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# Sums and Products of Weighted Shifts

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*Abstract.* In this article it is shown that every bounded linear operator on a complex, infinite dimensional, separable Hilbert space is a sum of at most eighteen unilateral (alternatively, bilateral) weighted shifts. As well, we classify products of weighted shifts, as well as sums and limits of the resulting operators.

## 1 Introduction

### 1.1 Preliminaries

Suppose that $\mathcal{H}$ is a complex Hilbert space, and by $\mathcal{B}(\mathcal{H})$ let us denote the set of bounded linear operators acting on $\mathcal{H}$. The literature is replete with examples of factorizations of elements in $\mathcal{B}(\mathcal{H})$ as sums and products of operators of a specified class. We cite but a few examples, [9, 11, 20, 22, 29, 30], and refer the reader to [31] and its references for an extensive account of these problems. In many classical cases [1, 23, 24], the underlying Hilbert space is finite dimensional. More recently, there has been an interest in asymptotic versions of these questions. For example, in [19] and [13], a study of the norm closure of products of $k$ positive invertible operators was undertaken, where $k$ is a fixed integer greater than or equal to 2.

In this note, our focus will be upon characterizing sums and products of weighted shifts, as well as the closures of these and associated sets.

### 1.2

From this point onward, we shall assume that $\mathcal{H}$ is infinite dimensional, but separable. The term "shift" appears often in the literature, where it is assigned different meanings, depending upon the context. For our purposes, a bounded operator $W$ will be called a *bilateral weighted shift* if there exists an orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ for $\mathcal{H}$ and a bounded sequence $\{w_n\}_{n\in\mathbb{Z}}$ of complex numbers (called the *weights* of $W$) such that $We_n = w_n e_{n+1}$ for all $n \in \mathbb{Z}$. Similarly, an operator $V$ is called a *unilateral (forward) shift* if there exists an orthonormal basis $\{f_n\}_{n=1}^{\infty}$ for $\mathcal{H}$ and a bounded sequence $\{v_n\}_{n=1}^{\infty}$ such that $Vf_n = v_n f_{n+1}$. If we can choose the orthonormal basis so that $w_n = 1$ for all $n \in \mathbb{Z}$, then we shall write $B$ instead of $W$ and refer to this as a *bilateral (unweighted) shift*. Analogously, if we can choose $v_n = 1$ for all $n \geq 1$, we write $S$ for $V$ and refer to this as a *unilateral (unweighted, forward) shift*. Unlike [14] or [5], *we will not allow our shifts to have multiplicity greater than* 1. In particular, neither $S^*$ nor $S \oplus S$ are shifts according to our definition. (In this connection it is

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worth mentioning that L. G. Brown [5] has shown that every contraction is a product of a forward (unweighted) shift of infinite multiplicity and a backward (unweighted) shift of infinite multiplicity.) The paper by X.H. Ding and G. Shi [7] also deals with related questions.

It is clear that the set $\mathcal{W}$ of bilateral weighted shifts and the set $\mathcal{V}$ of unilateral weighted shifts are invariant under unitary equivalence (—recall that two operators $A$ and $B$ are *unitarily equivalent* if there exists a unitary operator $U$ such that $A = U^*BU$, in which case we write $A \simeq B$). Given a set $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ and an integer $k \geq 1$, let us write $\mathcal{S}_k$ (resp. $\sum_k \mathcal{S}$) for the set $\{S_1S_2\cdots S_k : S_i \in \mathcal{S}, 1 \leq i \leq k\}$ (resp. $\{S_1 + S_2 + \cdots + S_k : S_i \in \mathcal{S}, 1 \leq i \leq k\}$). It is understood that $\mathcal{S} = \mathcal{S}_1$. It then follows from the above observation that $\sum_j \mathcal{W}_k$ and $\sum_j \mathcal{V}_k$ are invariant under unitary conjugation as well, as are their closures and even $\sum_j \overline{\mathcal{W}_k}$ and $\sum_j \overline{\mathcal{V}_k}$ for each $j, k \geq 1$.

**1.3**

Before proceeding, we would like to record a few simple but useful facts about $\mathcal{V}_k$ and $\mathcal{W}_k$ which will be repeatedly used below.

(i) By a *finite* weighted shift, we shall mean an operator $X \in \mathcal{B}(\mathbb{C}^n)$ (where $n$ is a positive integer) for which we can find an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ for $\mathbb{C}^n$ and scalars $x_i$, $1 \leq i \leq n$ so that $Xe_j = x_je_{j+1}$, $1 \leq j \leq n-1$, and $Xe_n = 0$. Observe that $X^*$ is then also a finite shift with weights $\{\bar{x}_{n-1}, \bar{x}_{n-2}, \ldots, \bar{x}_1\}$ with respect to the orthonormal basis $\{e_n, e_{n-1}, \ldots, e_1\}$.

Given a family $\{X_j\}_{j=1}^{\infty}$ of finite shifts, $V = \bigoplus_{j=1}^{\infty} X_j$ and $V^*$ both lie in $\mathcal{V}_1$. Indeed, $Z, Z^* \in \mathcal{V}_1$ if and only if $Z$ is a direct sum of finite weighted shifts.

Furthermore, if $\kappa\colon \mathbb{Z} \to \mathbb{N}$ is a bijection, then $W = \bigoplus_{j\in\mathbb{Z}} X_{\kappa(j)} (\simeq V)$ also defines a bilateral weighted shift.

(ii) The set $\mathcal{W}$ is self-adjoint: if $We_n = w_ne_{n+1}$ for all $n \in \mathbb{Z}$, then $W^*f_n = \bar{w}_{-(n+1)}f_{n+1}$ for all $n \in \mathbb{Z}$, where $f_n = e_{-n}$. It follows that $\mathcal{W}_k$, $\sum_k \mathcal{W}$ and their closures are again self-adjoint.

(iii) Let $\mathcal{D}$ denote the set of *diagonalizable* operators on $\mathcal{H}$; that is, those operators for which there exists an orthonormal basis $\{e_n\}$ (indexed by $\mathbb{N}$ or by $\mathbb{Z}$) and a bounded sequence $\{d_n\}$ so that $De_n = d_ne_n$ for all $n$. Then $\mathcal{D} \subseteq \mathcal{W}_2$. Indeed, let $D = \operatorname{diag}\{d_n\}_{n\in\mathbb{Z}}$ be any diagonal operator with respect to the basis $\{e_n\}_{n\in\mathbb{Z}}$. Let $We_n = d_ne_{n+1}$ and $Xe_n = e_{n-1}$, $n \in \mathbb{Z}$. By (ii), $X \simeq B^* \in \mathcal{W}_1$. Then $XWe_n = d_ne_n = De_n$ for all $n \in \mathbb{Z}$, so that $\mathcal{W}_2$ contains all diagonal operators.

Also, $\mathcal{V}_2 \supseteq \{0^{(\infty)} \oplus D' : D' \in \mathcal{D}\}$. Let $\{f_n\}_{n=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$ and suppose $D'f_n = d'_nf_n$, $n \geq 1$. Define $Vf_n = v_nf_{n+1}$, $n \geq 1$ and $Xf_n = x_nf_{n-1}$, where $x_n = 1$ if $n$ is even, and $x_n = 0$ if $n$ is odd. Then $X \simeq \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]^{(\infty)} \simeq \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]^{(\infty)} \in \mathcal{V} \cap \mathcal{V}^*$. Thus $T = XV \in \mathcal{V}_2$, where $T = \operatorname{diag}\{v_1, 0, v_3, 0, \ldots\}$. Let $v_{2k-1} = d'_k$ for each $k \geq 1$ to find that $T \simeq 0^{(\infty)} \oplus D'$.

