COMPOSITIO MATHEMATICA

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Compositio Math. **150** (2014), 729–748

doi:10.1112/S0010437X13007653

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY

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Abstract

As the simplest case of Langlands functoriality, one expects the existence of the symmetric power $S^n(\pi)$, where $\pi$ is an automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ and $\mathbb{A}$ denotes the adeles of a number field $F$. This should be an automorphic representation of $\mathrm{GL}(N, \mathbb{A})$ $(N = n + 1)$. This is known for $n = 2, 3$ and 4. In this paper we show how to deduce the general case from a recent result of J.T. on deformation theory for 'Schur representations', combined with expected results on level-raising, as well as another case (a particular tensor product) of Langlands functoriality. Our methods assume $F$ totally real, and the initial representation $\pi$ of classical type.

## 1. Introduction

The purpose of this paper is to shed new light on functoriality for regular, algebraic automorphic representations over CM fields which satisfy a self-duality condition. We formulate three conjectures. The first, Conjecture 3.1, asserts that one can find congruences between algebraic modular forms on unitary groups of a certain type. This is a natural generalization of several results going back to a theorem of Ribet (see [Rib84]) concerning elliptic modular forms, and is closely related to the conjectural 'Ihara's lemma' of [CHT08].

The other two conjectures are specific instances of Langlands functoriality, essentially the tensor product $\mathrm{GL}_2 \times \mathrm{GL}_n \to \mathrm{GL}_{2n}$ and the symmetric power $\mathrm{GL}_2 \to \mathrm{GL}_{n+1}$ for the automorphic representations under consideration here; see Conjectures 3.2 and 3.3, respectively. Our main theorem (Theorem 3.4) gives a specific relation between this family of conjectures. As a particular application, we can prove the following theorem.

Theorem 1.1. *Let $\pi$ be a regular algebraic automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$, which is not automorphically induced from a quadratic extension. Then (cf. § 3.4 below):*

(1) *Assume Conjecture 3.1 below. Then for each odd integer $1 \leqslant n \leqslant 25$, the $n$th symmetric power lifting of $\pi$ exists, as an automorphic representation of $\mathrm{GL}_{n+1}(\mathbb{A}_{\mathbb{Q}})$.*

(2) *Assume Conjectures 3.1 and 3.2 below. Then for each integer $n \geqslant 1$, the $n$th symmetric power lifting of $\pi$ exists, as an automorphic representation of $\mathrm{GL}_{n+1}(\mathbb{A}_{\mathbb{Q}})$.*

We refer the reader to § 3 for a detailed description of our results. We begin in § 2 by recalling some background material on automorphic representations and their attached Galois representations. The proof of Theorem 3.4 occupies §§ 4–5.

Received 19 March 2013, accepted in final form 22 August 2013, published online 26 March 2014.
*2010 Mathematics Subject Classification* 11F03, 11F66, 11F80 (primary).
*Keywords:* modular and automorphic functions, Langlands $L$-functions, Galois representations.
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In a sequel to this paper [CT], we will prove some cases of level-raising, closely related to Conjecture 3.1, and apply this to the automorphy of symmetric powers, following the program outlined here. Since these two papers are essentially self-contained, the reader will find some unavoidable duplication in the material. On the other hand, we have have also referred to some proofs from the second paper, in particular in § 2. We apologise for the possible inconvenience.

## 2. Automorphic forms

### 2.1 $\mathrm{GL}_n$

Let $p$ be a prime, and let $K$ be a finite extension of $\mathbb{Q}_p$. Let $\Omega$ denote an algebraically closed field of characteristic zero. Denote by $\mathrm{Adm}_\Omega \mathrm{GL}_n(K)$ the set of isomorphism classes of irreducible admissible representations of this group over $\Omega$, and by $\mathrm{WD}^n_\Omega W_K$ the set of Frobenius-semisimple Weil–Deligne representations $(r, N)$ of $W_K$ valued in $\mathrm{GL}_n(\Omega)$. There is a bijection

$$\mathrm{rec}_K : \mathrm{Adm}_{\mathbb{C}} \mathrm{GL}_n(K) \leftrightarrow \mathrm{WD}^n_{\mathbb{C}} W_K,$$

characterized by a certain equality of $\epsilon$- and $L$-factors on either side; cf. [Hen02, HT01]. We define $\mathrm{rec}^T_K(\pi) = \mathrm{rec}_K(\pi|\cdot|^{(1-n)/2})$. This is the normalization of the local Langlands correspondence with good rationality properties; in particular, for any $\sigma \in \mathrm{Aut}(\mathbb{C})$ and any $\pi \in \mathrm{Adm}_{\mathbb{C}} \mathrm{GL}_n(K)$ there is an isomorphism

$$\mathrm{rec}^T_K({}^\sigma\pi) \cong {}^\sigma\mathrm{rec}^T_K(\pi).$$

This can be seen using, for example, the characterization of $\mathrm{rec}_K$ and the description given in [Tat79, § 3] of the action of Galois on local $\epsilon$- and $L$-factors. It follows that for any $\Omega$ we can define a canonical bijection

$$\mathrm{rec}^T_K : \mathrm{Adm}_\Omega \mathrm{GL}_n(K) \leftrightarrow \mathrm{WD}^n_\Omega W_K.$$

Suppose instead that $K$ is a finite extension of $\mathbb{R}$. Write $\mathrm{Adm}_{\mathbb{C}} \mathrm{GL}_n(K)$ for the set of infinitesimal equivalence classes of irreducible admissible representations of $\mathrm{GL}_n(K)$ and $\mathrm{Rep}^n_{\mathbb{C}} W_K$ for the set of continuous semisimple representations of $W_K$ into $\mathrm{GL}_n(\mathbb{C})$. Then there is a bijection (Langlands' normalization):

$$\mathrm{rec}_K : \mathrm{Adm}_{\mathbb{C}} \mathrm{GL}_n(K) \leftrightarrow \mathrm{Rep}^n_{\mathbb{C}} W_K.$$

We define $\mathrm{rec}^T_K(\pi) = \mathrm{rec}_K(\pi|\cdot|^{(1-n)/2})$.

Now suppose that $E$ is an imaginary CM field with totally real subfield $F$, and let $c \in \mathrm{Gal}(E/F)$ denote the non-trivial element.

Definition 2.1. (1) We say that an automorphic representation $\pi$ of $\mathrm{GL}_n(\mathbb{A}_E)$ is RACSDC (regular algebraic, conjugate self-dual, cuspidal) if it satisfies the following conditions.

- It is conjugate self-dual: $\pi^c \cong \pi^\vee$.
- It is cuspidal.
- It is regular algebraic. By definition, this means that for each place $v|\infty$ of $E$, the representation $\mathrm{rec}^T_{E_v}(\pi_v)$ is a direct sum of pairwise distinct algebraic characters.

(2) We say that a pair $(\pi, \chi)$ consisting of an automorphic representation $\pi$ of $\mathrm{GL}_n(\mathbb{A}_E)$ and a character $\chi : F^\times\backslash\mathbb{A}^\times_F \to \mathbb{C}^\times$ is RAECSDC (regular algebraic, essentially conjugate self-dual, cuspidal) if it satisfies the following conditions.

- It is essentially conjugate self-dual: $\pi^c \cong \pi^\vee \otimes \chi \circ \mathbb{N}_{E/F}$.
- $\pi$ is cuspidal.

- $\pi$ is regular algebraic.
- $\chi$ is an algebraic character such that $\chi_v(-1) = (-1)^n$ for each place $v|\infty$.

(3) We say that a pair $(\pi, \chi)$ consisting of an automorphic representation $\pi$ of $\mathrm{GL}_n(\mathbb{A}_F)$ and a character $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ is RAESDC (regular algebraic, essentially self-dual, cuspidal) if it satisfies the following conditions.

- It is essentially self-dual: $\pi \cong \pi^\vee \otimes \chi$.
- $\pi$ is cuspidal.
- $\pi$ is regular algebraic. By definition, this means that for each place $v|\infty$, the representation $\mathrm{rec}^T_{F_v}(\pi_v)|_{\mathbb{C}^\times}$ is a direct sum of pairwise distinct algebraic characters.
- $\chi$ is an algebraic character such that $\chi_v(-1)$ is independent of the place $v|\infty$.

If $\pi$ is a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$, then for each embedding $\tau : E \hookrightarrow \mathbb{C}$, we are given a representation $r_\tau : \mathbb{C}^\times \to \mathrm{GL}_n(\mathbb{C})$, induced by $\mathrm{rec}_{E_v}(\pi_v)$, where $v$ is the infinite place induced by $\tau$, and the isomorphism $E_v^\times \cong \mathbb{C}^\times$ induced by $\tau$. There exists (cf. [Clo90, Lemma 4.9]) an integer $w \in \mathbb{Z}$ such that this representation has the form

$$r_\tau(z) = (z^{a_{\tau,1}} \overline{z}^{w-a_{\tau,1}}, \dots, z^{a_{\tau,n}} \overline{z}^{w-a_{\tau,n}}),$$

where $a_{\tau,i} \in (n-1)/2 + \mathbb{Z}$ and $a_{\tau,1} > \dots > a_{\tau,n}$. (Note that $w = 0$ if and only if $\pi$ is unitary; this will be the case if $\pi$ is conjugate self-dual.) We will refer to the tuple $\mathbf{a} = (a_{\tau,1}, \dots, a_{\tau,n})_{\tau \in \mathrm{Hom}(E,\mathbb{C})}$ as the infinity type of $\pi$. We also define a tuple $\boldsymbol{\lambda} = (\lambda_\tau)_{\tau \in \mathrm{Hom}(E,\mathbb{C})} = (\lambda_{\tau,1}, \dots, \lambda_{\tau,n})_{\tau \in \mathrm{Hom}(E,\mathbb{C})}$, which we call the weight of $\pi$, by the formula $\lambda_{\tau,i} = -a_{\tau,n+1-i} + (n-1)/2 - (n-i)$. Then for each $\tau : E \hookrightarrow \mathbb{C}$, we have $\lambda_{\tau,1} \geqslant \dots \geqslant \lambda_{\tau,n}$, and the irreducible admissible representation of $\mathrm{GL}_n(\mathbb{C})$ corresponding to $r_\tau$ has the same infinitesimal character as the dual of the algebraic representation of $\mathrm{GL}_n(\mathbb{C})$ with highest weight $\lambda_\tau$. If $\pi$ is a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, then for each embedding $F \hookrightarrow \mathbb{C}$, we get a representation $r_\tau = \mathrm{rec}_{F_v}(\pi_v)|_{\mathbb{C}^\times}$, where $v$ is the place of $F$ corresponding to $\tau$. In this case we use the same formulae to define the infinity type and the weight of $\pi$.

