Arith.* **138**(3) (2009), 259–268.

[Wa] L. C. Washington, *Introduction to Cyclotomic Fields* (Springer, New York, 1982).

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