(iv) The classes $\mathcal{V}_k$, $\mathcal{W}_k$ are invariant under scalar multiplication for all $k \geq 1$. We will use this implicitly in our proofs—often we will only show that an operator $X$ of norm 1 lies in $\mathcal{V}_k$ (for example), from which we may automatically deduce that all multiples of $X$ also lie there.

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### 1.4

Some of our estimates are based on results of P. Y. Wu and of C. K. Fong and P. Y. Wu on sums and products of normal operators. Let us also record these, as well as the Weyl-von Neumann-Berg-Sikonia Theorem for use below.

***Theorem 1.5*** **(Wu [30])** *Let $\mathcal{N}$ denote the set of normal operators in $\mathcal{B}(\mathcal{H})$. For $T \in \mathcal{B}(\mathcal{H})$, the following are equivalent:*

*(i)* $T \in \bigcup_{k=1}^{\infty} \mathcal{N}_k$;
*(ii)* $T \in \mathcal{N}_3$;
*(iii)* $\operatorname{nul} T = \operatorname{nul} T^*$ *or* $\operatorname{ran} T$ *is not closed.*

***Theorem 1.6*** **(Fong-Wu [12])**

*(i) Every operator is a sum of three diagonal operators; that is,* $\mathcal{B}(\mathcal{H}) = \sum_3 \mathcal{D}$.
*(ii)* $\mathcal{D}_5 = \mathcal{N}_3$.

Recall that two operators $A$ and $B$ are said to be *approximately unitarily equivalent* and we write $A \simeq_a B$ if for all $\varepsilon > 0$ there exists $U$ unitary and $K$ compact with $\|K\| < \varepsilon$ such that $A = U^*BU + K$. Denote by $\mathcal{K}(\mathcal{H})$ the set of all compact operators on $\mathcal{H}$, and by $\mathcal{Q}(\mathcal{H})$ the *Calkin algebra* $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Set $\pi\colon \mathcal{B}(\mathcal{H}) \to \mathcal{Q}(\mathcal{H})$ to be the canonical quotient map. The *essential spectrum* of $A$ is $\sigma_e(A) := \sigma\big(\pi(A)\big)$ in $\mathcal{Q}(\mathcal{H})$. By $\sigma_{\mathrm{lre}}(A)$ we denote the *left-right essential spectrum* of $A$, and by $\mathcal{N}$ the set of normal operators. The decomposition of an operator into its real and imaginary parts implies the trivial result that $\mathcal{B}(\mathcal{H}) = \sum_2 \mathcal{N}$.

***Theorem 1.7*** **(Weyl-von Neumann-Berg-Sikonia [2, 27])** *Given $N$ normal and $\varepsilon > 0$, there exists a diagonalizable operator $D$ and a compact operator $K$ with $\|K\| < \varepsilon$ such that $N = D + K$.*

***Corollary 1.8*** *Two normal operators $N$ and $M$ are approximately equivalent if and only if $\sigma_e(N) = \sigma_e(M)$, $\sigma(N) = \sigma(M)$, and any isolated eigenvalues of $N$ and $M$ appear with equal multiplicities.*

## 2 Products of Weighted Shifts

***Lemma 2.1*** *Let $X \in \mathcal{B}(\mathcal{H})$.*

*(i) If $X \in \overline{\mathcal{V}_k}$ for some $k \geq 1$ and $X$ is semi-Fredholm, then $X$ is Fredholm of index $-k$.*
*(ii) If $X \in \overline{\mathcal{W}_k}$ for some $k \geq 1$ and $X$ is semi-Fredholm, then $X$ is Fredholm of index* $0$.

**Proof** (i) First choose $\varepsilon > 0$ so that $\pi(Y) \in \mathcal{Q}(\mathcal{H})$ and $\|\pi(Y) - \pi(X)\| < \varepsilon$ implies $\pi(Y)$ is semi-Fredholm and $\operatorname{ind} Y = \operatorname{ind} X$. This is possible because the set of semi-Fredholm operators is open, and the index function is continuous. Next, choose $V_i, 1 \leq i \leq k \in \mathcal{V}_1$ so that

$$\|X - V_1 V_2 \cdots V_k\| < \varepsilon.$$

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Clearly $\|\pi(X) - \pi(V_1)\pi(V_2)\cdots\pi(V_k)\| < \varepsilon$ as well. If $\pi(X)$ is left invertible, then so is $\pi(V_k)$, and so $V_k$ is semi-Fredholm. But then the weight sequence $\{v_j(k)\}_{j=1}^{\infty}$ for $V_k$ is essentially bounded below, from which it follows that $V_k$ is Fredholm, and $\operatorname{ind} V_k = -1$.

Multiplying on the right by $\pi(V_k)^{-1}$, we find that $\pi(V_1)\pi(V_2)\cdots\pi(V_{k-1})$ is left invertible, and so we may repeat the argument to find that each $V_i$ is Fredholm with index $-1$. Finally,

$$\operatorname{ind} X = \operatorname{ind}(V_1 V_2 \cdots V_k) = \sum_{j=1}^{k} \operatorname{ind} V_j = -k.$$

If $\pi(X)$ is right invertible, then $\pi(V_1)$ is right invertible, and the proof follows in an analogous manner.

(ii) The fact that $\mathcal{W}_1$ is self-adjoint means that the situation is symmetric with respect to left and right invertibility. The only change to the above argument is that if we choose $W_j \in \mathcal{W}_1$ to play the analogous rôle to that played by $V_j$, then $W_j$ Fredholm implies that $\operatorname{ind} W_j = 0$, and hence $\operatorname{ind} X = \sum_{j=1}^{k} \operatorname{ind} W_j = 0$. ■

***Lemma 2.2***

*(i)* $\mathcal{N} \subseteq \overline{\mathcal{W}_2}$*, whence* $\overline{\mathcal{N}_k} \subseteq \overline{\mathcal{W}_{2k}}$ *for each* $k \geq 1$*;*
*(ii)* $\mathcal{N} \cap \overline{\mathcal{V}_2} = \{N \text{ normal} : 0 \in \sigma_e(N)\}$.

**Proof** (i) As observed in paragraph 1.3, $\mathcal{D} \subseteq \mathcal{W}_2$. Since $\mathcal{D}$ is dense in $\mathcal{N}$ by the Weyl-von Neumann-Berg-Sikonia Theorem, we are done. The second statement is a trivial consequence of the first.

(ii) Suppose first that $N$ is normal and $0 \in \sigma_e(N)$. As before, we may use Corollary 1.8 to replace $N$ by an (approximately unitarily equivalent) diagonal operator $D = \operatorname{diag}\{d_n\}_{n=1}^{\infty}$ satisfying $d_{2n} = 0$, $n \geq 1$ and acting on the basis $\{e_n\}_{n=1}^{\infty}$. By 1.3(iii), $D \in \mathcal{V}_2$, and so $N \in \overline{\mathcal{V}_2}$.