We will also have cause to consider representations which are not cuspidal. Suppose that $\sigma_1, \sigma_2$ are conjugate self-dual cuspidal automorphic representations of $\mathrm{GL}_{n_1}(\mathbb{A}_E), \mathrm{GL}_{n_2}(\mathbb{A}_E)$, respectively, and that $\Sigma = \sigma_1 \boxplus \sigma_2$ is regular algebraic. Then the representations $\sigma_i|\cdot|^{(n_i-n)/2}$ are regular algebraic. We call a representation $\Sigma$ arising in this way a RACSD sum of cuspidal representations. In this case, define $\mathbf{a}^i = (a^i_\tau)_{\tau \in \mathrm{Hom}(E,\mathbb{C})}$ by the requirement that $(a^i_{\tau,1} + (n_i - n)/2, \dots, a^i_{\tau,n_i} + (n_i - n)/2)$ equal the infinity type of $\sigma_i|\cdot|^{(n_i-n)/2}$, and define $\mathbf{b} = (b_\tau)_{\tau \in \mathrm{Hom}(E,\mathbb{C})}$ by the formula

$$(b_{\tau,1}, \dots, b_{\tau,n}) = (a^1_{\tau,1}, \dots, a^1_{\tau,n_1}, a^2_{\tau,1}, \dots, a^2_{\tau,n_2}).$$

Let $\mathfrak{S}_n$ denote the symmetric group on $\{1, \dots, n\}$. There is a unique tuple $\mathbf{w} = (w_\tau)_{\tau \in \mathrm{Hom}(E,\mathbb{C})} \in \mathfrak{S}_n^{\mathrm{Hom}(E,\mathbb{C})}$ such that for each $\tau \in \mathrm{Hom}(E,\mathbb{C})$,

$$b_{\tau,w_\tau(1)} > \dots > b_{\tau,w_\tau(n)}.$$

The infinity type of $\Sigma$ is defined to be $(b_{\tau,w_\tau(1)}, \dots, b_{\tau,w_\tau(n)})_{\tau \in \mathrm{Hom}(E,\mathbb{C})}$.

Theorem 2.2. (1) *Let $\pi$ be a RACSD sum of cuspidals or a RAECSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$, and fix an isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$. Then there exists a continuous semisimple representation*

$$r_\iota(\pi) : G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$$

*satisfying the following property: for every finite place $v$ of $E$ not dividing $l$, there is an isomorphism*

$$\mathrm{WD}(r_\iota(\pi)|_{G_{E_v}})^{F\text{-}ss} \cong \mathrm{rec}^T_{E_v}(\iota^{-1}\pi_v).$$

*For each place $v$ of $E$ dividing $l$, $r_\iota(\pi)|_{G_{E_v}}$ is de Rham, and if $\tau : E_v \hookrightarrow \overline{\mathbb{Q}}_l$ is an embedding then the Hodge–Tate weights with respect to this embedding are*

$$\mathrm{HT}_\tau(r_\iota(\pi)) = \{-a_{\iota^{-1}\tau,1} + (n-1)/2, \dots, -a_{\iota^{-1}\tau,n} + (n-1)/2\}.$$

*(2) Let $(\pi, \chi)$ be a RAESDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, and fix an isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$. Then there exists a continuous semisimple representation*

$$r_\iota(\pi) : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$$

*satisfying the following property: for every finite place $v$ of $F$ not dividing $l$, there is an isomorphism*

$$\mathrm{WD}(r_\iota(\pi)|_{G_{F_v}})^{F\text{-}ss} \cong \mathrm{rec}^T_{F_v}(\iota^{-1}\pi_v).$$

*For each place $v$ of $F$ dividing $l$, $r_\iota(\pi)|_{G_{F_v}}$ is de Rham, and if $\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_l$ is an embedding then the Hodge–Tate weights with respect to this embedding are*

$$\mathrm{HT}_\tau(r_\iota(\pi)) = \{-a_{\iota^{-1}\tau,1} + (n-1)/2, \dots, -a_{\iota^{-1}\tau,n} + (n-1)/2\}.$$

*Proof.* This theorem is due to many people, including Kottwitz, L.C., Harris, Taylor and Shin. We give references for the case of a RACSDC automorphic representation $\pi$, from which the others can be deduced. In this case the existence of the representation $r_\iota(\pi)$ is proved in [CH13, Theorem 3.2.3]. The strong form of local-global compatibility is proved in [Car12]. See also [BGGT14, Theorem 2.1.1]. □

Lemma 2.3. *Let $\pi$ be one of the above types of automorphic representations, and fix an isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$. Let $\sigma$ be a continuous automorphism of $\overline{\mathbb{Q}}_l$. Then ${}^{\iota\sigma\iota^{-1}}\pi$ is defined, by [Clo90, Theorem 3.13]. There are isomorphisms*

$$r_\iota({}^{\iota\sigma\iota^{-1}}\pi) \cong r_{\iota\sigma}(\pi) \cong {}^\sigma r_\iota(\pi).$$

*Proof.* This follows from local-global compatibility, the rationality of the local Langlands correspondence for $\mathrm{GL}_n$, and the Chebotarev density theorem. □

We will use the following convention for residual representations. If $L$ is a number field and $\rho : G_L \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ is a continuous representation, then after choosing an invariant lattice, defined over a finite extension of $\mathbb{Q}_l$, we obtain by reduction modulo $l$ a residual representation valued in $\mathrm{GL}_n(\overline{\mathbb{F}}_l)$. By the Brauer–Nesbitt principle, the semisimplification of this representation depends, up to isomorphism, only on $\rho$, and will be denoted $\overline{\rho} : G_L \to \mathrm{GL}_n(\overline{\mathbb{F}}_l)$.

### 2.2 Ordinary forms

We recall that deformation theory in the context of ordinary, conjugate self-dual automorphic representations has been studied by Geraghty [Ger]. Let $L = E$ or $F$. If $\pi$ is a regular algebraic automorphic representation of $\mathrm{GL}_n(\mathbb{A}_L)$ of infinity type $\mathbf{a}$ and weight $\boldsymbol{\lambda}$, we define Hecke operators $U^j_{\boldsymbol{\lambda},v}$ as follows at primes $v$ above $l$. They depend on a choice of isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, which we fix for the rest of this section, as well as a choice of uniformizer $\varpi_v$ of $\mathcal{O}_{L_v}$. Let $\mathrm{Iw}_c(v) \subset \mathrm{GL}_n(\mathcal{O}_{L_v})$ be the subgroup of matrices whose reduction modulo $\varpi_v^c$ is an upper-triangular matrix with 1's on the diagonal. Define a matrix

$$\alpha_v^j = \operatorname{diag}(\underbrace{\varpi_v, \dots, \varpi_v}_{j}, \underbrace{1, \dots, 1}_{n-j}),$$

and set

$$U_{\boldsymbol{\lambda},v}^j = \prod_\tau \iota^{-1}\tau(\varpi_v)^{-(\lambda_{\tau,n}+\dots+\lambda_{\tau,n+1-j})}[\mathrm{Iw}_c(v)\alpha_v^j \,\mathrm{Iw}_c(v)].$$

Here the product runs over embeddings $\tau : L \hookrightarrow \mathbb{C}$ such that $\iota^{-1}\tau$ induces the place $v$ of $L$. If $\pi_v$ is an admissible representation of $\mathrm{GL}_n(L_v)$ over $\mathbb{C}$, then the operator $U_{\lambda,v}^j$ acts on $\iota^{-1}\pi_v^{\mathrm{Iw}_c(v)}$. We note that by [Ger, Lemma 2.3.3], the Hecke operators $U_{\boldsymbol{\lambda},v}^j$ commute with the inclusions $\iota^{-1}\pi_v^{\mathrm{Iw}_c(v)} \to \iota^{-1}\pi_v^{\mathrm{Iw}_{c'}(v)}$ when $c' \geqslant c$. It therefore makes sense to omit $c$ from the notation defining $U_{\boldsymbol{\lambda},v}^j$. We also write $T_c(v) \subset \mathrm{Iw}_c(v)$ for the group of diagonal matrices with integral entries which are congruent to 1 modulo $\varpi_v^c$, $e_v$ for the absolute ramification index of $L_v$, $f_v$ for the absolute residue degree, and $\mathrm{val} : \overline{\mathbb{Q}}_l^\times \to \mathbb{Q}$ for the valuation such that $\mathrm{val}(l) = 1$.

Definition 2.4. Let $\pi$ be a regular algebraic automorphic representation of $\mathrm{GL}_n(\mathbb{A}_L)$ of weight $\boldsymbol{\lambda}$. We say that $\pi$ is $\iota$-ordinary if for each place $v$ of $L$ dividing $l$, there exist an integer $c \geqslant 1$ and a line inside $\iota^{-1}\pi_v^{\mathrm{Iw}_c(v)}$ which is invariant under each operator $U_{\boldsymbol{\lambda},v}^j$, and such that the eigenvalues of these operators on this line are all $l$-adic units.

Lemma 2.5. *Let $\pi$ be a regular algebraic automorphic representation of $\mathrm{GL}_n(\mathbb{A}_L)$, and let $v$ be a place of $L$ dividing $l$.*

(1) *If $L = E$ and $\pi$ is RACSDC, then the eigenvalues of $U_{\boldsymbol{\lambda},v}^j$ on $\iota^{-1}\pi_v^{\mathrm{Iw}_c(v)}$ are integral.*

(2) *Let $\pi_{v,N}$ denote the normalized Jacquet module with respect to the standard Borel subgroup, and suppose that $\pi_{v,N}^{T_c(v)} \neq 0$. Then $\iota^{-1}\pi_v$ is a subquotient of a representation $\sigma = \text{n-Ind}_B^{\mathrm{GL}_n} \alpha_1 \otimes \dots \otimes \alpha_n$, for some characters $\alpha_i : L_v^\times \to \overline{\mathbb{Q}}_l^\times$ such that $\mathrm{val}(\alpha_1(\varpi_v)) \leqslant \mathrm{val}(\alpha_2(\varpi_v)) \leqslant \dots \leqslant \mathrm{val}(\alpha_n(\varpi_v))$. If $\pi$ is $\iota$-ordinary, then $\mathrm{val}(\alpha_1(\varpi_v)) < \dots < \mathrm{val}(\alpha_n(\varpi_v))$ and $\iota^{-1}\pi_v$ is the unique generic subquotient of $\sigma$.*

*Conversely, if $\mathrm{val}(\alpha_1(\varpi_v)) < \dots < \mathrm{val}(\alpha_n(\varpi_v))$ and $\pi_v$ is generic, let $u_{\boldsymbol{\lambda},v}^j$ denote an eigenvalue of $U_{\boldsymbol{\lambda},v}^j$ with smallest valuation. Then $u_{\boldsymbol{\lambda},v}^j \neq 0$ is unique and there is a unique line inside $\iota^{-1}\pi_v^{\mathrm{Iw}_c(v)}$ where $U_{\boldsymbol{\lambda},v}^j$ acts with eigenvalue $u_{\boldsymbol{\lambda}}^j$, $j = 1, \dots, n$. Finally, we have*

$$\mathrm{val}(\alpha_j(\varpi_v)) = \mathrm{val}(u_{\boldsymbol{\lambda}}^j / u_{\boldsymbol{\lambda}}^{j-1}) - 1/e_v \sum_\tau a_{\tau,j},$$

*the sum being over embeddings $\tau : L \hookrightarrow \mathbb{C}$ such that $\iota^{-1}\tau$ induces the place $v$ of $L$.*

*Proof.* The first part follows from Proposition 2.9, Theorem 2.7, and Lemma 2.10. For the second part, we note as in the proof of [Ger, Lemma 5.1.3] that there is, for any admissible representation $\sigma$ of $\mathrm{GL}_n(L_v)$ over $\overline{\mathbb{Q}}_l$, a surjection $p_\sigma : \sigma^{\mathrm{Iw}_c(v)} \to \sigma_N^{T_c(v)}$, where $\sigma_N$ denotes the normalized Jacquet module. The kernel of this map is given by the subspace where some operator $U_{\boldsymbol{\lambda},v}^j$ does not act invertibly, and we have the formula for all $x \in \sigma^{\mathrm{Iw}_c(v)}$:

$$p_\sigma(U_{\boldsymbol{\lambda},v}^j x) = q_v^{\sum_{i=1}^j (n-1)/2-(i-1)} \prod_\tau \iota^{-1}\tau(\varpi_v)^{-\sum_{i=1}^j \lambda_{\tau,n+1-i}} p_\sigma(x).$$

In particular, if $\pi_{v,N}^{T_c(v)} \neq 0$, as in the statement of the lemma, then $\iota^{-1}\pi_v$ is a subquotient of a representation $\sigma = \text{n-Ind}_B^{\mathrm{GL}_n} \alpha_1 \otimes \dots \otimes \alpha_n$, and $\sigma_N^{\mathrm{ss}} = \oplus_{w \in \mathfrak{S}_n} \alpha_{w(1)} \otimes \dots \otimes \alpha_{w(n)}$. The characters $\alpha_1, \dots, \alpha_n$ are uniquely determined, up to permutation, and we suppose that they have been chosen so that $\mathrm{val}(\alpha_1(\varpi)) \leqslant \cdots \leqslant \mathrm{val}(\alpha_n(\varpi))$.