Conversely, suppose $N \in \mathcal{V}_2 \cap \mathcal{N}$. If $N$ is semi-Fredholm, then by Lemma 2.1, $N$ is Fredholm and $\operatorname{ind} N = -2$. But this contradicts the fact that $\ker N = \ker N^*$ for all normal operators. Thus $N$ is not semi-Fredholm, *i.e.* $0 \in \sigma_{\mathrm{lre}}(N) = \sigma_e(N)$. ■

***Lemma 2.3*** *Let $B$ denote the bilateral shift. Then* $\mathcal{N}_3 B = \mathcal{N}_3$.

**Proof** Note that $B, B^* \in \mathcal{N}$. Using Theorem 1.5, we obtain:

$$\mathcal{N}_3 = \mathcal{N}_3 B^* B \subseteq \mathcal{N}_4 B = \mathcal{N}_3 B \subseteq \mathcal{N}_4 = \mathcal{N}_3.$$ ■

***Theorem 2.4***

*(i)* $\overline{\mathcal{W}_k} = \mathcal{N}_3$ *when* $k \geq 6$.
*(ii)* $\mathcal{W}_k = \mathcal{N}_3$ *when* $k \geq 10$.

**Proof** (i) First observe that $\overline{\mathcal{W}_k} \subseteq \mathcal{N}_3$ for all $k \geq 1$. Indeed, suppose $T \in \overline{\mathcal{W}_k}$, but $T \notin \mathcal{N}_3$. Then ran $T$ is closed, and at least one of nul $T$ and nul $T^*$ must be finite. In other words, $T$ must be semi-Fredholm. By Lemma 2.1, $T$ is Fredholm of index zero, contradicting the fact that $T \notin \mathcal{N}_3$. It remains to prove the reverse inclusion when $k \geq 6$.

Since $I \in \mathcal{W}_2$ by 1.3(iii), we have $\overline{\mathcal{W}_k} \subseteq \overline{\mathcal{W}_{k+2}}$ for all $k \geq 1$. Thus $\mathcal{N}_3 \subseteq \overline{\mathcal{W}_6} \subseteq \overline{\mathcal{W}_{6+2j}}$ for all $j \geq 1$. Also, $\mathcal{N}_3 \subseteq \overline{\mathcal{W}_6}$ implies, using Lemma 2.3, that

$$\mathcal{N}_3 = \mathcal{N}_3 B \subseteq \overline{\mathcal{W}_6} B \subseteq \overline{\mathcal{W}_7} \subseteq \overline{\mathcal{W}_{7+2j}} \quad \text{for all } j \geq 1.$$

This completes the proof.

(ii) By Theorem 1.6(ii), $\mathcal{N}_3 = \mathcal{D}_5$. Since $\mathcal{D}_1 \subseteq \mathcal{W}_2$, the result easily follows. ■

***Theorem 2.5*** *For each $k \geq 7$, $\overline{\mathcal{V}_k} = \{T \in \mathcal{B}(\mathcal{H}) : T$ is not semi-Fredholm, or $T$ is Fredholm of index $-k\}$.*

**Proof** One inclusion is Proposition 2.1 (i). We now consider the other inclusion. Let $k \geq 7$ be a fixed integer.

There are two ways that $T$ can fail to be semi-Fredholm. Either the range of $T$ is not closed, or nul $T =$ nul $T^* = \infty$.

Let us first consider the case where nul $T =$ nul $T^* = \infty$. Consider $\mathcal{H}_1 = \ker T$, $\mathcal{H}_2 = \mathcal{H}_1^{\perp} \cap \overline{\operatorname{ran} T}$, and $\mathcal{H}_3 = \mathcal{H}_1^{\perp} \cap (\overline{\operatorname{ran} T})^{\perp}$. Then $\mathcal{H}_1$ is infinite dimensional and $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. With respect to this decomposition of $\mathcal{H}$, we may write

$$T = \begin{bmatrix} 0 & T_1 & T_2 \\ 0 & T_3 & T_4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Suppose $\dim \mathcal{H}_3 < \infty$. Let $\mathcal{M} = \ker T \cap \ker T^*$. Since nul $T^* = \infty$, $\dim \mathcal{M} = \infty$. With respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$, $T$ decomposes as $0^{(\infty)} \oplus T_0$. By Corollary 2.3 of [12], $T \in \mathcal{D}_3$, say $T = D_1 D_2 D_3$. But then $T \simeq 0^{(\infty)} \oplus T = (0^{(\infty)} \oplus D_1)(0^{(\infty)} \oplus D_2)(0^{(\infty)} \oplus D_3) \in \mathcal{V}_6$, by 1.3(iii). By Lemma 5 of [15] (or by [21, 25]), $S \simeq_a S \oplus I$, and hence $T \simeq (0^{(\infty)} \oplus T) = (0^{(\infty)} \oplus T)(S \oplus I) \in \mathcal{V}_6 \overline{\mathcal{V}_1} \subseteq \overline{\mathcal{V}_7}$.

Secondly, suppose that $\dim \mathcal{H}_3 = \infty = \dim \mathcal{H}_2$. As above, from [12] we see that $X_1 = \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & T_1 & T_2 \\ 0 & T_3 & T_4 \end{smallmatrix}\right]$ lies in $\mathcal{V}_6$. Since $X_2 = \left[\begin{smallmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{smallmatrix}\right] \simeq \left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right]^{(\infty)} \in \mathcal{V}_1$, we obtain $T = X_2 X_1 \in \mathcal{V}_7$.

If $\dim \mathcal{H}_3 = \infty$ but $\dim \mathcal{H}_2 < \infty$, then we can decompose $\mathcal{H}_1$ as $\mathcal{H}_{1a} \oplus \mathcal{H}_{1b}$, each of infinite dimension. With respect to the decomposition $\mathcal{H} = \mathcal{H}_{1a} \oplus (\mathcal{H}_{1b} \oplus \mathcal{H}_2) \oplus \mathcal{H}_3$, we see that

$$T \simeq \begin{bmatrix} 0 & 0 & T_{1a} & T_{2a} \\ 0 & 0 & T_{1b} & T_{2b} \\ 0 & 0 & T_3 & T_4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \simeq \begin{bmatrix} 0 & T_1' & T_{2a} \\ 0 & T_3' & T_4' \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{c} \mathcal{H}_{1a} \\ \mathcal{H}_{1b} \oplus \mathcal{H}_2. \\ \mathcal{H}_3 \end{array}$$

But by the argument in the previous paragraph, every operator of this form lies in $\mathcal{V}_7$.

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So far we have shown that if nul $T = $ nul $T^* = \infty$, then $T \in \overline{\mathcal{V}_7}$. Now, as we have seen, $L = S \oplus I \in \overline{\mathcal{V}_1}$, and so $T = TL^j \in \overline{\mathcal{V}_{7+j}}$ for all $j \geq 1$. Let $j = k - 7$ to obtain the precise statement of the theorem.

Next, if ran $T$ is not closed, then $0 \in \sigma_{\mathrm{lre}}(T)$ as $T$ is not semi-Fredholm. But then $T$ is the limit of operators $T_n$, each of which has infinite dimensional kernel and cokernel. By the previous paragraph, $T_n \in \overline{\mathcal{V}_k}$ for all $n \geq 1$, from which we conclude that $T \in \overline{\mathcal{V}_k}$.

Finally, suppose that $T$ is Fredholm of index $-k$. Consider the polar decomposition $T = U|T|$ of $T$, where $U$ is a partial isometry of index $-k$, and $\ker U = \ker |T| = \ker T$. Let $\mathcal{H}_1 = \ker T$ and $\mathcal{H}_0 = \mathcal{H}_1^{\perp}$. With respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0$, we may write $|T| = 0 \oplus |T|_0$. Furthermore, given $\varepsilon > 0$, we can use the Weyl-von Neumann-Berg-Sikonia Theorem to find an invertible operator $D_0 \in \mathcal{B}(\mathcal{H}_0)$ so that $D_0 - |T|_0$ is compact and has norm less than $\varepsilon$. Let $\{e_1, e_2, \dots, e_s\}$ be a basis for $\mathcal{H}_1$, and choose a basis $\{e_{s+1}, e_{s+2}, \dots\}$ for $\mathcal{H}_0$ which diagonalises $D_0$—say $D_0 e_i = d_i e_i$ for all $i \geq s + 1$. Set $D = 0 \oplus D_0 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_0)$.

Next, let $S \in \mathcal{B}(\mathcal{H})$ be the unilateral shift satisfying $Se_n = e_{n+1}$ for all $n \geq 1$. Then $V_1 := SD \in \mathcal{V}_1$, and if $R = US^*$ and $Y = RV_1$, then

$$\begin{aligned} \|T - Y\| &= \big\| U|T| - US^*SD \big\| \\ &= \big\| \, |T| - D \big\| < \varepsilon. \end{aligned}$$

(The difference $T - Y$ is also seen to be compact.) We claim that $R$ is a partial isometry. Indeed, since $U$ is a partial isometry, $Q_1 = U^*U$ and $Q_2 = UU^*$ are both projections. Now $RR^* = US^*SU^* = UU^* = Q_2$ is clearly a projection. As well, $R^*R$ is self adjoint and $(R^*R)^2 = (SU^*US^*)^2 = (SQ_1S^*)^2 = SQ_1^2S^* = SQ_1S^* = R^*R$, proving that $R^*R$ is also idempotent, as required.

Furthermore, $\ker R = \operatorname{span}\{e_1, S(\ker U)\} = \operatorname{span}\{e_1, e_2, \dots, e_{s+1}\}$, and ind $R =$ ind $U +$ ind $S^* = -k+1$. Let $Z$ be any partial isometry whose initial space is $\ker R$ and whose final space is contained in $(\operatorname{ran} R)^{\perp}$. Then $R + Z$ is an *isometry*, $\operatorname{ind}(R + Z) =$ ind $R$ (as $Z$ is finite rank, for example), and $(R+Z)V_1 = (R+Z)SD = RSD + ZSD = Y + 0$.

By the Wold decomposition ([6], Theorem V.2.1), $(R + Z)$ is unitarily equivalent to $S^{(k-1)} \oplus A$, where $A$ is a (possibly vacuous) unitary operator. Once again, we may apply Lemma 5 of [15] to obtain $S^{(k-1)} \oplus A \simeq S^{(k-2)} \oplus S \oplus A \simeq_a S^{(k-2)} \oplus S \simeq S^{(k-1)} \simeq S^{k-1}$. Hence $R + Z \in \overline{\mathcal{V}_{k-1}}$.

It follows that $Y = (R+Z)V_1 \in \overline{\mathcal{V}_k}$. Since $\varepsilon > 0$ was arbitrary, $T \in \overline{\mathcal{V}_k}$, completing the proof. ■

It is clear that a dual result holds for backward shifts by considering adjoints.

## 3 Sums of (Products of) Weighted Shifts

***Lemma 3.1***

*(i)* $\mathcal{V}_1 \subseteq \mathcal{W}_1 + \mathcal{W}_1$.
*(ii)* $\mathcal{W}_1 \subseteq \mathcal{V}_1 + \mathcal{V}_1$.

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**Proof** (i) Suppose $Ve_n = v_n e_{n+1}$ for all $n \geq 1$. Let $X$ and $Y$ be weighted shifts with respect to $\{e_n\}_{n=1}^{\infty}$ with weight sequences $\{v_1, 0, v_3, 0, v_5, 0, \dots\}$ for $X$ and $\{0, v_2, 0, v_4, 0, v_6, \dots\}$ for $Y$. Clearly $V = X + Y$ and $X, Y \in \mathcal{W}_1$, by 1.3(i).

(ii) Similar to the above, as operators of the form $\bigoplus_{n\in\mathbb{Z}} \left[\begin{smallmatrix} 0 & 0 \\ * & 0 \end{smallmatrix}\right]$ lie in $\mathcal{V}_1$ as well. ■

***Proposition 3.2***

*(i)* $\mathcal{B}(\mathcal{H}) = \sum_6 \overline{\mathcal{V}_1} = \overline{\sum_6 \mathcal{V}_1}$.
*(ii)* $\mathcal{B}(\mathcal{H}) = \sum_6 \overline{\mathcal{W}_1} = \overline{\sum_6 \mathcal{W}_1}$.
*(iii)* $\mathcal{B}(\mathcal{H}) = \sum_{18} \mathcal{V}_1 = \sum_{18} \mathcal{W}_1$.

**Proof** (i) Using a technique due to I. D. Berg [3], it was shown by D. A. Herrero [17], Corollary 5.3 that there exists a normal operator $M_0$ with $\sigma(M_0) = \bar{\mathbb{D}}$ which is a limit of nilpotent weighted shifts $Y_n$. Since each nilpotent weighted shift is a direct sum of finite weighted shifts, $Y_n \in \mathcal{V}_1$ for each $n \geq 1$. By Corollary 1.8, it follows that every normal $M$ with $\sigma(M) = \bar{\mathbb{D}}$ lies in $\overline{\mathcal{V}_1}$.

Suppose $N$ is normal, $\|N\| \leq 1$ and write $N = N_1 \oplus N_2$ where both summands act on infinite dimensional subspaces. (That such a decomposition is possible follows from the Spectral Theorem.)

Now $M \simeq_a -M \simeq_a M \oplus M \simeq_a -M \oplus -M$, again, by the Weyl-von Neumann-Berg-Sikonia Theorem. Furthermore, the same result shows that $M \simeq_a N_1 \oplus M \simeq_a M \oplus N_2$. Hence $X_1 = N_1 \oplus M$, $X_2 = -M \oplus -M$, and $X_3 = M \oplus X_2$ are all approximately unitarily equivalent to $M$ and hence lie in $\overline{\mathcal{V}_1}$. It follows that $N = X_1 + X_2 + X_3 \in \sum_3 \overline{\mathcal{V}_1}$, from which we deduce that $\mathcal{B}(\mathcal{H}) = \sum_6 \overline{\mathcal{V}_1} = \overline{\sum_6 \mathcal{V}_1}$.

(ii) We remark that the above proof also works for bilateral weighted shifts, since $Y_n \in \mathcal{W}_1$ for all $n \geq 1$.