We may decompose $\iota^{-1}\pi_v^{\mathrm{Iw}_c(v)}$ under the algebra $\overline{\mathbb{Q}}_l[U^1_{\boldsymbol{\lambda},v}, \dots, U^n_{\boldsymbol{\lambda},v}]$ as the direct sum of the simultaneous generalized eigenspaces of these operators; the sum of the eigenspaces corresponding to a tuple of non-zero eigenvalues is mapped isomorphically onto $\iota^{-1}\pi_{v,N}^{T_c(v)}$. The tuples appearing in $\sigma_N^{\mathrm{ss}}$ have the form

$$\left(\prod_{i=1}^{j}\left[q_v^{(n-1)/2-(i-1)}\prod_{\tau}\iota^{-1}\tau(\varpi_v)^{-\lambda_{\tau,n+1-i}}\alpha_{w(i)}(\varpi_v)\right]\right)_{j=1}^{n},$$

the $j$th entry having valuation $\sum_{i=1}^{j}(\mathrm{val}(\alpha_{w(i)}(\varpi_v)) + 1/e_v\sum_\tau a_{\tau,i})$. If $\pi$ is $\iota$-ordinary then there exists $w \in \mathfrak{S}_n$ such that $\sum_{i=1}^{j}(\mathrm{val}(\alpha_{w(i)}(\varpi_v)) + 1/e_v\sum_\tau a_{\tau,i}) = 0$ for each $j = 1, \dots, n$, and hence $\mathrm{val}(\alpha_{w(j)}(\varpi_v)) = -1/e_v\sum_\tau a_{\tau,j}$ for each $j = 1, \dots, n$. This implies that $w = 1$ and the $\mathrm{val}(\alpha_j(\varpi_v))$ are distinct.

Suppose now that $\pi$ is not necessarily $\iota$-ordinary, but that the $\mathrm{val}(\alpha_j(\varpi_v))$ are distinct. After the Zelevinsky classification [Zel80], the Jordan–Hölder factors of the representation $\sigma$ appear with multiplicity one, and $\sigma$ has a unique generic subquotient $\rho$, characterized by the following condition: $\rho_N^{\mathrm{ss}}$ is a direct sum of those characters $\alpha_{r_1} \otimes \cdots \otimes \alpha_{r_n}$ such that if $\alpha_{r_i} = |\cdot|\alpha_{r_j}$, some $1 \leqslant i, j \leqslant n$, then $i < j$. If $\alpha_i = |\cdot|\alpha_j$ then $\mathrm{val}(\alpha_i(\varpi_v)) = -f_v + \mathrm{val}(\alpha_j(\varpi_v))$, and hence $i < j$. This certainly holds for the character $\alpha_1 \otimes \cdots \otimes \alpha_n$, which shows that if $\pi$ is $\iota$-ordinary then $\pi_v$ is generic. Conversely, if $\pi_v$ is generic then the character $\alpha_1 \otimes \cdots \otimes \alpha_n$ appears in $\iota^{-1}\pi_{v,N}$. It follows that there is a unique line inside $\iota^{-1}\pi_v^{\mathrm{Iw}_c(v)}$ where the operators $U^j_{\boldsymbol{\lambda},v}$ act with their eigenvalue $u^j_{\boldsymbol{\lambda},v}$ of minimal valuation

$$\mathrm{val}(u^j_{\boldsymbol{\lambda},v}) = \sum_{i=1}^{j}\left(\mathrm{val}(\alpha_i(\varpi_v)) - 1/e_v\sum_\tau a_{\tau,i}\right).$$

This completes the proof of the lemma. □

Lemma 2.6. *Suppose that $\pi_1, \pi_2$ are cuspidal conjugate self-dual automorphic representations of $\mathrm{GL}_{n_1}(\mathbb{A}_E)$ and $\mathrm{GL}_{n_2}(\mathbb{A}_E)$, respectively, where $n_1 + n_2 = n$. Suppose that $\Pi = \pi_1 \boxplus \pi_2$ is regular algebraic. Then the representations $\pi_i|\cdot|^{(n_i-n)/2}$ are regular algebraic, and $\Pi$ is $\iota$-ordinary if and only if $\pi_1|\cdot|^{(n_1-n)/2}, \pi_2|\cdot|^{(n_2-n)/2}$ are $\iota$-ordinary and the following condition on infinity types holds: let $\mathbf{w} = (w_\tau)_{\tau\in\mathrm{Hom}(E,\mathbb{C})} \in \mathfrak{S}_n^{\mathrm{Hom}(E,\mathbb{C})}$ be the element defined in the paragraph before Theorem 2.2. Then $w_\tau$ depends only on the place $v$ of $E$ dividing $l$ induced by the embedding $\iota^{-1}\tau : E \hookrightarrow \overline{\mathbb{Q}}_l$.*

*Proof.* We first establish some notation. Let $v|l$ be a place of $E$. Suppose that $\iota^{-1}\pi_{1,v}$ is the generic subquotient of the representation $\text{n-Ind}_B^{\mathrm{GL}_{n_1}}\beta_1 \otimes \cdots \otimes \beta_{n_1}$, and that $\iota^{-1}\pi_{2,v}$ is the generic subquotient of the representation $\text{n-Ind}_B^{\mathrm{GL}_{n_2}}\gamma_1 \otimes \cdots \otimes \gamma_{n_2}$, where $\mathrm{val}(\beta_1(\varpi_v)) \leqslant \cdots \leqslant \mathrm{val}(\beta_{n_1}(\varpi_v))$ and $\mathrm{val}(\gamma_1(\varpi_v)) \leqslant \cdots \leqslant \mathrm{val}(\gamma_{n_2}(\varpi_v))$. Let $\delta_1, \dots, \delta_n = \beta_1, \dots, \beta_{n_1}, \gamma_1, \dots, \gamma_{n_2}$. Since $\pi_{1,v}$ and $\pi_{2,v}$ are unitary and generic, $\Pi_v$ is the generic subquotient of a representation $\text{n-Ind}_B^{\mathrm{GL}_n}\alpha_1 \otimes \cdots \otimes \alpha_n$, $\mathrm{val}(\alpha_1(\varpi_v)) \leqslant \cdots \leqslant \mathrm{val}(\alpha_n(\varpi_v))$ and $\{\alpha_1, \dots, \alpha_n\} = \{\delta_1, \dots, \delta_n\}$.

Similarly, let $\mathbf{b}, \mathbf{c}$ denote the infinity types of $\pi_1$ and $\pi_2$ respectively, and define $\mathbf{d}$ by $d_{\tau,1}, \dots, d_{\tau,n} = b_{\tau,1}, \dots, b_{\tau,n_1}, c_{\tau,1}, \dots, c_{\tau,n_2}$. If $\tau : E \hookrightarrow \mathbb{C}$ is an embedding such that $\iota^{-1}\tau$ induces the place $v$ of $E$, then the Weyl group element $w_\tau$ is defined by the condition that $d_{\tau,w_\tau(i)} = a_{\tau,i}$, where $\mathbf{a}$ is the infinity type of $\Pi$.

We now come to the proof of the lemma. Suppose first that $\Pi$ is $\iota$-ordinary. Then the $\mathrm{val}(\alpha_i(\varpi_v)) = -1/e_v \sum_\tau a_{\tau,i}$ are distinct. We can therefore define a permutation $w_v$, uniquely determined by $\pi_1$ and $\pi_2$, by the formula $\delta_{w_v(i)} = \alpha_i$. We show that $w_v = w_\tau$ for each $\tau$ as above. Suppose for contradiction that $w_v \neq w_\tau$ for some $\tau$, and let $j$ be minimal with the property that $w_v(j+1) \neq w_\tau(j+1)$ for some $\tau$. Suppose that

$$\{\delta_{w_v(1)}, \dots, \delta_{w_v(j)}\} = \{\beta_1, \dots, \beta_r, \gamma_1, \dots, \gamma_s\}.$$

Since $\Pi$ is $\iota$-ordinary,

$$\begin{aligned}\min(\mathrm{val}(\beta_{r+1}(\varpi_v)), \mathrm{val}(\gamma_{s+1}(\varpi_v))) = \mathrm{val}(\delta_{w_v(j+1)}) &= -1/e_v \sum_\tau d_{\tau, w_\tau(j+1)} \\ &= -1/e_v \sum_\tau \max(b_{\tau,r+1}, c_{\tau,s+1}).\end{aligned}$$

Suppose that $\mathrm{val}(\beta_{r+1}(\varpi_v)) < \mathrm{val}(\gamma_{s+1}(\varpi_v))$. We have $\mathrm{val}(\beta_i(\varpi_v)) = -1/e_v \sum_\tau b_{\tau,i}$ for each $i = 1, \dots, r$, so the previous lemma implies that $\mathrm{val}(\beta_{r+1}(\varpi_v)) \geqslant -1/e_v \sum_\tau b_{\tau,r+1}$, and hence $\sum_\tau \max(b_{\tau,r+1}, c_{\tau,s+1}) \leqslant \sum_\tau b_{\tau,r+1}$. Since $\Pi$ is regular algebraic, for each $\tau$ we have $b_{\tau,r+1} \neq c_{\tau,s+1}$ so equality holds and $w_\tau(j+1) = w_v(j+1) = r+1$. Similarly, if $\mathrm{val}(\beta_{r+1}(\varpi_v)) > \mathrm{val}(\gamma_{s+1}(\varpi_v))$ then we deduce that $w_\tau(j+1) = w_v(j+1) = s+1$, a contradiction.

We therefore have $w_v = w_\tau$ for each $\tau$, and $\mathrm{val}(\beta_j(\varpi_v)) = -1/e_v \sum_\tau b_{\tau,j}$, $j = 1, \dots, n_1$, and $\mathrm{val}(\gamma_j(\varpi_v)) = -1/e_v \sum_\tau c_{\tau,j}$, $j = 1, \dots, n_2$. This implies that $\pi_1|\cdot|^{(n_1-n)/2}, \pi_2|\cdot|^{(n_2-n)/2}$ are $\iota$-ordinary.

Suppose conversely that $\pi_1|\cdot|^{(n_1-n)/2}, \pi_2|\cdot|^{(n_2-n)/2}$ are $\iota$-ordinary and that the condition on infinity types holds. We see that for each $j = 1, \dots, n$, $\mathrm{val}(\alpha_j(\varpi_v)) = \mathrm{val}(\delta_{w_v(j)}) = -1/e_v \sum_\tau d_{\tau, w_\tau(j)} = -1/e_v \sum_\tau a_{\tau,j}$. It now follows from Lemma 2.5 that $\Pi$ is also $\iota$-ordinary. ☐

### 2.3 Soluble base change for $\mathrm{GL}_n$

Let $E$ be an imaginary CM field with totally real subfield $F$. We suppose that $L/E$ is a soluble CM extension. Recall that the base change $\pi_L$ of a cuspidal representation $\pi$ of $\mathrm{GL}_n(\mathbb{A}_E)$, an automorphic representation, always exists. We also fix a prime $l$ and an isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$.