(iii) We shall prove that $\mathcal{D}_1 \subseteq \sum_6 \mathcal{V}_1$. Since $\mathcal{B}(\mathcal{H}) = \sum_3 \mathcal{D}$ by 1.6(i), this is sufficient to prove our claim. For each $\lambda \in \mathbb{C}$, consider the matrix $M_\lambda = \left[\begin{smallmatrix} 0 & \lambda \\ \lambda & 0 \end{smallmatrix}\right]$. Then $M_\lambda$ is normal, $\sigma(M_\lambda) = \{\lambda, -\lambda\}$, and so $M_\lambda \simeq \operatorname{diag}\{-\lambda, \lambda\} \simeq \operatorname{diag}\{\lambda, -\lambda\} \simeq -M_\lambda$. Now given $D = \operatorname{diag}\{d_n\}_{n=1}^{\infty}$, we let $D_1 = \operatorname{diag}\{d_{2n-1}\}_{n=1}^{\infty}$ and $D_2 = \operatorname{diag}\{d_{2n}\}_{n=1}^{\infty}$. Clearly $D \simeq D_1 \oplus D_2$. Consider next

$$L_0 := \bigoplus_{n=1}^{\infty} M_{d_n} = \begin{bmatrix} 0 & d_1 & & & & & \\ d_1 & 0 & 0 & & & & \\ & 0 & 0 & d_2 & & & \\ & & d_2 & 0 & 0 & & \\ & & & 0 & 0 & d_3 & \\ & & & & d_3 & 0 & \ddots \\ & & & & & \ddots & \ddots \end{bmatrix}.$$

Clearly $L_0 \simeq D \oplus -D \simeq -L_0 \in \mathcal{V}_1 + \mathcal{V}_1$. Indeed, the same proof shows that for any diagonal operator $C$, $C \oplus -C \in \sum_2 \mathcal{V}_1$. Since $C = D^{(\infty)}$ is just another diagonal operator, we find that $L = L_0^{(\infty)} \in \sum_2 \mathcal{V}_1$. It is not hard to see that $L \simeq L \oplus L \simeq -L \oplus -L \simeq -L$, and that $L \simeq D_1 \oplus L \simeq L \oplus D_2$.

Thus $X_1 = D_1 \oplus L$, $X_2 = -L \oplus -L$, and $X_3 = L \oplus D_2$ are all unitarily equivalent to $L$ and hence lie in $\sum_2 \mathcal{V}_1$. As such, their sum $D_1 \oplus D_2 \simeq D$ lies in $\sum_6 \mathcal{V}_1$, as required.

As for bilateral shifts, simply note that $L_0$ also lies in $\sum_2 \mathcal{W}_1$, and so the same argument applies. ∎

***Remark 3.3*** In part (i) of the above theorem, we showed that $\mathcal{N}_1 \subseteq \sum_3 \overline{\mathcal{V}_1}$. That this is the best possible estimate for this inclusion is demonstrated by the following argument. For $X \in \mathcal{B}(\mathcal{H})$, we denote the spectral radius of $X$ by $\operatorname{spr}(X)$, and we let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

The key to the argument is that if $Z \in \bar{\mathcal{V}}_1$ and $\alpha \in \sigma(Z)$ with $|\alpha| = \operatorname{spr}(Z)$, then $\lambda\alpha \in \sigma(Z)$ for all $\lambda \in \mathbb{T}$, *i.e.* $\operatorname{spr}(Z)\mathbb{T} \subseteq \sigma(Z)$.

We shall show that this is the case by demonstrating that if $\mathcal{Y} \subseteq \mathcal{B}(\mathcal{H})$ and if $X \in \mathcal{Y}$ implies $X \simeq_a \lambda X$ for all $\lambda \in \mathbb{T}$, then every $Y \in \bar{\mathcal{Y}}$ has spectrum with circular symmetry; *i.e.*, $\alpha \in \sigma(Y)$ implies $\alpha\mathbb{T} \subseteq \sigma(Y)$.

To see this, suppose $Y = \lim_{n\to\infty} X_n$ with $\{X_n\} \subseteq \mathcal{Y}$. If $\alpha \in \sigma(Y)$, then either there exists an infinite subsequence $\{X_{n_k}\}$ of $\{X_n\}$ so that $\alpha \in \sigma(X_{n_k})$ for all $k \geq 1$, or there exists $n_0$ so that $n \geq n_0$ implies $\alpha \notin \sigma(X_n)$.

Let $\lambda \in \mathbb{T}$. In the first case, $\lambda\alpha \in \sigma(X_{n_k})$, since $X_{n_k} \simeq_a \lambda X_{n_k}$. Since the invertibles are open in $\mathcal{B}(\mathcal{H})$, we conclude that $\lambda\alpha \in \sigma(Y)$. In the second case, $\alpha \in \sigma(Y)$ but $\alpha \notin \sigma(X_n)$ for $n \geq n_0$ implies that $\{(\alpha I - X_n)^{-1}\}_{n \geq n_0}$ is not bounded in norm. Since $\|(\alpha I - X_n)^{-1}\| = \|(\alpha I - \bar{\lambda} X_n)^{-1}\|$ for all $n \geq n_0$ (as $X_n \simeq_a \bar{\lambda} X_n$), it follows that $\{\|(\alpha\lambda I - X_n)^{-1}\|\}_{n \geq n_0}$ is not bounded. Since $(\alpha\lambda I - X_n)$ converges to $(\alpha\lambda I - Y)$, it follows that $\alpha\lambda \in \sigma(Y)$. Since $\lambda \in \mathbb{T}$ was arbitrary, we are done with this part of the proof.

Now suppose that $\mathcal{Z}$ is any subset of $\mathcal{B}(\mathcal{H})$ for which $Z \in \mathcal{Z}$ implies $\operatorname{spr}(Z)\mathbb{T} \subseteq \sigma(Z)$. We claim that $I \notin \sum_2 \mathcal{Z}$. Indeed, suppose $I = Z_1 + Z_2$ with $Z_1, Z_2 \in \mathcal{Z}$, and let $r_i = \operatorname{spr}(Z_i)$, $i = 1, 2$. We may assume without loss of generality that $0 \leq r_1 \leq r_2$. Since $\partial\big(\sigma(X)\big) \subseteq \sigma_a(X)$ for all $X \in \mathcal{B}(\mathcal{H})$, we can find a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ so that $\|(Z_2 + r_2 I)x_n\| \to 0$. Let us write $0 = \big((Z_1 + Z_2) - I\big)\, x_n = (Z_2 + r_2 I)x_n + \big(Z_1 - (r_2 + 1)I\big)\, x_n$. Since $\lim_{n\to\infty}(Z_2 + r_2 I)x_n = 0$, it follows that

$$\|Z_1 x_n - (r_2 + 1)x_n\| \to 0.$$

But then $\operatorname{spr}(Z_1) = r_1 < r_2 + 1 \in \sigma_a(Z_1)$, a contradiction.

Since $\mathcal{V}_1$ has the property that $V \in \mathcal{V}_1$ implies $V \simeq \lambda V$ for all $\lambda \in \mathbb{T}$, we conclude that $I \notin \sum_2 \bar{\mathcal{V}}_1$. The same argument also shows that $I \notin \sum_2 \bar{\mathcal{W}}_1$.

***Proposition 3.4***

*(i)* $\mathcal{B}(\mathcal{H}) = \sum_3 \mathcal{W}_k$ *for all* $k \geq 2$.
*(ii)* $\mathcal{B}(\mathcal{H}) = \sum_2 \mathcal{W}_k$ *for all* $k \geq 6$.
*(iii)* $\mathcal{B}(\mathcal{H}) = \sum_6 \mathcal{V}_k$ *for all* $k \geq 2$.

**Proof** (i) From 1.3(iii), $\mathcal{D}_1 \subseteq \mathcal{W}_2$. In particular, $I \in \mathcal{W}_2$, and so $\mathcal{D}_1 \subseteq \mathcal{W}_{2k}$ for all $k \geq 1$. Thus $\mathcal{B}(\mathcal{H}) = \sum_3 \mathcal{D}_1 = \sum_3 \mathcal{W}_{2k}$ for all $k \geq 1$.

From this we see that if $X \in \mathcal{B}(\mathcal{H})$, then $XB \in \sum_3 \mathcal{W}_{2k}$, and hence $X = (XB)B^* \in \sum_3 \mathcal{W}_{2k+1}$ for all $k \geq 1$.

(ii) By Lemma 4.3 of [12], every normal operator is a product of 3 diagonal operators. Since each diagonal is in turn a product of two bilateral weighted shifts, $\mathcal{N}_1 \subseteq \mathcal{W}_6$, from which we obtain $\mathcal{B}(\mathcal{H}) = \mathcal{W}_6 + \mathcal{W}_6$. Again, since $I \in \mathcal{W}_2$, it follows that $\mathcal{B}(\mathcal{H}) = \mathcal{W}_{2k} + \mathcal{W}_{2k}$ for all $k \geq 3$. Finally, as in part (i) above, if $X \in \mathcal{B}(\mathcal{H})$, then $XB \in \mathcal{W}_{2k} + \mathcal{W}_{2k}$ implies that $X = (XB)B^* \in \mathcal{W}_{2k+1} + \mathcal{W}_{2k+1}$ for all $k \geq 3$.

(iii) Again, from 1.3(ii), we see that if $D$ is a diagonal operator, then $D \oplus 0 \in \mathcal{V}_2$. Thus $I \oplus 0 \in \mathcal{V}_2$, and therefore $D \oplus 0 = (D \oplus 0)(I \oplus 0)^j \in \mathcal{V}_{2+2j}$ for all $j \geq 1$. Since every diagonal operator is the sum of two diagonal operators, each unitarily equivalent to one of the form $D \oplus 0$, and since every operator is the sum of three diagonal operators, it follows that $\mathcal{B}(\mathcal{H}) = \sum_6 \mathcal{V}_{2k}$ for all $k \geq 1$.

Next, in an argument similar to that above, let $X \in \mathcal{B}(\mathcal{H})$. Then $XS^* \in \sum_6 \mathcal{V}_{2k}$ and so $X = (XS^*)S \in \sum_6 \mathcal{V}_{2k+1}$ for each $k \geq 1$. ■

***Proposition 3.5***

*(i)* $\mathcal{B}(\mathcal{H}) = \overline{\mathcal{W}_k} + \overline{\mathcal{W}_k}$ *for every* $k \geq 2$.
*(ii)* $\mathcal{B}(\mathcal{H}) = \sum_4 \overline{\mathcal{V}_k}$ *for each* $k \geq 2$.
*(iii)* $\mathcal{B}(\mathcal{H}) = \overline{\mathcal{V}_k + \mathcal{V}_k}$ *for each* $k \geq 6$.

**Proof** (i) By Lemma 2.2(i), $\mathcal{N}_1 \subseteq \overline{\mathcal{W}_2}$, and so $\mathcal{B}(\mathcal{H}) = \sum_2 \overline{\mathcal{W}_2} \subseteq \sum_2 \overline{\mathcal{W}_{2+2j}}$ for all $j \geq 1$. As in the previous proof, $X \in \mathcal{B}(\mathcal{H})$ implies $XB^* \in \sum_2 \overline{\mathcal{W}_{2k}}$ and hence $X = (XB^*)B \in \sum_2 \overline{\mathcal{W}_{2k+1}}$ for all $k \geq 1$.

(ii) Now for all normal operators $N$, we have seen that $N \oplus 0 \in \overline{\mathcal{V}_2}$. Recall that $I \oplus S \in \overline{\mathcal{V}_1}$, and hence $N \oplus 0 = (N \oplus 0)(I \oplus S)^j \in \overline{\mathcal{V}_{2+j}}$ for all $j \geq 0$. Given $N \in \mathcal{N}_1$, write $N = (N_1 \oplus 0) + (0 \oplus N_2)$ to get $\mathcal{N}_1 \subseteq \overline{\mathcal{V}_{2+j}} + \overline{\mathcal{V}_{2+j}}$. Then $\mathcal{B}(\mathcal{H}) = \sum_4 \overline{\mathcal{V}_k}$ for all $k \geq 2$.

(iii) By Corollary 2.3 of [12], every operator of the form $T \oplus 0^{(\infty)}$ is a product of three diagonal operators. From this it easily follows that $T \oplus 0^{(\infty)} \oplus 0^{(\infty)}$ can be written as $(D_1 \oplus 0^{(\infty)}) + (D_2 \oplus 0^{(\infty)}) + (D_3 \oplus 0^{(\infty)})$ where each $D_i$ is diagonal. Since $D_i \oplus 0^{(\infty)} \in \mathcal{V}_2$, we obtain $T \oplus 0^{(\infty)} \in \mathcal{V}_6$ for any $T \in \mathcal{B}(\mathcal{H})$.

We can then apply Voiculescu's Theorem [28] to write $X \simeq_a X \oplus \rho\big(\pi(X)\big)^{(\infty)}$, where $\rho\colon C^*\big(\pi(X)\big) \to \mathcal{B}(\mathcal{H}_\rho)$ is a faithful representation of the $C^*$-algebra generated by $\pi(X)$ in the Calkin algebra. Since $X \oplus 0^{(\infty)}$ and $0^{(\infty)} \oplus \rho\big(\pi(X)\big)^{(\infty)}$ both lie in $\mathcal{V}_6$, it follows that $X \in \overline{\mathcal{V}_6 + \mathcal{V}_6}$. The usual trick now shows that $\mathcal{B}(\mathcal{H}) = \overline{\mathcal{V}_k + \mathcal{V}_k}$ for each $k \geq 6$. ■

## 4 Miscellanea

### 4.1

We shall finish this note by pointing out some open questions as well as a couple of miscellaneous results on sums of Laurent and Toeplitz operators. Recall that an operator $N$ is called a *Laurent operator* if $N$ is unitarily equivalent to a multiplication operator $M_f$ on $L^2(\mathbb{T}, dm)$, where $dm$ represents normalized Lebesgue arclength measure and $f \in L^\infty(\mathbb{T}, dm)$; equivalently, $N$ admits a matrix form $N = [n_{i,j}]_{i,j\in\mathbb{Z}}$

for which $n_{i,j} = n_{i+1,j+1}$ for all $i, j$. If $P$ denotes the projection of $L^2(\mathbb{T}, dm)$ onto the Hardy space $H^2(\mathbb{T}, dm)$, and if $f \in L^\infty(\mathbb{T}, dm)$, then the corresponding *Toeplitz operator* is defined by $T_f(\varphi) = P(f\varphi)$ for each $\varphi \in H^2$. Equivalently, $T$ defines a Toeplitz operator if $T$ admits a matrix $[t_{i,j}]_{i,j=1}^\infty$ with $t_{i,j} = t_{i+1,j+1}$ for all $i, j$. We denote the set of Laurent and Toeplitz operators by $\mathcal{L}$ and $\mathcal{T}$ respectively. A useful characterization (for our purposes) of Laurent operators was given by A. Brown and P. R. Halmos.