Theorem 2.7. (1) *Let $\pi$ be a RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$, and suppose that $r_\iota(\pi)|_{G_L}$ is irreducible. Then there exists a RACSDC automorphic representation $\pi_L$ of $\mathrm{GL}_n(\mathbb{A}_L)$ such that $r_\iota(\pi)|_{G_L} \cong \pi_L$.*

(2) *Suppose that $\rho : G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ is a continuous representation such that $\rho|_{G_L}$ is irreducible, and that there exists a RACSDC automorphic representation $\Pi$ of $\mathrm{GL}_n(\mathbb{A}_L)$ such that $\rho|_{G_L} \cong r_\iota(\Pi)$. Then there exists a RACSDC automorphic representation $\pi$ of $\mathrm{GL}_n(\mathbb{A}_E)$ such that $\Pi \cong \pi_L$.*

(3) *Let $\pi$ be a RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ such that $\pi_L$ is cuspidal. Then $\pi$ is $\iota$-ordinary if and only if $\pi_L$ is $\iota$-ordinary.*

*Proof.* For the first part, the existence of $\pi_L$ follows from [AC89, Theorem 4.2]. To see that $\pi_L$ is cuspidal, we reduce to the case $L/E$ cyclic of prime order. If $\pi_L$ fails to be cuspidal then there is an isomorphism $\pi \otimes \epsilon \cong \pi$, where $\epsilon$ is an Artin character associated to $L/E$. This implies that $r_\iota(\pi)|_{G_L}$ is reducible, a contradiction. The second part follows from [BGHT11, Lemma 1.4]. The third part follows from [Ger, Lemma 5.1.6]. ☐

### 2.4 Definite unitary groups

Let $E$ be an imaginary CM field with totally real subfield $F$. We now suppose that $E/F$ is everywhere unramified and that $[F:\mathbb{Q}]$ is even. Let $G$ be a unitary group in $n$ variables associated to the extension $E/F$, quasi-split at every finite place, such that $G(\mathbb{R})$ is compact. Such a group exists since $[F:\mathbb{Q}]$ is even, and is uniquely determined up to isomorphism. We can choose the matrix algebra $B = M_n(E)$ and an involution $\dagger$ of $B$ of the second kind, so that $G$ is defined by

$$G(R) = \{g \in (B \otimes_F R) \mid g^\dagger g = 1\}$$

for any $F$-algebra $R$. We may choose an order $\mathcal{O}_B \subset B$, stable under $\dagger$, so that $\mathcal{O}_{B,w}$ is maximal for any place $w$ of $E$ split over $F$. This defines an integral model of $G$ over $\mathcal{O}_F$, and for any place $v$ of $F$ split as $v = ww^c$ in $E$, we can choose an isomorphism

$$\mathcal{O}_B \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v} \cong M_n(\mathcal{O}_{E_w}) \times M_n(\mathcal{O}_{E_{w^c}}),$$

such that $\dagger$ acts as $(g_1, g_2) \mapsto (g_2, {}^t g_1)$. Projection onto the first factor induces an isomorphism $\iota_w : G(F_v) \to \mathrm{GL}_n(E_w)$ such that $\iota_w(G(\mathcal{O}_{F_v})) = \mathrm{GL}_n(\mathcal{O}_{E_w})$.

Let $l$ be a prime, and suppose that every prime of $F$ above $l$ splits in $E$. Let $S_l$ denote the set of primes of $F$ above $l$. We choose a prime $\widetilde{v}$ of $E$ above $v$ for each $v \in S_l$, and let $\widetilde{S}_l$ denote the set of these primes. Then, as above, we are given an isomorphism $\iota_{\widetilde{v}} : G(F_v) \to \mathrm{GL}_n(E_{\widetilde{v}})$. We write $I_l$ for the set of embeddings $F \hookrightarrow \overline{\mathbb{Q}}_l$, and $\widetilde{I}_l$ for the set of embeddings $E \hookrightarrow \overline{\mathbb{Q}}_l$ inducing an element of $\widetilde{S}_l$. These two sets are therefore in canonical bijection.

Let $K \subset \overline{\mathbb{Q}}_l$ be a finite extension of $\mathbb{Q}_l$, with ring of integers $\mathcal{O}$ and residue field $k$. We suppose that $K$ contains the image of $E$ under every embedding $E \hookrightarrow \overline{\mathbb{Q}}_l$. To a tuple $\boldsymbol{\lambda} = (\lambda_{\tau,1}, \ldots, \lambda_{\tau,n})_{\tau \in \widetilde{I}_l}$ of dominant weights of $\mathrm{GL}_n$, we associate a representation $M_{\boldsymbol{\lambda}}$ of the group $\prod_{v \in S_l} G(\mathcal{O}_{F_v})$ as in [Ger, Definition 2.2.3]. It is an $\mathcal{O}$-lattice inside the representation $W_{\boldsymbol{\lambda}} = \otimes_{\tau \in \widetilde{I}_l} (W_{\lambda_\tau} \otimes_{F_v,\tau} K)$, where $W_{\lambda_\tau}$ is the algebraic representation of $\mathrm{GL}_n(F_v)$ of highest weight $\lambda_\tau$, and $v$ is the place of $F$ induced by $\tau$.

Fix $\boldsymbol{\lambda}$ and an open compact subgroup $U = \prod_v U_v \subset G(\mathbb{A}_F^\infty)$, such that $U_v \subset G(\mathcal{O}_{F_v})$ for each $v \in S_l$. Let $A$ be an $\mathcal{O}$-algebra. We can then define a space of automorphic forms with $A$-coefficients as follows. By definition, $S_{\boldsymbol{\lambda}}(U, A)$ is the set of functions $f : G(F)\backslash G(\mathbb{A}_F^\infty) \to M_{\boldsymbol{\lambda}} \otimes_{\mathcal{O}} A$ such that for all $u \in U$, we have $f(gu) = u_l^{-1} \cdot f(g)$. Here $u_l$ denotes the projection of $u$ to its $\prod_{v \in S_l} G(\mathcal{O}_{F_v})$-component. The relation with classical automorphic forms is given by the following result. Let $\mathcal{A}$ denote the space of automorphic forms on $G(F)\backslash G(\mathbb{A})$, and let $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$ be an isomorphism. There is an algebraic representation $W_{\iota\boldsymbol{\lambda}}$ of $G(F \otimes_{\mathbb{Q}} \mathbb{R})$, defined by the formula $\otimes_{\tau \in \widetilde{I}_l} W_{\lambda_\tau} \otimes_{F_v, \iota\tau} \mathbb{C}$.

Proposition 2.8. *There is a canonical isomorphism*

$$\left(\varinjlim_{U} S_{\boldsymbol{\lambda}}(U, K)\right) \otimes_{K,\iota} \mathbb{C} \cong \mathrm{Hom}_{G(F \otimes_{\mathbb{Q}} \mathbb{R})}(W_{\iota\boldsymbol{\lambda}}^\vee, \mathcal{A}).$$

*In particular, for any irreducible subrepresentation $\sigma \subset \mathcal{A}$, we have a canonical subspace $\iota^{-1}(\sigma^\infty)^U \subset S_{\boldsymbol{\lambda}}(U, \overline{\mathbb{Q}}_l)$.*

*Proof.* This can be proved exactly as in the proof of [CHT08, Proposition 3.3.2]. □

If $\pi$ is an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ and $\sigma$ is an automorphic representation of $G(\mathbb{A}_F)$, we say that $\pi$ is the base change of $\sigma$ if for any finite place $w$ of $E$, the following condition is satisfied.

- If $w$ is split over the place $v$ of $F$, then $\pi_w = \sigma_v \circ \iota_w$ is the standard base change of $\sigma_v$.
- If $w$ is inert over the place $v$ of $F$ and $\sigma_v$ is unramified, then $\pi_w$ is the standard unramified base change of $\sigma_v$; cf. [Mín11, § 4.1].

Proposition 2.9. (1) *Suppose that $\sigma$ is an automorphic representation of $G(\mathbb{A}_F)$. Then there exist a partition $n = n_1 + \cdots + n_s$ and, for each $i = 1, \ldots, s$, a discrete, conjugate self-dual representation $\pi_i$ of $\mathrm{GL}_{n_i}(\mathbb{A}_E)$ such that $\pi = \pi_1 \boxplus \cdots \boxplus \pi_s$ is the base change of $\sigma$ in the above sense.*

(2) *Suppose that $\pi$ is a RACSDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ such that if $\pi_w$ is ramified, then $w$ is split over $F$. Then there exists an automorphic representation $\sigma$ of $G(\mathbb{A}_F)$ such that $\pi$ is the base change of $\sigma$ in the above sense.*

(3) *Suppose that $\pi = \pi_a \boxplus \pi_b$ is a RACSD automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$, where $\pi_a$, $\pi_b$ are cuspidal, conjugate self-dual automorphic representations of $\mathrm{GL}_a(\mathbb{A}_E)$ and $\mathrm{GL}_b(\mathbb{A}_E)$, respectively. We assume the following hypotheses.*

- *Let $\mathbf{w} = (w_\tau)_{\tau \in \mathrm{Hom}(E,\mathbb{C})}$ denote the Weyl group element associated to the infinity types of $\pi_a$, $\pi_b$. For each place $v|\infty$ of $F$, choose an embedding $\tau(v) : E \hookrightarrow \mathbb{C}$ inducing $v$. Then $\prod_v \mathrm{sgn}(w_{\tau(v)}) = 1$.*
- *$ab$ is even and $a \neq b$.*
- *If $\pi_w$ is ramified then $w$ is split over $F$.*

*Then there exists an automorphic representation $\sigma$ of $G(\mathbb{A}_F)$ such that $\pi$ is the base change of $\sigma$ in the above sense.*

*Proof.* The first part follows from [Lab11, Corollaire 5.3]. The second part follows from [Lab11, Théorème 5.4]. We now prove the third part. We will use arguments given in more detail in the companion paper [CT], to which we refer in part.

First consider the quasi-split inner form $G^*$ of $G$. By [Mok, Theorem 2.5.2], there are representations $\sigma = \otimes_v \sigma_v$ of $G^*(\mathbb{A}_F)$ occurring in $L^2_{\mathrm{disc}}(G^*(F)\backslash G^*(\mathbb{A}_F))$ such that at any finite place $\sigma_v$ is associated (by standard, i.e. stable, base change) to $\pi_v = \otimes_{w|v} \pi_w$. Each such representation $\sigma$ occurs with multiplicity 1.

In fact, note that $\pi_\infty = \otimes_{w|\infty} \pi_w$ determines, for each place $v|\infty$ of $F$, an $L$-packet $\Pi_v$ of discrete series of $G^*(F_v)$. On the other hand, the datum $(\pi_a, \pi_b)$ defines a parameter $\psi$ in the sense of Arthur and Mok [Mok, § 2.3]. There is an associated finite group $\mathcal{S}_\psi$ (see [Mok, Definition 1.4.8]); in our case, it is equal to $\{\pm 1\}$. There is a pairing, at all Archimedean primes $v$, between $\mathcal{S}_\psi$ and $\Pi_v$, which determines a sign $\varepsilon_\psi(\sigma_v)$, $\sigma_v \in \Pi_v$ (see [Mok, Theorem 2.5.1]).