***Theorem 4.2*** **(Brown-Halmos [4])** *A normal operator $N$ is a Laurent operator if and only if $N$ has no eigenvalues of finite multiplicity.*

***Proposition 4.3*** $\mathcal{N}_1 \subseteq \sum_3 \mathcal{L}$. *Thus* $\mathcal{B}(\mathcal{H}) = \sum_6 \mathcal{L}$.

**Proof** Choose $N \in \mathcal{N}_1$, and write $N = N_1 \oplus N_2$ where each summand acts on an infinite dimensional space. Let $M = N^{(\infty)} \oplus -N^{(\infty)}$ and observe that $M \in \mathcal{L}$ by Theorem 4.2. Now $M \simeq N_1 \oplus M \simeq -M \oplus -M \simeq M \oplus N_2$, and therefore $N \in \mathcal{L}_3$, being the sum of three operators in $\mathcal{L}$. The second statement is trivial. ■

***Proposition 4.4*** *If $N$ is normal, then $N \in \overline{\mathcal{L} + \mathcal{L}}$. Hence $\mathcal{B}(\mathcal{H}) = \overline{\sum_4 \mathcal{L}}$.*

**Proof** By the Weyl-von Neumann-Berg-Sikonia Theorem, given $\varepsilon > 0$ we can perturb $N$ by a compact operator $K$ with $\|K\| < \varepsilon$ to obtain $N + K \simeq D_1 \oplus D_2^{(\infty)}$, where $D_1$ and $D_2$ are diagonal and act on infinite dimensional spaces.

Then $X_1 = D_1 \oplus D_1^{(\infty)} \oplus 0^{(\infty)}$ and $X_2 = 0^{(\infty)} \oplus (D_2 - D_1)^\infty \oplus D_2^{(\infty)} \in \mathcal{L}$ by Theorem 4.2. As such, $N + K \simeq X_1 + X_2 \in \mathcal{L} + \mathcal{L}$, and hence $\mathcal{N}_1 \in \overline{\mathcal{L} + \mathcal{L}}$. From this the result follows. ■

We now need an auxiliary result, not unrelated to the Marriage Lemma.

***Lemma 4.5*** **(Ménage-à-trois)** *Let $\{d_n\}_{n=1}^\infty$ be an enumeration of the rational numbers in $[-1, 1]$ with each number appearing infinitely often in the sequence. Let $\{h_n\}_{n=1}^\infty$ be any sequence of rational numbers in $[-1, 1]$. Then there exist three permutations $\alpha$, $\beta$, and $\gamma$ of $\mathbb{N}$ so that*

$$h_n = d_{\alpha(n)} + d_{\beta(n)} + d_{\gamma(n)} \quad \text{for all } n \geq 1.$$

**Proof** Choose $\alpha(1) = 1$. Select $\beta(1), \gamma(1) \in \mathbb{N}$ so that $d_{\beta(1)} = -d_{\alpha(1)}$ and $d_{\gamma(1)} = h_1$. Next, choose $\beta(2) = \min\bigl(\mathbb{N} \setminus \{\beta(1)\}\bigr)$. Choose $\gamma(2) \in \mathbb{N} \setminus \{\gamma(1)\}$ and $\alpha(2) \in \mathbb{N} \setminus \{\alpha(1)\}$ so that $d_{\gamma(2)} = -d_{\beta(2)}$ and $d_{\alpha(2)} = h_2$. Thirdly, choose $\gamma(3) = \min\bigl(\mathbb{N} \setminus \{\gamma(i)\}_{i=1}^2\bigr)$. Choose $\alpha(3) \in \mathbb{N} \setminus \{\alpha(i)\}_{i=1}^2$, $\beta(3) \in \mathbb{N} \setminus \{\beta(i)\}_{i=1}^2$ so that $d_{\alpha(3)} = -d_{\gamma(3)}$ and $d_{\beta(3)} = h_3$.

In general, given $k \geq 1$, suppose that $\alpha(j)$, $\beta(j)$, and $\gamma(j)$ have been chosen, $1 \leq j \leq 3k$.

Choose $\alpha(3k+1) = \min\bigl(\mathbb{N} \setminus \{\alpha(j)\}_{j=1}^{3k}\bigr)$. Select $\beta(3k+1) \in \mathbb{N} \setminus \{\beta(j)\}_{j=1}^{3k}$ and $\gamma(3k+1) \in \mathbb{N} \setminus \{\gamma(j)\}_{j=1}^{3k}$ so that $d_{\beta(3k+1)} = -d_{\alpha(3k+1)}$ and $d_{\gamma(3k+1)} = h_{3k+1}$. Choose $\beta(3k+2) = \min\bigl(\mathbb{N} \setminus \{\beta(j)\}_{j=1}^{3k+1}\bigr)$. Select $\gamma(3k+2) \in \mathbb{N} \setminus \{\gamma(j)\}_{j=1}^{3k+1}$

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and $\alpha(3k+2) \in \mathbb{N} \setminus \{\alpha(j)\}_{j=1}^{3k+1}$ so that $d_{\gamma(3k+2)} = -d_{\beta(3k+2)}$ and $d_{\alpha(3k+2)} = h_{3k+2}$. Choose $\gamma(3k+3) = \min\bigl(\mathbb{N} \setminus \{\gamma(j)\}_{j=1}^{3k+2}\bigr)$. Select $\alpha(3k+3) \in \mathbb{N} \setminus \{\alpha(j)\}_{j=1}^{3k+2}$ and $\beta(3k+3) \in \mathbb{N} \setminus \{\beta(j)\}_{j=1}^{3k+2}$ so that $d_{\alpha(3k+3)} = -d_{\gamma(3k+3)}$ and $d_{\beta(3k+3)} = h_{3k+3}$.

That $\alpha$, $\beta$, and $\gamma$ are injective is clear from the construction. That they are surjective follows from the fact that $\alpha(3k+1) = \min\bigl(\mathbb{N} \setminus \{\alpha(j)\}_{j=1}^{3k}\bigr) \geq k+1$, $\beta(3k+2) = \min\bigl(\mathbb{N} \setminus \{\beta(j)\}_{j=1}^{3k+1}\bigr) \geq k+1$, and $\gamma(3k+3) = \min\bigl(\mathbb{N} \setminus \{\gamma(j)\}_{j=1}^{3k+2}\bigr) \geq k+1$ for all $k \geq 1$. ■

Using what is essentially the same argument as in Remark 3.3, we can show that we can not do this with only two permutations. Let $h_n = \frac{1}{2}$ for all $n \geq 1$. If $h_n = d_{\alpha(n)} + d_{\beta(n)}$ for each $n \geq 1$, then for some $m \geq 1$, $d_{\alpha(m)} = -1$, so that $d_{\beta(m)} = \frac{3}{2}$, a contradiction.

***Theorem 4.6*** *If $H$ is a self-adjoint operator, then $H \in \overline{\sum_3 \mathcal{T}}$. Thus $\mathcal{B}(\mathcal{H}) = \overline{\sum_6 \mathcal{T}}$.*

**Proof** As always, we may assume that $\|H\| \leq 1$. By Theorem 1.7, given $\varepsilon > 0$, we can approximate $H$ by a self-adjoint diagonal operator $H_1 = \operatorname{diag}\{h_n\}_{n=1}^{\infty}$ satisfying $h_n \in [-1,1] \cap \mathbb{Q}$ for all $n \geq 1$, and $\|H - H_1\| < \varepsilon$.