Now $\sigma = \sigma_\infty \otimes \sigma^\infty$ occurs, with multiplicity 1, if and only if

$$\prod_{v|\infty} \varepsilon_\psi(\sigma_v) = 1.$$

(We have used the fact that $\varepsilon_\psi(\sigma_v) = 1$ for $v$ finite and $\sigma_v$ unramified; cf. [Mok, Theorem 2.5.1]. Recall that $E/F$ is, by assumption, everywhere unramified.)

We now want to transfer (some) $\sigma$ to $G(\mathbb{A})$. The proof is similar to the proof of [CT, Theorem 3.11], but simpler as we do not have to obtain specific local components (different from the unramified ones) at finite places. Let $f^* = \otimes_v f^*_v$, $f = \otimes_v f_v$ be decomposed, smooth, compactly supported functions on $G^*(\mathbb{A}_F)$, $G(\mathbb{A}_F)$. By [Art03], we have identities

$$T^{G^*}_{\mathrm{disc}}(f^*) = \sum_{\mathcal{E}} \iota(G^*, \mathcal{E}) ST^{\mathcal{E}}_{\mathrm{disc}}(f^*_{\mathcal{E}}), \tag{1}$$

$$T^{G}_{\mathrm{disc}}(f) = \sum_{\mathcal{E}} \iota(G, \mathcal{E}) ST^{\mathcal{E}}_{\mathrm{disc}}(f_{\mathcal{E}}), \tag{2}$$

where $\mathcal{E}$ runs over the endoscopic data for $G$ and $G^*$. Recall that these are simply the products $U(n_1) \times U(n_2)$ where $n = n_1 + n_2$ and $U(m)$ is the quasi-split unitary group of rank $m$. In fact, by a simple argument of separation of eigenvalues for the Hecke algebra, only the groups $G^*$ and $H = U(a) \times U(b)$ intervene in the calculation; cf. [CT, § 3.8].

The terms on the right are stable traces; the functions $f, f^*$ determine functions $f_{G^*}, f_H$ and $f^*_H$ on $G^*, H$. Again, after separation of the Hecke eigenvalues associated to $\psi$, it is easy to see that the left-hand side of (1) reduces to the trace in the ($\psi$-part of) $L^2_{\mathrm{disc}}(G^*(F)\backslash G^*(\mathbb{A}_F))$; this is trivial for the $\mathbb{R}$-anisotropic form.

Since $G(F_v) \cong G^*(F_v)$ at finite places, we of course take $f_v = f^*_v = (f_{G^*})_v$ at these places. The assignments $f^* \rightsquigarrow f^*_H$, $f \rightsquigarrow f_H$ depend on a choice of transfer factors. At a finite place $v$, in order to use Mok's results, we must use the 'Whittaker-normalized' factors; see [Mok] as well as [CT, § 3.5]. These are the Langlands–Shelstad factors of [LS87], multiplied by a sign $\varepsilon(V, \psi)$. Here (see [CT, § 3.5]), $V = V_G - V_H$ is a virtual representation of $\mathrm{Gal}(\overline{F}_v/F_v)$. If $ab$ is even it is easy to see that $V_G = V_H$. The functions $f_H$, $f^*_H$ coincide at finite places.

Consider now the functions $f_{H,\infty}$ and $f^*_{H,\infty}$. The datum $(\pi_a, \pi_b)$ determines by descent an $L$-packet $\Pi_v(H)$ of discrete series for $H(F_v)$, for each $v|\infty$; after separation of Hecke eigenvalues, only representations of $H(F_\infty)$ of this type occur in the right-hand side of (1) and (2).

Since $G(F_\infty)$ is compact, the datum of $(\pi_a, \pi_b)$ (with $\pi_a \boxplus \pi_b$ RACSD) also determines a unique irreducible representation $\sigma'_\infty = \otimes_v \sigma'_v$ of $G(F_\infty)$. Let $f_\infty$ be a coefficient of $\sigma'_\infty$, with $\operatorname{tr} \sigma'(f_\infty) = 1$; let $\sigma = \otimes_v \sigma_v$ be a representation of $G^*(F_\infty)$ in the pertinent $L$-packet, and $f_\sigma$ a pseudo-coefficient of $\sigma$. As $[F : \mathbb{Q}]$ is even, $f_\infty$ and $\otimes_{v|\infty} f_\sigma$ are associated. Moreover, the functions $f_{H,\infty}, f^*_{H,\infty}$ satisfy the identities

$$\langle \operatorname{tr} \Pi_\infty(H), f_{H,\infty} \rangle = \prod_v \operatorname{sgn}(w_{\tau(v)}) \langle \operatorname{tr} \sigma'_\infty, f_\infty \rangle = 1,$$

by our first assumption in the third part of the proposition. This follows from [Clo11] (cf. [CT, § 3.6]), on checking that the product of the Langlands–Shelstad transfer factors coincides with the product of the Kottwitz transfer factors: use the fact that $ab$ and $[F : \mathbb{Q}]$ are even.

The other identity is

$$\langle \operatorname{tr} \Pi_\infty(H), f^*_{H,\infty} \rangle = \prod_v \varepsilon_\psi(\sigma_v);$$

see [Mok, Theorem 3.2.1]. Now assume that $\sigma_\infty = \otimes_v \sigma_v$ is a possible factor for $G^*$. In the spaces cut out by $\psi$, the right-hand sides of (1), (2) then coincide term by term, and therefore $\sigma_G = \sigma'_\infty \otimes (\otimes_{v\nmid\infty} \sigma_v)$ occurs for $G$. □

Let $U = \prod_v U_v$ be an open compact subgroup as above, and suppose that there exists an integer $c \geqslant 1$ such that for each $v \in S_l$, $U_v = \iota^{-1}_{\widetilde{v}} \operatorname{Iw}_c(\widetilde{v})$. For each prime $v \in S_l$, fix a uniformizer $\varpi_{\widetilde{v}}$ of $\mathcal{O}_{E_{\widetilde{v}}}$, and define the matrix

$$\alpha^j_v = \operatorname{diag}(\underbrace{\varpi_{\widetilde{v}}, \dots, \varpi_{\widetilde{v}}}_{j}, \underbrace{1, \dots, 1}_{n-j}).$$

We define an endomorphism $U^j_{\boldsymbol{\lambda},v}$ of the space $S_{\boldsymbol{\lambda}}(U,\mathcal{O})$ by the formula

$$U^j_{\boldsymbol{\lambda},v} = \prod_{\tau} \iota^{-1}\tau(\varpi_{\widetilde{v}})^{-(\lambda_{\tau,n}+\cdots+\lambda_{\tau,n+1-j})}\iota_{\widetilde{v}}^{-1}[\mathrm{Iw}_c(\widetilde{v})\alpha^j_v \,\mathrm{Iw}_c(\widetilde{v})],$$

the product running over the embeddings $\tau : E \hookrightarrow \mathbb{C}$ such that $\iota^{-1}\tau$ induces the place $\widetilde{v}$ of $E$. These operators obviously act on $S_{\boldsymbol{\lambda}}(U,K)$. In fact, they preserve the integral lattice $S_{\boldsymbol{\lambda}}(U,\mathcal{O})$, by the remark after [Ger, Definition 2.3.1].

LEMMA 2.10. *If $\sigma$ is an irreducible subrepresentation of $\mathcal{A}$ such that $(\sigma^\infty)^U \neq 0$ and $\sigma_\infty \cong W^\vee_{\iota\boldsymbol{\lambda}}$, then the eigenvalues of $U^j_{\boldsymbol{\lambda},v}$ on $\iota^{-1}(\sigma^\infty)^U$ are integral.*

*Proof.* This follows immediately from the above remarks. □

## 3. Congruences and functoriality

In this section we formulate some conjectures about automorphic forms which are related to conjugate self-dual Galois representations. Since we mostly take the point of view of Galois representations, rather than automorphic forms, we formulate these using a Galois-theoretic language, rather than, for example, the automorphic language of [Clo90].

The conjectures below are stated in the context of an imaginary CM field $E$ with totally real subfield $F$, and automorphic representations $\pi_1, \pi_2, \ldots$. When we state later that we will assume that a given conjecture holds, we mean that it holds for all choices of $E/F$ and automorphic representations satisfying the given conditions.

### 3.1 Level raising

We put ourselves in the situation of § 2.4. Thus $G$ is a definite unitary group in $n$ variables associated to a CM extension $E/F$. Fix an irreducible $G(\mathbb{A}_F)$-subrepresentation $\sigma$ of the space $\mathcal{A}$ with $\sigma_\infty \cong W^\vee_{\iota\boldsymbol{\lambda}}$ for some dominant weight $\boldsymbol{\lambda}$ and isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$. By [Gue11, Theorem 2.3], there exists a continuous semisimple representation $r_\iota(\sigma) : G_E \to \overline{\mathbb{Q}}_l$ satisfying the relation $\mathrm{WD}(r_\iota(\sigma)|_{G_{E_w}})^{\text{F-ss}} \cong \mathrm{rec}^T_{E_w}(\iota^{-1}\sigma_v \circ \iota_w)$ for every place $w$ of $E$ split over $F$. Let $w_0$ be such a place, and let $v_0$ be the place of $F$ below it. If $\sigma_{v_0} \circ \iota_{w_0}$ has an Iwahori-fixed vector and $w_0$ does not divide $l$, then $r_\iota(\sigma)|^{\mathrm{ss}}_{G_{E_{w_0}}}$ is unramified. We say that $\sigma$ satisfies the level-raising congruence at $w_0$ if the eigenvalues $\alpha_1, \ldots, \alpha_n$ of $r_\iota(\sigma)|^{\mathrm{ss}}_{G_{E_{w_0}}}(\mathrm{Frob}_{w_0})$ satisfy $\alpha_i \equiv \alpha_1 q^{n-i}_{w_0} \bmod \mathfrak{m}_{\overline{\mathbb{Z}}_l}$, up to reordering, where $\mathfrak{m}_{\overline{\mathbb{Z}}_l} \subset \overline{\mathbb{Z}}_l$ is the unique maximal ideal.

CONJECTURE 3.1 ($\mathrm{LR}_n$). Suppose that $\sigma$ is $\iota$-ordinary and that the irreducible constituents of the residual representation $\overline{r_\iota(\sigma)}$ have pairwise distinct dimensions. Suppose further that $\sigma$ satisfies the level-raising congruence at the place $w_0$.

Let $U = \prod_v U_v \subset G(\mathbb{A}^\infty_F)$ be an open compact subgroup with $(\sigma^\infty)^U \neq 0$, and such that for some finite place $v$ of $F$, $U_v$ contains no non-trivial elements of finite order. Then there exists a second automorphic representation $\sigma_1$ of $G(\mathbb{A}_F)$ satisfying the following.

- $\sigma_{1,\infty} \cong W^\vee_{\iota\boldsymbol{\lambda}}$.
- $\overline{r_\iota(\sigma)} \cong \overline{r_\iota(\sigma_1)}$.
- $\sigma_1$ is $\iota$-ordinary and $(\sigma_1^\infty)^U \neq 0$.
- $\sigma_{1,v_0} \circ \iota_{w_0}$ is an unramified twist of the Steinberg representation.

This conjecture is closely related to Ihara's lemma (see, for example, [CHT08, Conjecture B]). It is known in some cases when $n \leqslant 3$, or when $l$ is a banal characteristic for $\mathrm{GL}_n(E_{w_0})$;

cf. [Thob]. We have chosen to restrict the statement to $\iota$-ordinary representations since this is all we require here for the application to symmetric power functoriality, and since we believe that this may be easier than the most general case. In fact, it would even suffice for our purposes to treat the case where $q_{w_0} \equiv 1 \bmod l$, $\sigma_{v_0}$ is unramified, and $\overline{r_\iota(\sigma)}|_{G_{E_{w_0}}}$ is trivial.

### 3.2 Automorphic tensor product

Let $n \geqslant 1$ be a positive integer, and suppose that $E$ is an imaginary CM field with totally real subfield $F$, and that $(\pi_1, \psi_1)$ and $(\pi_2, \psi_2)$ are RAECSDC automorphic representations of $\mathrm{GL}_2(\mathbb{A}_E)$ and $\mathrm{GL}_n(\mathbb{A}_E)$, respectively. We will state here a version of the conjectural $\mathrm{GL}_2 \times \mathrm{GL}_n \to \mathrm{GL}_{2n}$ lifting that we hope will be accessible through Galois-theoretic methods.

Conjecture 3.2 ($\mathrm{TP}_n$). Fix a prime $l$ and an isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$. Suppose that the representation $r_\iota(\pi_1) \otimes r_\iota(\pi_2)$ is irreducible and Hodge–Tate regular. Then there exists a RAECSDC automorphic representation $(\pi, \chi)$ of $\mathrm{GL}_{2n}(\mathbb{A}_E)$ such that $r_\iota(\pi) \cong r_\iota(\pi_1) \otimes r_\iota(\pi_2)$.

This conjecture is known to be true if $n = 2$ or if $n = 3$ (see [Ram00, KS02], respectively). In addition, a 'potential' version of this conjecture follows in many cases from potential automorphy theorems; cf. [BGGT14].

### 3.3 Automorphic symmetric power

Now suppose that $F$ is a totally real field, and that $(\pi, \chi)$ is a RAESDC automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$, without CM, i.e. not induced from an algebraic Grössencharacter of a CM quadratic extension of $F$. Let $n \geqslant 2$ be an integer, and let $\mathbf{K} = \{K_1, \ldots, K_s\}$ be a set of finite Galois extensions of $\mathbb{Q}$.

Conjecture 3.3 ($\mathrm{SP}_{n+1}(\mathbf{K})$). Suppose that $F$ does not contain $K_i$, for any $i = 1, \ldots, s$. Then the $n$th symmetric power lifting of $\pi$ exists, in the following sense: there exists a RAESDC automorphic representation $(\Pi, \psi)$ of $\mathrm{GL}_{n+1}(\mathbb{A}_F)$ such that for any isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, there is an isomorphism $\mathrm{Sym}^n r_\iota(\pi) \cong r_\iota(\Pi)$.

We remark that if $\mathbf{K} \subset \mathbf{K}'$, then $\mathrm{SP}_{n+1}(\mathbf{K}) \Rightarrow \mathrm{SP}_{n+1}(\mathbf{K}')$. This conjecture is known to be true with $\mathbf{K} = \emptyset$ if $n = 2, 3$ or $4$ (see [GJ78, Kim03, KS02], respectively). The 'potential' version follows from potential automorphy theorems. (See [BGG11] for the final result, following earlier work by L.C., Harris, Shepherd-Barron and Taylor. The reason for introducing the set $\mathbf{K}$ here is that the automorphy lifting theorems to be used later require supplementary hypotheses on the presence of roots of unity in the base field $F$.)

### 3.4 Main theorem

Let $\mathbf{K} = \{K_1, \ldots, K_s\}$ be a set of finite Galois extensions of $\mathbb{Q}$. We write $\mathbb{Q}(\zeta_l)^+$ for the totally real subfield of $\mathbb{Q}(\zeta_l)$.

Theorem 3.4. *Let $l \geqslant 5$ be prime, and let $0 < r < l$ be an integer. Suppose that $\mathbb{Q}(\zeta_l)^+ \in \mathbf{K}$. Then the following implication holds:*

$$\mathrm{SP}_{l-r}(\mathbf{K}) + \mathrm{SP}_r(\mathbf{K}) + \mathrm{TP}_r + \mathrm{LR}_{l+r} \Rightarrow \mathrm{SP}_{l+r}(\mathbf{K}).$$

The proof of this theorem will be given in §§ 4–5.

Corollary 3.5. *Suppose that $\mathrm{TP}_r$ and $\mathrm{LR}_{r+1}$ hold for all integers $r \geqslant 1$. Let $F$ be a totally real field, and let $(\pi, \chi)$ be a RAESDC automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$, not automorphically induced from a quadratic CM extension.*

*Suppose that if $l \geqslant 5$ is prime, then $[F(\zeta_l) : F] > 2$. Then the symmetric $r$th power lifting of $\pi$ exists for all integers $r \geqslant 1$.*

*Proof of Corollary 3.5.* If $r \geqslant 1$ is an integer, let $\mathbf{K}_r$ denote the set of fields $\mathbb{Q}(\zeta_l)^+$, as $l$ runs over primes $5 \leqslant l \leqslant r$. Under the assumption of hypotheses $\mathrm{TP}_r$ and $\mathrm{LR}_{r+1}$, the above theorem simply gives the implication (whenever $l \geqslant 5$ is prime and $0 < s < l$, and $\mathbb{Q}(\zeta_l)^+ \in \mathbf{K}$)

$$\mathrm{SP}_{l-s}(\mathbf{K}) + \mathrm{SP}_s(\mathbf{K}) \Rightarrow \mathrm{SP}_{l+s}(\mathbf{K}).$$

To prove the corollary, it suffices to prove $\mathrm{SP}_{r+1}(\mathbf{K}_r)$ for all $r \geqslant 1$. We prove this by induction on $r \geqslant 1$. It is already known to hold for $1 \leqslant r \leqslant 4$. For general $r$, note that by Bertrand's postulate there exists a prime $l$ satisfying $(r+1)/2 < l < r+1$, and hence $l < r+1 < 2l$. Writing $r+1 = l+s$, we therefore have $0 < s < l$. The above implication now implies that $\mathrm{SP}_{l+s}(\mathbf{K}_r) = \mathrm{SP}_{r+1}(\mathbf{K}_r)$ holds.

COROLLARY 3.6. *Suppose that $\mathrm{LR}_{r+1}$ holds for all $1 \leqslant r \leqslant 26$. Then $\mathrm{SP}_{r+1}(\mathbf{K}_{25})$ holds for all integers $1 \leqslant r \leqslant 9$, and for all odd integers $1 \leqslant r \leqslant 25$.*

*Proof of Corollary 3.6.* The deduction of this corollary is similar, using the fact that $\mathrm{TP}_r$ and $\mathrm{SP}_r(\emptyset)$ are already known to hold for $1 \leqslant r \leqslant 3$. Indeed, we now have the implications (under $\mathrm{LR}_{r+1}$, and $\mathbb{Q}(\zeta_l)^+ \in \mathbf{K}$)

$$\mathrm{SP}_{l-1}(\mathbf{K}) \Rightarrow \mathrm{SP}_{l+1}(\mathbf{K}), \quad \mathrm{SP}_{l-2}(\mathbf{K}) \Rightarrow \mathrm{SP}_{l+2}(\mathbf{K}) \quad \text{and} \quad \mathrm{SP}_{l-3}(\mathbf{K}) \Rightarrow \mathrm{SP}_{l+3}(\mathbf{K}).$$

The result follows on using the primes $5, 7, \ldots, 23$. □

### 3.5 Lemmas about ordinariness

In certain situations, the functorial operations above preserve the property of being ordinary. This is the content of the results of this section.

LEMMA 3.7. *In the situation of conjecture $\mathrm{TP}_n$, suppose the following.*

- *$\pi_1$ and $\pi_2$ are $\iota$-ordinary.*
- *Let $\mathbf{a}$ and $\mathbf{b}$ denote the infinity types of $\pi_1$ and $\pi_2$, respectively. Then $a_\tau$ and $b_\tau$ depend only on the place of $E$ induced by the embedding $\iota^{-1}\tau : E \hookrightarrow \overline{\mathbb{Q}}_l$.*

*Then $\pi$ is $\iota$-ordinary.*

*Proof.* Let $v|l$ be a place of $E$, and suppose that $\iota^{-1}\pi_{1,v}$ is a subquotient of $\text{n-Ind}_B^{\mathrm{GL}_2} \alpha_1 \otimes \alpha_2$ and $\iota^{-1}\pi_{2,v}$ is a subquotient of $\text{n-Ind}_B^{\mathrm{GL}_n} \beta_1 \otimes \cdots \otimes \beta_n$, where $\mathrm{val}(\alpha_1(\varpi_v)) < \mathrm{val}(\alpha_2(\varpi_v))$ and $\mathrm{val}(\beta_1(\varpi_v)) < \cdots < \mathrm{val}(\beta_n(\varpi_v))$. Since $\pi_1$ and $\pi_2$ are $\iota$-ordinary, we have by Lemma 2.5 the equalities

$$\mathrm{val}(\alpha_i(\varpi_v)) = -1/e_v \sum_\tau a_{\tau,i} \quad \text{and} \quad \mathrm{val}(\beta_j(\varpi_v)) = -1/e_v \sum_\tau b_{\tau,j},$$

the sum being over embeddings $\tau : E \hookrightarrow \mathbb{C}$ such that $\iota^{-1}\tau$ induces the place $v$. In particular, since $r_\iota(\pi_1) \otimes r_\iota(\pi_2)$ is Hodge–Tate regular, the quantities $\mathrm{val}(\alpha_i(\varpi_v)\beta_j(\varpi_v))$ are distinct as $i, j$ vary, and the permutation required to put these quantities in strictly increasing order is the same as the permutation required, for each $\tau$, to put the quantities $a_{\tau,i} + b_{\tau,j}$ in strictly decreasing order. The same argument as in the proof of Lemma 2.6 now gives the conclusion. □

LEMMA 3.8. *In the situation of conjecture $\mathrm{SP}_{n+1}(\mathbf{K})$, suppose that $\pi$ is $\iota$-ordinary. Then $\Pi$ is $\iota$-ordinary.*

*Proof.* The proof is essentially the same as the proof of Lemma 3.7. □

## 4. Construction of a special automorphic representation

Let $E$ be an imaginary CM field with totally real subfield $F$ such that $E/F$ is everywhere unramified and $[F:\mathbb{Q}]$ is even. Suppose that $\pi$ is a RACSDC automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$ of weight $\boldsymbol{\lambda}=0$. Let $l \geqslant 5$ be a prime, and let $0<r<l$. Set $n=l+r$.

We fix a choice of isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$. In order to reduce notation, we now write $\rho = r_\iota(\pi)$. We suppose that the following hypotheses are in effect.

- Every prime of $F$ dividing $l$ or above which $\pi$ is ramified is split in $E$.
- $\pi$ is $\iota$-ordinary.
- The residual representation $\overline{\rho} : G_E \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$ is irreducible, and its image contains $\mathrm{SL}_2(\mathbb{F}_{l^a})$ up to conjugation, for some $a>1$.
- There exist RACSDC automorphic representations $\Pi_1$, $\Pi_2$ of the groups $\mathrm{GL}_r(\mathbb{A}_E)$ and $\mathrm{GL}_{l-r}(\mathbb{A}_E)$, respectively, such that $r_\iota(\Pi_1) \cong \mathrm{Sym}^{r-1}\rho$ and $r_\iota(\Pi_2) \cong \mathrm{Sym}^{l-r-1}\rho$. (These Galois representations are irreducible, by the previous hypothesis.)
- There exists a place $w_0$ of $E$, split over $F$ and coprime to $l$, such that $\pi_{w_0}$ is an unramified twist of the Steinberg representation. We write $v_0$ for the place of $F$ below $w_0$.