Let $\{d_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers in $[-1,1]$ with each number appearing infinitely often, and set $D = \operatorname{diag}\{d_n\}_{n=1}^{\infty}$. Let $f \in L^{\infty}(\mathbb{T}, dm)$ be the function $f(z) = (z + \bar{z})/2$. Then $f = \bar{f}$ and $\operatorname{ran} f = [-1,1]$. Thus $T_f = T_{\bar{f}} = T_f^*$ and by the Hartman-Wintner Theorem ([8], Theorem 7.20), $\sigma(T_f) = [-1,1]$. Once again, Theorem 1.7 implies that $T_f \simeq_a D$, and hence $D \in \bar{\mathcal{T}}$. The permutations $\alpha$, $\beta$ and $\gamma$ of Lemma 4.5 give rise to three unitary operators $U_1$, $U_2$ and $U_3$ so that $D_1 := U_1^* D U_1 = \operatorname{diag}\{d_{\alpha(n)}\}_{n=1}^{\infty}$, $D_2 := U_2^* D U_2 = \operatorname{diag}\{d_{\beta(n)}\}_{n=1}^{\infty}$, and $D_3 := U_3^* D U_3 = \operatorname{diag}\{d_{\gamma(n)}\}_{n=1}^{\infty}$.

Thus $T_f \simeq_a D_i \in \bar{\mathcal{T}}$, $1 \leq i \leq 3$. Furthermore, $H_1 = D_1 + D_2 + D_3 \in \sum_3 \bar{\mathcal{T}}$. As such, $\|H - (D_1 + D_2 + D_3)\| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $H \in \overline{\sum_3 \mathcal{T}}$. ■

## 4.7 Open Questions

A number of open questions remain. Among the most interesting is one posed by Halmos [14] and again by Herrero [18]: what is the closure of the set of weighted shifts? Again, the answer depends upon one's notion of "weighted shift", so we propose this question for both unilateral shifts and bilateral shifts as defined in this note.

A second question is to find optimal bounds for the estimates obtained here. In particular, can every operator be expressed as a sum of fewer than 18 unilateral weighted shifts?

Proposition 4.3 for Laurent operators clearly suggests tackling the same question for Toeplitz operators. The lack of an analogue of the Brown-Halmos Theorem for Toeplitz operators provides an impediment. On the other hand, the fact that every operator is a finite sum of Toeplitz operators follows simply from the fact that this latter set is closed under unitary conjugation as well as scalar multiplication. The set of finite sums of Toeplitz operators therefore coincides with the linear span of this set, and forms a unitarily invariant linear manifold of $\mathcal{B}(\mathcal{H})$. Since it is not contained in

$\mathbb{C}I + \mathcal{K}(\mathcal{H})$, a result of Fong, Miers and Sourour [10] implies that it coincides with $\mathcal{B}(\mathcal{H})$.

There is nothing special about the Toeplitz operators in this regard. Let $T$ be any operator which is not of the form scalar plus compact. Let $\mathcal{U}(T)$ denote the set of all operators unitarily equivalent to $T$. The same result implies that $\operatorname{span}\mathcal{U}(T) = \mathcal{B}(\mathcal{H})$. (Note: this fact was independently observed by P. Y. Wu [32].) Given $X \in \mathcal{B}(\mathcal{H})$, let $\beta_T(X) = \min\{k \in \mathbb{N} : X \in \sum_k \mathbb{C}\mathcal{U}(T)\}$; that is, $\beta_T(X)$ denotes the minimum number $k$ such that $X = \sum_{j=1}^k \lambda_j T_j$ for some $\lambda_j \in \mathbb{C}$ and $T_j \in \mathcal{U}(T)$, $1 \leq j \leq k$. Set $\beta_T = \sup_{X\in\mathcal{B}(\mathcal{H})} \beta_T(X)$. For which $T \in \mathcal{B}(\mathcal{H})$ is $\beta_T$ finite? There is also a topological version of this problem, namely: set $\bar{\beta}_T(X) = \min\{k \in \mathbb{N} : X \in \overline{\sum_k \mathbb{C}\mathcal{U}(T)}\}$ and $\bar{\beta}_T = \sup_{X\in\mathcal{B}(\mathcal{H})} \bar{\beta}_T(X)$. Since $\operatorname{span}\mathcal{U}(T) = \mathcal{B}(\mathcal{H})$ from above, it follows that $\mathcal{B}(\mathcal{H}) = \bigcup_{k=1}^\infty \overline{\sum_k \mathbb{C}\mathcal{U}(T)}$. The Baire Category Theorem says that $\mathcal{B}(\mathcal{H})$ is not a countable union of closed, nowhere dense sets, and hence there exists $k_0 \in \mathbb{N}$ so that $\overline{\sum_{k_0} \mathbb{C}\mathcal{U}(T)}$ has interior. It follows that $\overline{\sum_{2k_0} \mathbb{C}\mathcal{U}(T)}$ has the zero operator in its interior, and thus $\overline{\sum_{2k_0} \mathbb{C}\mathcal{U}(T)}$ coincides with $\mathcal{B}(\mathcal{H})$. In particular, $\bar{\beta}_T$ is finite for all operators $T \in \mathcal{B}(\mathcal{H})$ which are not of the form scalar plus compact.

The proof of Theorem 4.6 shows that if $R = R^*$ and $\sigma(R) = [-1, 1]$, then $\bar{\beta}_R \leq 6$. We conclude with an example of an operator $T$ for which $\bar{\beta}_T = 3$.

***Example 4.7*** Let $T = \bigoplus_{n=1}^\infty \bigoplus_{j=1}^\infty (T_{n,j} \oplus -T_{n,j})^{(\infty)}$, where $\{T_{n,j}\}_{j=1}^\infty$ is a dense subset of the unit ball of $\mathbb{M}_n(\mathbb{C})$ for all $n \geq 1$. If follows from [16], Corollary 4.2 that if $Y \in \mathcal{B}(\mathcal{H})$, $\|Y\| \leq 1$, then $T \simeq_a T \oplus Y$. We may now recycle the proof of Proposition 3.5.

Given $X \in \mathcal{B}(\mathcal{H})$, $\|X\| \leq 1$, we can apply Voiculescu's Theorem to obtain a faithful representation $\rho$ of $C^*\big(\pi(X)\big)$ so that $X \simeq_a X \oplus \rho\big(\pi(X)\big)^{(\infty)}$. Since $T^{(\infty)} \simeq T \simeq -T$, we have $Z_1 = -T \oplus -T$, $Z_2 = X \oplus T$, and $Z_3 = T \oplus \rho\big(\pi(X)\big)^{(\infty)}$ which are all approximately unitarily equivalent to $T$. Then $X \simeq_a Z_1 + Z_2 + Z_3 \in \overline{\sum_3 \mathbb{C}\mathcal{U}(T)}$. Since $X \in \mathcal{B}(\mathcal{H})$ was arbitrary, $\bar{\beta}_T \leq 3$.

It is not difficult to see that $\sigma(T) = \bar{\mathbb{D}}$, and that for any $\lambda \in \mathbb{T}$, $T \simeq_a \lambda T$. The argument used in Remark 3.3 implies that $I \notin \sum_2 \mathbb{C}\overline{\mathcal{U}(T)}$, and hence $\bar{\beta}_T \geq 3$. In conclusion, $\bar{\beta}_T = 3$.

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