In this case we note that there is an isomorphism of residual representations

$$(\mathrm{Sym}^{l+r-1}\overline{\rho})^{\mathrm{ss}} \cong ({}^{\varphi}\overline{\rho} \otimes \mathrm{Sym}^{r-1}\overline{\rho}) \oplus \overline{\chi}^r \,\mathrm{Sym}^{l-r-1}\overline{\rho},$$

where $\varphi$ denotes a lift to $\overline{\mathbb{Q}}_l$ of the arithmetic Frobenius, and $\chi = \det \rho$. (This follows from the corresponding identity of representations of $\mathrm{GL}_2(\overline{\mathbb{F}}_l)$, which can be seen by calculating the trace on either side of an upper-triangular element.) The two summands here are irreducible, and each of different dimension, prime to $l$. We remark that ${}^{\varphi}\overline{\rho}$ is already the residual representation of a RACSDC automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$ of weight zero, by Lemma 2.3 and [Clo90, Proposition 4.12], which describes the action of Galois on infinity types.

Proposition 4.1. *Suppose that conjecture* $\mathrm{TP}_r$ *holds. Then there exist cuspidal conjugate self-dual automorphic representations* $\sigma_1, \sigma_2$ *of* $\mathrm{GL}_{2r}(\mathbb{A}_E)$ *and* $\mathrm{GL}_{l-r}(\mathbb{A}_E)$*, respectively, and satisfying the following.*

- $\Sigma = \sigma_1 \boxplus \sigma_2$ *is regular algebraic and* $\iota$*-ordinary of weight zero.*
- *The representation* $\Sigma_{w_0}$ *has an Iwahori-fixed vector.*
- *There is an isomorphism of residual representations*

$$\overline{r_\iota(\Sigma)} \cong (\mathrm{Sym}^{l+r-1}\overline{\rho})^{\mathrm{ss}}.$$

- *If* $\Sigma_w$ *is ramified then* $\pi_w$ *is ramified.*

*Proof.* Consider the following conditions on a RAECSDC automorphic representation $(\pi', |\cdot|^{1-l})$.

- $(\pi', |\cdot|^{1-l})$ is $\iota$-ordinary.
- $\overline{r_\iota(\pi')} \cong \overline{\rho}$.
- $r_\iota(\pi')^c \cong r_\iota(\pi')^\vee \epsilon^{-l}$.
- $\pi'_{w_0}$ has an Iwahori fixed vector.
- If $\pi'_w$ is ramified then $\pi_w$ is ramified.
- For all embeddings $\tau : E \hookrightarrow \overline{\mathbb{Q}}_l$, we have $\mathrm{HT}_\tau(r_\iota(\pi')) = \{0, l\}$.

By [Tho12, Theorem 10.2], the deformation ring $R$ associated to the corresponding local conditions on a representation of $G_E$ is a finite $\mathcal{O}$-module. By [CHT08, Corollary 2.1.5], its Krull dimension is strictly positive. Thus there is a homomorphism $R \to \overline{\mathbb{Q}}_l$, and hence a lift of $\overline{\rho}$ satisfying these local conditions. By applying [Tho12, Theorem 9.1] we see that this lift is automorphic, and conclude that there exists a $\iota$-ordinary RAECSDC automorphic representation $(\pi', |\cdot|^{1-l})$ of $\mathrm{GL}_2(\mathbb{A}_E)$ satisfying the specified conditions.

By $\mathrm{TP}_r$, there exists a RAECSDC automorphic representation $(\Pi_2', \chi')$ of $\mathrm{GL}_{2r}(\mathbb{A}_E)$ such that $r_\iota(\Pi_2') \cong {}^\varphi r_\iota(\pi') \otimes r_\iota(\Pi_2)$. In fact, we have $\chi' = |\cdot|^{(r-l)}$, and $\Pi_2'$ is $\iota$-ordinary, by Lemma 3.7. Let $\sigma_2 = \Pi_2'|\cdot|^{(l-r)/2}$. Then $\sigma_2$ is conjugate self-dual and cuspidal, and $\sigma_{2,w_0}$ has an Iwahori fixed vector. Let $\psi = (\epsilon\chi)^r$. Then $\psi\psi^c = 1$ and $\sigma_1 = \Pi_1 \otimes \iota\psi$ is RACSDC, $\iota$-ordinary, and has an Iwahori-fixed vector.

We claim that $\Sigma = \sigma_1 \boxplus \sigma_2$ is regular algebraic. To see this it suffices to calculate the infinity types of the constituent cuspidal representations at each embedding $\tau : E \hookrightarrow \mathbb{C}$. These are independent of $\tau$; for $\sigma_2$ we have the infinity type $((l-r-1)/2, \ldots, (r+1-l)/2)$, and for $\sigma_1$ we have

$$((l+r-1)/2, \ldots, (l-r+1)/2, (r-l-1)/2, \ldots, (1-r-l)/2).$$

The representation $\Sigma_{w_0}$ has an Iwahori-fixed vector, and is $\iota$-ordinary by Lemma 2.6. It satisfies the third and last points by construction. This concludes the proof. □

THEOREM 4.2. *With hypotheses as above, assume conjectures* $\mathrm{TP}_r$ *and* $\mathrm{LR}_{l+r}$. *Then there exists a RACSDC automorphic representation* $\Pi$ *of* $\mathrm{GL}_{l+r}(\mathbb{A}_E)$ *satisfying the following.*

- $\Pi$ *is* $\iota$*-ordinary.*
- *The representation* $\Pi_{w_0}$ *is an unramified twist of the Steinberg representation.*
- *There is an isomorphism*
$$\overline{r_\iota(\Pi)} \cong (\mathrm{Sym}^{l+r-1}\,\overline{\rho})^{ss}.$$

*Proof.* Let $\Sigma$ denote the automorphic representation constructed in Proposition 4.1. Let $G$ be the definite unitary group of § 2.4, with $n = l+r$. By Proposition 2.9, there exists an automorphic representation $\Sigma_1$ of $G(\mathbb{A}_F)$ such that $\Sigma$ is the base change of $\Sigma_1$. Applying conjecture $\mathrm{LR}_{l+r}$ to $\Sigma_1$, we deduce the existence of an $\iota$-ordinary automorphic representation $\Sigma_2$ of $G(\mathbb{A}_F)$ of the same weight, such that $\Sigma_{2,v_0} \circ \iota_{w_0}$ is an unramified twist of the Steinberg representation. Let $\Pi$ denote the base change of $\Sigma_2$ to $\mathrm{GL}_{l+r}(\mathbb{A}_E)$, which exists, again by Proposition 2.9. Since $\Pi_{w_0}$ is an unramified twist of the Steinberg representation, $\Pi$ must be cuspidal. This completes the proof. □

## 5. Proof of Theorem 3.4

In this section we give the proof of Theorem 3.4. We therefore suppose throughout that $l \geqslant 5$ is a prime, and that $0 < r < l$. We fix a set $\mathbf{K} = \{K_1, \ldots, K_s\}$ of finite Galois extensions of $\mathbb{Q}$, and suppose that $\mathbb{Q}(\zeta_l)^+ \in \mathbf{K}$. We also assume that conjectures $\mathrm{SP}_{l-r}(\mathbf{K})$, $\mathrm{SP}_r(\mathbf{K})$, $\mathrm{LR}_{l+r}$, and $\mathrm{TP}_r$ hold. The linchpin in the proof is the following special case, which asserts that we can deduce the existence of the $(l+r-1)$th symmetric power lifting of a Hilbert modular form when certain local hypotheses are in play.

Let Sd denote the standard representation of $\mathrm{SL}_2(\overline{\mathbb{F}}_l)$ on $\overline{\mathbb{F}}_l^2$. By [Gur, Theorem 1.2], there exists an integer $a > 1$ such that the representations ${}^\varphi\mathrm{Sd} \otimes \mathrm{Sym}^{r-1}\,\mathrm{Sd}$ and $\mathrm{Sym}^{l-r-1}\,\mathrm{Sd}$ of $\mathrm{SL}_2(\mathbb{F}_{l^b})$ are adequate (in the sense of [Tho12, § 2]) whenever $b \geqslant a$. We recall that by [Gur, Lemma 1.4], any finite subgroup of $\mathrm{GL}_2(\overline{\mathbb{F}}_l)$ containing an adequate subgroup as a normal subgroup of index prime to $l$ is adequate.

Proposition 5.1. *Let $F$ be a totally real field, and fix an isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$. Suppose that $(\pi, \chi)$ is a RAESDC automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ satisfying the following hypotheses.*

- *$\pi$ is $\iota$-ordinary of weight $\boldsymbol{\lambda} = 0$. (In more classical language, $\pi$ has parallel weight two.)*
- *The image of the residual representation $\overline{r_\iota(\pi)}$ contains $\mathrm{SL}_2(\mathbb{F}_{l^b})$ for some integer $b \geqslant a$.*
- *There exists a place $v_0 \nmid l$ of $F$ such that $\pi_{v_0}$ is an unramified twist of the Steinberg representation.*

*Then the $(l + r - 1)$th symmetric power lifting of $\pi$ exists: there exists an $\iota$-ordinary RAESDC automorphic representation $\Pi$ of $\mathrm{GL}_{l+r}(\mathbb{A}_F)$ such that $r_\iota(\Pi) \cong \mathrm{Sym}^{l+r-1} r_\iota(\pi)$.*

*Proof.* We deduce the theorem from [Thoa, Theorem 7.1]. After replacing $F$ by a soluble extension not containing any $K_i$, we can assume that there exists a quadratic CM imaginary extension $E/F$, linearly disjoint over $F$ from the extension of $F(\zeta_l)$ cut out by $\overline{r_\iota(\pi)}$ and satisfying the hypotheses of § 4. Arguing as in the proof of [CHT08, Theorem 4.4.3], we see that there exists an algebraic character $\psi : G_E \to \overline{\mathbb{Q}}_l^\times$ such that $\chi|_{G_E} = \psi\psi^c$, and (if $\pi_E$ denotes the base change of $\pi$ to $E$) $\pi' = \pi_E \otimes \psi^{-1}$ is RACSDC. Replacing $F$ again by a soluble extension, we can arrange that the hypotheses of § 4 apply to $\pi'$, so by Theorem 4.2 there exists a RACSDC automorphic representation $\Pi$ of $\mathrm{GL}_{l+r}(\mathbb{A}_E)$ such that

$$\overline{r_\iota(\Pi)} \cong (\mathrm{Sym}^{l+r-1} \overline{r_\iota(\pi')})^{\mathrm{ss}},$$

and moreover that $\Pi$ is $\iota$-ordinary and $\Pi_{w_0}$ is an unramified twist of the Steinberg representation, for some place $w_0$ of $E$ above $v_0$. The result follows from [Thoa, Theorem 7.1] and Theorem 2.7, on checking the following remaining hypotheses of [Thoa, Theorem 7.1].

- The element $\zeta_l$ is not fixed by $\ker \mathrm{ad}\, \overline{r_\iota(\Pi)}$.
- Each irreducible constituent of $\overline{r_\iota(\Pi)}|_{G_{E(\zeta_l)}}$ is adequate.

The first point holds because, on the one hand, $[E(\zeta_l) : E] > 2$ and the extension of $E$ cut out by $\mathrm{ad}\, \overline{r_\iota(\Pi)}$ is contained inside the extension cut out by $\mathrm{ad}\, \overline{r_\iota(\pi)}$, while, on the other hand, the projective image of $\overline{r_\iota(\pi)}$ contains a simple normal subgroup of index at most 2 (by the classification of finite subgroups of $\mathrm{PGL}_2(\overline{\mathbb{F}}_l)$). The second point follows from our hypothesis on the image of $\overline{r_\iota(\pi)}$ and [Gur, Lemma 1.4]. □

We now reduce the general case of $\mathrm{SP}_{l+r}(\mathbf{K})$ to this one by using a chain of congruences. Let $F$ be a totally real field not containing $K_i$, $i = 1, \ldots, s$, and let $(\pi, \chi)$ denote a RAESDC automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ without CM. We must show that the symmetric $(l + r - 1)$th power lifting of $\pi$ exists.

Proposition 5.2. *There exist a prime $p \neq l$, an isomorphism $\iota_p : \overline{\mathbb{Q}}_p \cong \mathbb{C}$, a soluble totally real extension $F'/F$ linearly disjoint from the extension of $F(\zeta_p)$ cut out by $\overline{r_{\iota_p}(\pi)}$ and not containing any field $K_i$, and a RAESDC automorphic representation $\pi'$ of $\mathrm{GL}_2(\mathbb{A}_{F'})$ satisfying the following.*

- *The image of the residual representation $\overline{r_{\iota_p}(\pi)}$ contains $\mathrm{SL}_2(\mathbb{F}_p)$, up to conjugation.*
- *$\pi'$ has weight zero, and for every prime $v|l$, $\pi'_v$ is an unramified twist of the Steinberg representation.*

- *There exists a place $v_0$ of $F'$, not dividing $pl$, and such that $\pi'_{v_0}$ is an unramified twist of the Steinberg representation.*
- *The symmetric $(l+r-1)$th power lifting of $\pi$ exists if and only if the symmetric $(l+r-1)$th power lifting of $\pi'$ exists.*

*Proof.* By [Dim05, Proposition 3.8], all but finitely many pairs $(p, \iota_p)$ satisfy the first bullet point. We can therefore choose $p > 2(l+r+1)$ and $\iota_p$ such that the first bullet point is satisfied and $p$ is unramified in $F$. We can, moreover, assume that for every embedding $\tau : F \hookrightarrow \overline{\mathbb{Q}}_p$, the Hodge–Tate weights $\mathrm{HT}_\tau(r_{\iota_p}(\pi))$ differ by at most $p-2$. For each place $v|p$ of $F$, $r_{\iota_p}(\pi)|_{G_{F_v}}$ is potentially diagonalizable, by [BGGT14, Lemma 1.4.1]. Let $F'$ be a soluble totally real extension of $F$, linearly disjoint from the extension of $F(\zeta_p)$ cut out by $\overline{r_{\iota_p}(\pi)}$, not containing any $K_i$, and such that for every prime $v|l$ of $F'$, $\overline{r_{\iota_p}(\pi)}|_{G_{F'_v}}$ is trivial and $q_v \equiv 1 \bmod p$. Choose a place $v_0 \nmid lp$ of $F'$ such that $\overline{r_{\iota_p}(\pi)}|_{G_{F'_{v_0}}}$ is trivial, and $q_{v_0} \equiv 1 \bmod p$. By [Gee11, Corollary 3.1.7], there exists a second RAESDC automorphic representation $\pi'$ of $\mathrm{GL}_2(\mathbb{A}_{F'})$ such that $\overline{r_{\iota_p}(\pi)}|_{G_{F'}} \cong \overline{r_{\iota_p}(\pi')}$, such that $r_{\iota_p}(\pi')|_{G_{F'_v}}$ is potentially diagonalizable of weight zero at every prime $v$ of $F'$ dividing $p$, and such that the representations $\pi'_{v_0}$ and $\pi'_v$ for each place $v|l$ of $F'$ are each an unramified twist of the Steinberg representation. Here we note [GK, Lemma 4.4.1], which states that a potentially Barsotti–Tate representation is also potentially diagonalizable.

It now follows that both representations $\mathrm{Sym}^{l+r-1} r_{\iota_p}(\pi)|_{G_{F'}}$ and $\mathrm{Sym}^{l+r-1} r_{\iota_p}(\pi')$ are potentially diagonalizable on restriction to any decomposition group at a place $v|p$ of $F'$. Moreover, their residual representations are irreducible and isomorphic, and adequate, even on restriction to $G_{F'(\zeta_p)}$, since $p > 2(l+r+1)$ (see the appendix to [Tho12]). We deduce immediately from [BGGT14, Theorem 4.2.1] that the automorphy of either one of these Galois representations is equivalent to that of the other. The final bullet point for $\pi'$ now follows on combining this with soluble base change [BGHT11, Lemma 1.3]. □

After replacing $F$ by $F'$ and $\pi$ by $\pi'$, we can suppose without loss of generality that the following hypotheses are in effect.

(1) $\pi$ has weight zero.
(2) For each place $v|l$, $\pi_v$ is an unramified twist of the Steinberg representation.
(3) There exists a place $v_0$ of $F$, not dividing $l$, such that $\pi_{v_0}$ is an unramified twist of the Steinberg representation.

With these assumptions, we have the following result.

PROPOSITION 5.3. *There exists a RAESDC automorphic representation $\pi'$ of $\mathrm{GL}_2(\mathbb{A}_F)$ satisfying the following hypotheses.*

- *$\pi'$ has weight zero, and for every prime $v|l$, $\pi'_v$ is an unramified twist of the Steinberg representation.*
- *$\pi'_{v_0}$ is an unramified twist of the Steinberg representation.*
- *For every isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, the image of the residual representation $\overline{r_\iota(\pi')}$ contains $\mathrm{SL}_2(\mathbb{F}_{l^b})$, up to conjugation, for some $b \geqslant a$.*
- *The symmetric $(l+r-1)$th power lifting of $\pi$ exists if and only if the symmetric $(l+r-1)$th power lifting of $\pi'$ exists.*

*Proof.* We use a trick inspired by Khare and Wintenberger's use of so-called 'good-dihedral' primes, in their proof of Serre's conjecture; cf. [KW09, Lemma 8.2]. Let $E \subset \mathbb{C}$ denote the coefficient field of $\pi$. As in the proof of Proposition 5.2, we choose a prime $p > \max(2(l + r + 1), \#\mathrm{GL}_2(\mathbb{F}_{l^a}))$ split in $E(\sqrt{-1})$ and such that $F$ and $\pi$ are unramified above $p$. We fix an isomorphism $\iota_p : \overline{\mathbb{Q}}_p \cong \mathbb{C}$. After conjugating we can assume that the image of the residual representation $\overline{\rho} = \overline{r_{\iota_p}(\pi)}$ is contained in $\mathrm{GL}_2(\mathbb{F}_p)$; we assume also that $p$ has been chosen so that the residual image contains $\mathrm{SL}_2(\mathbb{F}_p)$.

Let $K/F$ denote the maximal abelian extension of exponent 2 which is ramified only at primes of $F$ where $\pi$ is ramified. Let $M/F$ denote the extension inside $\overline{F}$ cut out by $\mathbb{P}\overline{\rho}$, the projective representation associated to $\overline{\rho}$. Thus $\mathrm{Gal}(M/F)$ is isomorphic to either $\mathrm{PGL}_2(\mathbb{F}_p)$ or $\mathrm{PSL}_2(\mathbb{F}_p)$, the index 2 simple subgroup. Let $L = M \cap F(\zeta_p)$. Then $M$ and $F(\zeta_p)$ are linearly disjoint over $L$, and $L/F$ is an extension of degree at most 2. Let $v$ be an infinite place of $F$, and let $c_v \in G_F$ denote a complex conjugation at this place. Since $p \equiv 1 \bmod 4$, $\overline{\rho}(c_v) \in \mathrm{Gal}(M/L)$. By linear disjointness, we can therefore choose a prime $u$ of $F$ coprime to $p$ at which $\overline{\rho}$ is unramified and such that $\mathbb{P}\overline{\rho}(\mathrm{Frob}_u) = \mathbb{P}\overline{\rho}(c_v)$, up to conjugation, and such that $u$ is split in $L$ and $q_u \equiv -1 \bmod p$. We can, moreover, assume that $u$ is split in $K$.

Applying [Gee11, Corollary 3.1.7] once more, we can find a RAESDC automorphic representation $\pi'$ of $\mathrm{GL}_2(\mathbb{A}_F)$ such that $\overline{r_{\iota_p}(\pi')} \cong \overline{\rho}$, such that the representations $\pi'_{v_0}$ and $\pi'_v$ for $v|l$ are each an unramified twist of the Steinberg representation, and such that $\pi'$ is of weight zero and $r_{\iota_p}(\pi')$ is potentially diagonalizable on restriction to any decomposition group at a place $v|p$ of $F$, and moreover such that there is an isomorphism

$$r_{\iota_p}(\pi')|_{I_{F_u}} \cong \begin{pmatrix} \psi & 0 \\ 0 & \psi^{q_u} \end{pmatrix},$$

with $\psi : I_{F_u} \to \overline{\mathbb{Z}}_p^\times$ a character of order $p$. In particular, $r_{\iota_p}(\pi')|_{G_{F_u}}$ is irreducible, and induced from a character. Moreover, we can suppose that, away from $u$, $\pi'$ is ramified only at those places of $F$ where $\pi$ is also ramified.

The representation $\pi'$ satisfies the final point above. This is proved in exactly the same manner as the same point for the representation $\pi'$ of Proposition 5.2. It remains to show that $\pi'$ satisfies the penultimate bullet point. Fix an isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, and consider the residual representation $\overline{r_\iota(\pi')}$. It is irreducible, since its restriction to $G_{F_u}$ is already irreducible, being induced from a character whose restriction to $I_{F_u}$ has order $p \nmid q_u - 1$. Since the projective image of $\overline{r_\iota(\pi')}$ contains an element of order $p > 5$, either $\overline{r_\iota(\pi')}$ contains a conjugate of $\mathrm{SL}_2(\mathbb{F}_{l^b})$ for some $b > 1$, or $\overline{r_\iota(\pi')}$ is induced from a character. In the first case, by choice of $p$ we obtain a conjugate with $b \geqslant a$.

It therefore remains to rule out the possibility that $\overline{r_\iota(\pi')} \cong \mathrm{Ind}_{K_0}^F \alpha$, for some quadratic extension $K_0/F$ and some character $\alpha : G_{K_0} \to \overline{\mathbb{F}}_l^\times$. The extension $K_0/F$ is ramified only at those places where $\overline{r_\iota(\pi')}$ is also ramified, hence at the places dividing $l$, $u$, or where $\pi$ is ramified. Since $\overline{r_\iota(\pi')}|_{G_{F_u}}$ is induced from a character of the unramified quadratic extension of $F_u$ by construction, we see that $K_0$ is unramified at $u$, and hence $K_0 \subset K$. But $u$ is split in $K$, hence in $K_0$, which implies that the representation $\overline{r_\iota(\pi')}|_{G_{F_u}}$ is a direct sum of two characters. This contradiction shows that $\overline{r_\iota(\pi')}$ must in fact have residual image containing $\mathrm{SL}_2(\mathbb{F}_{l^b})$, for some $b \geqslant a$, and therefore concludes the proof. □

After replacing $\pi$ by $\pi'$ we may therefore suppose that $\pi$ satisfies, in addition to the above three points, the following hypothesis.

(4) For every isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, the image of the residual representation $\overline{r_\iota(\pi)}$ contains $\mathrm{SL}_2(\mathbb{F}_{l^b})$, up to conjugation, for some $b \geqslant a$.

We claim that $\pi$ now satisfies the hypotheses of Theorem 5.1. Indeed, it remains to check only that $\pi$ is $\iota$-ordinary, for some choice of $\iota$. This follows immediately from points (1) and (2) above, by [Ger, Lemma 5.1.5]. We deduce that the $(l + r - 1)$th symmetric power lifting of $\pi$ exists. This concludes the proof.

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